Research Article

Linearization of Impulsive Differential Equations with Ordinary Dichotomy

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This paper presents a linearization theorem for the impulsive differential equations when the linear system has ordinary dichotomy. We prove that when the linear impulsive system has ordinary dichotomy, the nonlinear system $\dot{x}(t) = A(t)x(t) + f(t,x)$, $t \neq t_k$, $\Delta x(t_k) = \widetilde{A}(t_k)x(t_k) + \widetilde{f}(t_k,x)$, $k \in \mathbb{Z}$, is topologically conjugated to $\dot{x}(t) = A(t)x(t)$, $t \neq t_k$, $\Delta x(t_k) = \widetilde{A}(t_k)x(t_k)$, $k \in \mathbb{Z}$, where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, represents the jump of the solution $x(t)$ at $t = t_k$. Finally, two examples are given to show the feasibility of our results.

1. Introduction

A basic linearization theorem is the famous Hartman-Grobman theorem (see [1, 2]). Then Palmer successfully generalized the standard Hartman-Grobman theorem to nonautonomous differential equations (see [3]). Then Fenner and Pinto [4] generalized Hartman-Grobman theorem to impulsive differential equations. Since they did not discuss the Hölder regularity of the topologically equivalent function $H(t,x)$, for this reason, recently, Xia et al. [5] gave a rigorous proof of the Hölder regularity. Xia et al. [6, 7] proved a version of generalized Hartman-Grobman theorem for dynamic systems on time scales. It should be noted that the abovementioned works are based on the linear differential equations with uniform exponential dichotomy. Therefore, motivated by [8], in this paper, we have a version of generalized Hartman-Grobman theorem for the impulsive differential equations when the linear system has ordinary dichotomy.

Our main objective in this paper is to prove that, when the impulsive linear system has an ordinary dichotomy, the nonlinear system

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + f(t,x), \quad t \neq t_k, \\
\Delta x(t_k) &= \widetilde{A}(t_k)x(t_k) + \widetilde{f}(t_k,x), \quad k \in \mathbb{Z},
\end{aligned}
\tag{1}
$$

is topologically conjugated to its linear part

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t), \quad t \neq t_k, \\
\Delta x(t_k) &= \widetilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z},
\end{aligned}
\tag{2}
$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, represents the jump of the solution $x(t)$ at $t = t_k$. Finally, two examples are given to show the feasibility of our results.

2. Definitions

Consider the linear nonautonomous system with impulses at times $\{t_k\}_{k \in \mathbb{Z}}$:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x, \quad t \neq t_k, \\
\Delta x(t_k) &= \widetilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z},
\end{aligned}
\tag{3}
$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, represents the jump of the solution $x(t)$ at $t = t_k$, $x \in \mathbb{R}^n$, and $A(t)$ and $\widetilde{A}(t)$ are $n \times n$ matrixes.

Definition 1. System (3) is said to be an *ordinary dichotomy*, if there exists a projection P ($P^2 = P$) and a constant $K > 0$ such that

$$\left\| \mathbf{U}(t) P \mathbf{U}^{-1}(s) \right\| \le K \quad (t \ge s),$$
$$\left\| \mathbf{U}(t)(I - P) \mathbf{U}^{-1}(s) \right\| \le K \quad (t \le s), \tag{4}$$

where $\mathbf{U}(t)$ is a fundamental matrix of linear system (3) and is given by

$$\mathbf{U}(t) = \Phi(t) \prod_{t_k \in [t_0, t)} \Phi^{-1}(t_k) \left(I + \widetilde{A}(t_k)\right) \Phi(t_k) \Phi^{-1}(t_0),$$
$$t \ge t_0, \tag{5}$$

where $\Phi(t)$ is a fundamental matrix of the system $\dot{x} = A(t)x$, provided that $\Phi(t_k)$ is invertible, for all $t_k \ge t_0$. In what follows, we will assume that $\mathbf{U}(t)$ is invertible for all $t \in \mathbb{R}$.

Definition 2. In Definition 1, if $\mathbf{U}(t)P \to 0$ as $t \to +\infty$, then system (3) is said to possess an *ordinary dichotomy with a positive asymptotically stable manifold*;

if $\mathbf{U}(t)(I - P) \to 0$ as $t \to -\infty$, then system (3) is said to possess an ordinary dichotomy with a negatively asymptotically stable manifold;

if both of them hold, then system (3) is said to possess an ordinary dichotomy with asymptotically stable manifolds.

3. Main Result and Proof

Consider the following nonautonomous impulse systems:

$$\dot{x}(t) = A(t)x, \quad t \ne t_k,$$
$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z}, \tag{6}$$

$$\dot{x}(t) = A(t)x + f(t, x), \quad t \ne t_k,$$
$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k) + \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z}, \tag{7}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, represents the jump of the solution $x(t)$ at $t = t_k$, $x \in \mathbb{R}^n$, and $A(t)$ and $\widetilde{A}(t)$ are $n \times n$ matrixes.

