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Research Article

Fractional Calculus of Fractal Interpolation Function on $[0, b](b > 0)$

XueZai Pan[1,2]

[1] *Faculty of Science, Jiangsu University, Zhenjiang 212013, China*
[2] *School of Mathematics, Nanjing Normal University Taizhou College, Taizhou 225300, China*

Correspondence should be addressed to XueZai Pan; xzpan1975@163.com

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The paper researches the continuity of fractal interpolation function's fractional order integral on $[0, +\infty)$ and judges whether fractional order integral of fractal interpolation function is still a fractal interpolation function on $[0, b](b > 0)$ or not. Relevant theorems of iterated function system and Riemann-Liouville fractional order calculus are used to prove the above researched content. The conclusion indicates that fractional order integral of fractal interpolation function is a continuous function on $[0, +\infty)$ and fractional order integral of fractal interpolation is still a fractal interpolation function on the interval $[0, b]$.

1. Introduction

Fractal geometry is a subject in which very irregular and complex phenomena and pictures in nature are researched. From the process of fractal development, there have been some effective methods used in studying fractals so far. For example, Mandelbrot [1–5] applied concept of fractal dimension to describe the roughness of fractal curves and fractal surfaces. Barnsley [6, 7] and Massopust [8, 9] proposed that fractal interpolation curve (refer to Figure 1) and fractal interpolation surface (refer to Figure 2) can be applied in fitting and analyzing the shape of the natural graphs. Feng et al. [10–12] proposed concept and principle of fractal variation and used it in estimating the Minkowski dimension of fractal surface and describing the roughness of fractal surface. Li and Wu [13] applied wavelet analysis in researching fractal geometry. Ran and Tan [14] and Mark [15] discussed the relationship between Fourier analysis and wavelets analysis. Generally, researchers always attempt to research fractals through classical integer order calculus. However, it is very rare that fractals are researched through fractional order calculus. Because classical integer order calculus researches smooth curves and surfaces, it almost cannot be applied in analyzing and dealing with fractal problems, while fractional calculus is regarded as an important and effective tool applied in researching fractal interpolation function.

In order to discuss property of fractal interpolation function's fractional order integral, the following content is discussed that the continuity of fractal interpolation function's fractional integral on $[0, +\infty)$ and judge whether fractional integral of fractal interpolation function is still a fractal interpolation function on $[0, b](b > 0)$ or not. So iterated function system, concepts, and theorems about Riemann-Liouville fractional order integral are used to prove the above problems. The results indicate that fractional order integral of self-affine transformation's fractal interpolation function is continuous on $[0, +\infty)$ and it is still a fractal interpolation function on $[0, b](b > 0)$.

2. Main Concepts and Lemmas

Definition 1 (see [16]). Let $\nu > 0$, and function f is a continuous function on interval $(0, +\infty)$ and can be integral on any bounded subinterval included in $[0, +\infty)$; then the following formula

$$I^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt \tag{1}$$

is called ν-order Riemann-Liouville fractional integral of f.

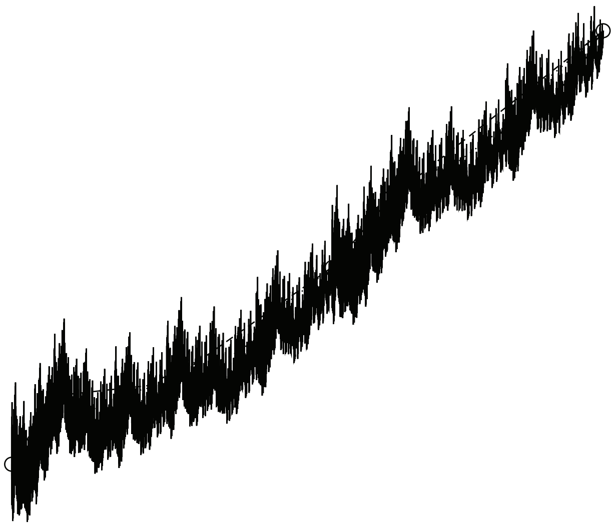

Figure 1: Fractal interpolation curve.