Definition 3. Suppose that there exists a function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

  (i) for each fixed t, $H(t, \cdot)$ is a homeomorphism of $\mathbb{R}^n$ into $\mathbb{R}^n$;

  (ii) $\|H(t, x) - x\|$ uniformly bounded with respect to t;

  (iii) assume that $G(t, \cdot) = H^{-1}(t, \cdot)$ has property (ii) also;

  (iv) if $x(t)$ is a solution of system (7), then $H(t, x(t))$ is a solution of system (6).

If such a map H exists, then system (7) is topologically conjugated to (6). H is an equivalent function.

Theorem 4. *Suppose that the impulsive linear system* (6) *has an ordinary dichotomy and for any* $x, x_1, x_2 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$ *one has*

  (H_1) $\|f(t, x)\| \le \psi(t)$,

  (H_2) $\|\widetilde{f}(t, x)\| \le \widetilde{\psi}(t)$,

  (H_3) $\|f(t, x_1) - f(t, x_2)\| \le r(t)\|x_1 - x_2\|$,

  (H_4) $\|\widetilde{f}(t, x_1) - \widetilde{f}(t, x_2)\| \le \widetilde{r}(t)\|x_1 - x_2\|$,

  (H_5) $\int_{-\infty}^{+\infty} \psi(t)dt + \sum_{t_k \in (-\infty, +\infty)} \widetilde{\psi}(t_k) < N$,

  (H_6) $\int_{-\infty}^{+\infty} r(t)dt + \sum_{t_k \in (-\infty, +\infty)} \widetilde{r}(t_k) < C$, $k \in \mathbb{Z}$,

where $\psi(t)$, $r(t)$ *are integrable functions and* $\widetilde{\psi}(t_k)$, $\widetilde{r}(t_k)$ *are summable functions in* $\mathbb{R}$, *and* C *and* N *are positive constants. Then system* (7) *is topologically conjugated to system* (6).

Remark 5 (pure continuous case). If impulsive jump operators are absent, then system (6) and (7) reduces to pure continuous systems. That is,

$$\dot{x} = A(t)x, \quad t \in \mathbb{R},$$
$$\dot{x} = A(t)x + f(t, x), \quad t \in \mathbb{R}. \tag{8}$$

Then Theorem 4 reduces to main results in [8].

Remark 6 (Pure discrete cases). A difference system, or a pure discrete-time system, is a special case of systems with impulses. Thus, instead of (6), we have only

$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z}, \tag{9}$$

or for the perturbed case (7),

$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k) + \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z}. \tag{10}$$

Now, $\Delta x(t_k)$ means $x(t_{k+1}) - x(t_k)$, so that we may write the linear system in the canonical way as

$$x(t_{k+1}) = \widehat{A}(t_k)x(t_k), \quad k \in \mathbb{Z}, \tag{11}$$

or similarly for the perturbed system,

$$x(t_{k+1}) = \widehat{A}(t_k)x(t_k) + \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z}, \tag{12}$$

where $\widehat{A} = (I + \widetilde{A})$. Then we have the following.

Corollary 7. *Suppose that* $\Delta x(t_k) = \widetilde{A}(t_k)x(t_k)$ *has an ordinary dichotomy, and for any* $x, x_1, x_2 \in \mathbb{R}^n$. *If* $f(t, x)$ *satisfies*

$$\left|\widetilde{f}(t, x)\right| \le \widetilde{\psi}(t),$$
$$\left|\widetilde{f}(t, x_1) - \widetilde{f}(t, x_2)\right| \le \widetilde{r}(t)\left|x_1 - x_2\right|,$$
$$\sum_{t_k \in (-\infty, +\infty)} \widetilde{\psi}(t_k) < N, \tag{13}$$
$$\sum_{t_k \in (-\infty, +\infty)} \widetilde{r}(t_k) < C,$$

then system (10) *is topologically equivalent to system* (9).

Remark 8. We point out that the conditions in Theorem 4 can be approached. For example, taking $\widetilde{\psi}(t) = \widetilde{r}(t) = 1/t^2$, if we assume that the interval $[n, n+1]$ contains finite number of sequences t_k, then

$$\sum_{t_k \in (-\infty, +\infty)} \widetilde{\psi}(t_k) = \sum_{t_k \in (-\infty, +\infty)} \widetilde{r}(t_k) < +\infty. \tag{14}$$

In particular, if $t = t_k = k$, $k = 1, 2, \ldots$, then

$$\sum_{t_k=1}^{+\infty} \widetilde{\psi}(t_k) = \sum_{t_k=1}^{+\infty} \widetilde{r}(t_k) = \frac{\pi^2}{6}. \tag{15}$$