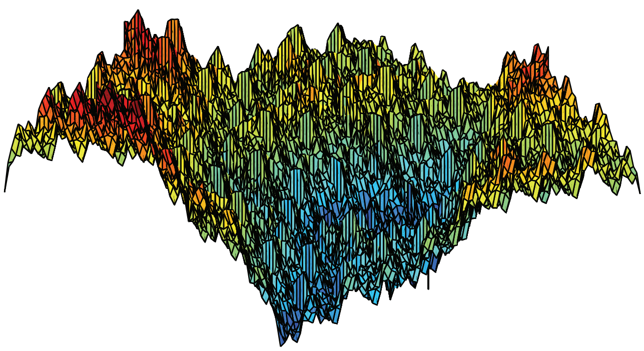

Figure 2: Fractal interpolation surface.

Definition 2 (see [17]). A "hyperbolic" iterated function system consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_i : X \to X$, with respective contractivity mappings factors s_i, for $i = 1, 2, \ldots, n$. The abbreviation "IFS" is used for "iterated function system." The notation for the IFS just announced is $\{X; w_i, i = 1, 2, \ldots, n\}$ and contractivity factor is $s = \max\{s_i : i = 1, 2, \ldots, n\}$.

Definition 3 (see [7]). Let $\{(x_i, y_i) \in R^2; i = 0, 1, 2, \ldots, n\}$ be a set of points, where $x_0 < x_1 < x_2 < \cdots < x_n$. An interpolation function corresponding to this set of data is a continuous function $f : [x_0, x_n] \to R$ such that

$$f(x_i) = y_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

The points (x_i, y_i) are called the interpolation points. It is called that the function of f interpolates the data and that the graph of f passes through the interpolation points.

Lemma 4 (see [17]). *Let n be a positive integer greater than 1. Let $\{R^2; w_i, i = 1, 2, \ldots, n\}$ denote the IFS defined above, associated with the data set*

$$\{(x_i, y_i) \in R^2; i = 0, 1, 2, \ldots, n\}. \tag{3}$$

Let the vertical scaling factor d_i obey $0 \le d_i < 1$ for $i = 1, 2, \ldots, n$. Then there is a metric d on R^2, equivalent to the Euclidean metric, such that the IFS is hyperbolic with respect to d. In particular, there is a unique nonempty compact set $G \subset R^2$, such that

$$G = \bigcup_{i=1}^{n} w_i(G). \tag{4}$$

In particular, an IFS of the form $\{R^2; w_i, i = 1, 2, \ldots, n\}$ is considered, where the mapping is an affine transformation of the special structure

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}. \tag{5}$$

The transformations are constrained by the data according to

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \quad w_i \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, n, \tag{6}$$

and a_i, e_i, c_i, f_i can be solved from (5) and (6) in terms of the data and vertical scaling factor d_i as follows:

$$\begin{aligned} a_i &= \frac{x_i - x_{i-1}}{x_n - x_0}, \qquad e_i = \frac{x_n x_{i-1} - x_0 x_i}{x_n - x_0}, \\ c_i &= \frac{y_i - y_{i-1}}{x_n - x_0} - \frac{d_i(y_n - y_0)}{x_n - x_0}, \\ f_i &= \frac{x_n y_{i-1} - x_0 y_i}{x_n - x_0} - \frac{d_i(x_n y_0 - x_0 y_n)}{x_n - x_0}. \end{aligned} \tag{7}$$

Lemma 5 (see [18]). *Suppose F is a set of continuous functions which satisfy $f : [x_0, x_n] \to R$ and $f(x_0) = y_0$, $f(x_n) = y_n$. The metric is defined by the following formula:*

$$d(f, g) = \max|f(x) - g(x)|, \quad x \in [x_0, x_n], \ \forall f, g \in F. \tag{8}$$

Then (F, d) is a complete metric space. Let the real numbers a_i, c_i, e_i, f_i, be defined by (7). Define a mapping $T : F \to F$ by

$$\begin{aligned} (Tf)(x) &= c_i L_i^{-1}(x) + d_i f(L_i^{-1}(x)) + f_i, \\ x &\in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n, \end{aligned} \tag{9}$$

where $L_i : [x_0, x_n] \to [x_{i-1}, x_i]$ is the invertible transformation

$$\begin{aligned} L_i(x) &= a_i x + e_i, \\ L_i^{-1}(x) &= \frac{x - e_i}{a_i}, \qquad L_i^{-1}(x_{i-1}) = x_0, \qquad L_i^{-1}(x_i) = x_n, \end{aligned} \tag{10}$$

and then Tf is continuous on the interval $[x_{i-1}, x_i]$ and T is a contraction mapping on (F, d), so T possesses a unique fixed point in F. That is, there exists a function $f \in F$ such that

$$Tf = f, \quad \forall f \in F. \tag{11}$$

3. The Continuity of Fractal Interpolation Function's Fractional Order Integral on the Interval $[0, +\infty)$

Lemma 6. *If f is a continuous function on the interval $[0, +\infty)$ and $0 < \nu < 1$, then $I^{-\nu} f(x)$ is a continuous function on $[0, +\infty)$ too.*

Proof. Since

$$\begin{aligned} & I^{-\nu} f(x+\Delta x) - I^{-\nu} f(x) \\ &= \frac{1}{\Gamma(\nu)} \int_0^{x+\Delta x} (x+\Delta x - t)^{\nu-1} f(t)\,dt \\ &\quad - \frac{1}{\Gamma(\nu)} \int_0^{x} (x-t)^{\nu-1} f(t)\,dt \\ &= \frac{1}{\Gamma(\nu)} \left(\int_0^{\Delta x} (x+\Delta x - t)^{\nu-1} f(t)\,dt \right. \\ &\qquad + \int_{\Delta x}^{x+\Delta x} (x+\Delta x - t)^{\nu-1} f(t)\,dt \\ &\qquad \left. - \int_0^{x} (x-t)^{\nu-1} f(t)\,dt \right) \\ &= \frac{1}{\Gamma(\nu)} \left(\int_0^{\Delta x} [x + (\Delta x - t)]^{\nu-1} f(t)\,dt \right. \\ &\qquad + \int_0^{x} (x-y)^{\nu-1} f(y+\Delta x)\,dy \\ &\qquad \left. - \int_0^{x} (x-t)^{\nu-1} f(t)\,dt \right), \end{aligned} \tag{12}$$

then

$$\begin{aligned} & \left| I^{-\nu} f(x+\Delta x) - I^{-\nu} f(x) \right| \\ &\leq \frac{1}{|\Gamma(\nu)|} \left(\left| \int_0^{\Delta x} x^{\nu-1} M dt \right| \right. \\ &\qquad \left. + \int_0^{x} (x-t)^{\nu-1} \left| f(t+\Delta x) - f(t) \right| dt \right) \\ &\leq \frac{1}{|\Gamma(\nu)|} \left(x^{\nu-1} M \Delta x + \varepsilon \int_0^{x} (x-t)^{\nu-1} dt \right) \\ &= \frac{1}{|\Gamma(\nu)|} \left(x^{\nu-1} M \Delta x + \frac{\varepsilon}{\nu} x^{\nu} \right) \longrightarrow 0, \end{aligned} \tag{13}$$

where $M = \max_{t \in [0, \Delta x]} |f(t)|$, so $I^{-\nu} f(x)$ is a continuous function on the interval $[0, +\infty)$. □

Corollary 7. *Suppose $f(x)$ is a fractal interpolation function on the interval $[0, +\infty)$; then $I^{-\nu} f(x)$ is continuous on $[0, +\infty)$ too.*

Proof. Since fractal interpolation function of affine transformation is a continuous function on $[0, +\infty)$, $f(x)$ is continuous on $[0, +\infty)$. According to Lemma 6, $I^{-\nu} f(x)$ is a continuous function on $[0, +\infty)$ too. □

Corollary 8. *Suppose $f(x)$ is a fractal interpolation function of affine transformation on the interval $[a, b] (0 < a < b < +\infty)$; then $I^{-\nu} f(x)$ can be integrated on $[a, b]$.*

Proof. From Corollary 7, and since continuous function on finite closed interval is an integrated function, the result of Corollary 8 is right. □

4. Judgement Theorem of Fractal Interpolation Function's Fractional Integral on $[0, b] (b > 0)$

Theorem 9. *If $f(x)$ is a fractal interpolation function of affine transformation on the interval $[0, b] (b > 0)$, then $I^{-\nu} f(x)$ is a fractal interpolation function of affine transformation on the interval $[0, b]$ too.*

Proof. For all $b \in (0, +\infty)$, consider the interval $[0, b]$, for $0 < \nu < 1$, $x \in [x_{i-1}, x_i] (x_i = (i/n) b$, $i = 1, 2, \ldots, n)$. Let iterated function system (IFS) be

$$\begin{aligned} L_i(x) &= \frac{1}{n} x + \frac{i-1}{n} b, \\ F_i(x, y) &= c_i x + d_i y + f_i, \end{aligned} \tag{14}$$

so

$$L_i^{-1}(x) = nx - (i-1) b,$$

$$\begin{aligned} I^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^{x} (x-t)^{\nu-1} f(t)\,dt \\ &= \frac{1}{\Gamma(\nu)} \sum_{i=1}^{k-1} \int_{x_{i-1}}^{x_i} (x-t)^{\nu-1} f(t)\,dt \\ &\quad + \frac{1}{\Gamma(\nu)} \int_{x_{k-1}}^{x} (x-t)^{\nu-1} f(t)\,dt \\ &= \frac{1}{\Gamma(\nu)} \sum_{i=1}^{k-1} \int_0^{b} \left(x - \frac{1}{n} y - \frac{i-1}{n} b \right)^{\nu-1} \\ &\qquad \times f[L_i(y)] \frac{1}{n} dy \\ &\quad (\text{Let } t = L_i(y)) \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^{L_k^{-1}(x)} \left(x - \frac{1}{n} y - \frac{k-1}{n} b \right)^{\nu-1} \\ &\qquad \times f[L_k(y)] \frac{1}{n} dy \\ &= \frac{1}{\Gamma(\nu) n^{\nu}} \sum_{i=1}^{k-1} \int_0^{b} [nx - y - (i-1) b]^{\nu-1} \\ &\qquad \times [c_i y + d_i f(y) + f_i]\,dy \\ &\quad + \frac{1}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} [nx - y - (k-1) b]^{\nu-1} \\ &\qquad \times (c_k y + d_k f(y) + f_k)\,dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu) n^{\nu}} \sum_{i=1}^{k-1} \int_0^b \left[L_i^{-1}(x) - y\right]^{\nu-1} \\
&\qquad \times \left[c_i y + d_i f(y) + f_i\right] dy \\
&\quad + \frac{d_k}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} \left[L_k^{-1}(x) - y\right]^{\nu-1} f(y)\, dy \\
&\quad + \frac{1}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} \left[L_k^{-1}(x) - y\right]^{\nu-1} \\
&\qquad \times \left(c_k y + f_k\right) dy \\
&= \frac{1}{\Gamma(\nu) n^{\nu}} \sum_{i=1}^{k-1} \int_0^b \left[L_i^{-1}(x) - y\right]^{\nu-1} \\
&\qquad \times \left[c_i y + d_i f(y) + f_i\right] dy \\
&\quad + \frac{1}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} \left[L_k^{-1}(x) - y\right]^{\nu-1} \\
&\qquad \times \left(c_k y + f_k\right) dy + \frac{d_k}{n^{\nu}} y, \\
&= \frac{1}{\Gamma(\nu) n^{\nu}} \\
&\quad \times \sum_{i=1}^{k-1} \Bigg[d_i \int_0^b \left(L_i^{-1}(x) - y\right)^{\nu-1} f(y)\, dy \\
&\qquad - \frac{b c_i}{\nu} \left(L_i^{-1}(x) - b\right)^{\nu} - \frac{c_i}{\nu(\nu+1)} \\
&\qquad \times \left[\left(L_i^{-1}(x) - b\right)^{\nu+1} - \left(L_i^{-1}(x)\right)^{\nu+1}\right] \\
&\qquad + b f_i \Bigg] \\
&\quad + \frac{d_k}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} \left(L_k^{-1}(x) - y\right)^{\nu-1} f(y)\, dy \\
&\quad + \frac{\left(L_k^{-1}(x)\right)^{\nu}}{\Gamma(\nu) n^{\nu}} \left[\frac{c_k}{\nu(\nu+1)} L_k^{-1}(x) + \frac{f_k}{\nu}\right].
\end{aligned} \tag{15}
$$

So $I^{-\nu} f(x)$ is a fractal interpolation function of affine transformation on the interval $[0, b]$ and its iterated function system (IFS):

$$
\begin{aligned}
L_i(x) &= \frac{1}{n} x + \frac{i-1}{n} b, \quad i = 1, 2, \ldots, n, \\
F_i(x, y) & \\
&= \frac{1}{\Gamma(\nu) n^{\nu}} \\
&\quad \times \sum_{i=1}^{k-1} \Bigg[d_i \int_0^b \left(L_i^{-1}(x) - y\right)^{\nu-1} f(y)\, dy \\
&\qquad - \frac{b c_i}{\nu} \left(L_i^{-1}(x) - b\right)^{\nu} - \frac{c_i}{\nu(\nu+1)} \\
&\qquad \times \left[\left(L_i^{-1}(x) - b\right)^{\nu+1} - \left(L_i^{-1}(x)\right)^{\nu+1}\right] + b f_i \Bigg] \\
&\quad + \frac{d_k}{\Gamma(\nu) n^{\nu}} \int_0^{L_k^{-1}(x)} \left(L_k^{-1}(x) - y\right)^{\nu-1} f(y)\, dy \\
&\quad + \frac{\left(L_k^{-1}(x)\right)^{\nu}}{\Gamma(\nu) n^{\nu}} \left[\frac{c_k}{\nu(\nu+1)} L_k^{-1}(x) + \frac{f_k}{\nu}\right],
\end{aligned} \tag{16}
$$

where $y = L_i(x)$, $x \in [0, b]$; then $\forall x \in [x_{i-1}, x_i]$,

$$
I^{-\nu} f(x) = F_i\left(L_i^{-1}(x), I^{-\nu} f\left(L_i^{-1}(x)\right)\right). \tag{17}
$$

So $I^{-\nu} f(x)$ is a fractal interpolation function on the interval $[0, b]$. □

5. Conclusion

There are three acquired results from the above content in this paper. Firstly, the fractional order integral of fractal interpolation function is continuous on the interval $[0, +\infty)$. Secondly, the fractional order integral of fractal interpolation function can be integrated on any closed interval $[a, b] \subset [0, +\infty)$. Finally, the fractional order integral of fractal interpolation function is still a fractal interpolation function on the interval $[0, b] (b > 0)$.

The fractional order integral's differentiability of fractal interpolation function and its boxing dimension will be researched in the future.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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