Before the proof of Theorem 4, let us make some discussions about ordinary dichotomy and introduce some lemmas. Note first that if system (6) has an ordinary dichotomy then a fundamental matrix can be chosen such that the projection $P = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$ in (4). In fact, for any projection P, there exists an invertible matrix T such that $T^{-1}PT = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$. If (4) holds for $\mathbf{U}(t)$ and P; then (4) holds for $\mathbf{U}(t)$ and $T^{-1}PT$, this implies that $T^{-1}PT = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$ is the required projection if $\mathbf{U}(t)T$ is chosen as a fundamental matrix. Furthermore, in (4), we can assume that $\mathbf{U}(t) = (\phi_1(t), \ldots, \phi_n(t))$ with $\phi_i(t)$ being unbounded on $\mathbb{R}$ for $i = 1, \ldots, r - s, r + t, \ldots, n$ and $\phi_j(t)$ bounded on $\mathbb{R}$ for $j = r - s + 1, \ldots, r + t - 1$ and $P = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} = P_1 + P_2$ with $P_1 = \begin{pmatrix} I_{r-s} & \\ & 0 \end{pmatrix}$, $I - P = P_3 + P_4$ with $P_4 = \begin{pmatrix} 0 & \\ & I_{n+1-r-t} \end{pmatrix}$. Then $|\mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(s)|$ is bounded on $\mathbb{R}$, $|\mathbf{U}(t)P_1\mathbf{U}^{-1}(s)|$ is bounded on $t \geq s$ (and unbounded on $t \leq s$), and $|\mathbf{U}(t)P_4\mathbf{U}^{-1}(s)|$ is bounded on $t \leq s$ (and unbounded on $t \geq s$).

In what follows, we assume that the assumptions in Theorem 4 always hold. Let $X(t, t_0, x_0)$ be a solution of (7) satisfying the initial condition $X(t_0) = x_0$ and $Y(t, t_0, y_0)$ a solution of (6) satisfying the initial condition $Y(t_0) = y_0$.

Lemma 9. *For each (τ, ξ), the system*

$$\dot{Z} = A(t) Z - f(t, X(t, (\tau, \xi))), \quad t \neq t_k,$$

$$\Delta Z(t_k) = \widetilde{A}(t_k) Z(t_k) - \widetilde{f}(t_k, X(t_k, (\tau, \xi))), \quad k \in \mathbb{Z}, \tag{16}$$

has a unique bounded solution $h(t, (\tau, \xi))$ with $(P_2 + P_3)\mathbf{U}^{-1}(0)h(0, (\tau, \xi)) = 0$.

Proof. For each (τ, ξ), the solution of system (16) satisfying $h(0, (\tau, \xi)) = x_0$ is

$$h(t, (\tau, \xi)) = \mathbf{U}(t)\mathbf{U}^{-1}(0) x_0 - \int_0^t \mathbf{U}(t)\mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi)))\, ds$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t)\mathbf{U}^{-1}(t_k^+)\widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$= \mathbf{U}(t)\mathbf{U}^{-1}(0) x_0 - \sum_{i=1}^{4}\int_0^t \mathbf{U}(t) P_i \mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi)))\, ds$$

$$- \sum_{i=1}^{4} \sum_{t_k \in [0, t)} \mathbf{U}(t) P_i \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$= \mathbf{U}(t)\mathbf{U}^{-1}(0) x_0 - \int_{-\infty}^{t} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi)))\, ds$$

$$+ \int_{-\infty}^{0} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi)))\, ds$$

$$- \int_0^t \mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi)))\, ds$$

$$- \left(\int_0^{+\infty} - \int_t^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) \right.$$

$$\left. \times f(s, X(s, (\tau, \xi)))\, ds \right)$$

$$- \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$+ \sum_{t_k \in (-\infty, 0)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+)\widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$- \left(\sum_{t_k \in [0, +\infty)} - \sum_{t_k \in [t, +\infty)} \right) \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi))). \tag{17}$$

Noting that

$$\int_{-\infty}^{0} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi)))\, ds$$

$$= \mathbf{U}(t) P_1 \mathbf{U}^{-1}(0) \int_{-\infty}^{0} \mathbf{U}(0) P_1 \mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi)))\, ds$$

$$\triangleq \mathbf{U}(t) P_1 \mathbf{U}^{-1}(0) x_1,$$

$$\int_0^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$= \mathbf{U}(t) P_4 \mathbf{U}^{-1}(0) \int_0^{+\infty} \mathbf{U}(0) P_4 \mathbf{U}^{-1}(s)$$

$$\times f(s, X(s, (\tau, \xi))) \, ds$$

$$\triangleq \mathbf{U}(t) P_4 \mathbf{U}^{-1}(0) x_2.$$

(18)

On the other hand,

$$\sum_{t_k \in (-\infty, 0)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$= \mathbf{U}(t) P_1 \mathbf{U}^{-1}(0) \sum_{t_k \in (-\infty, 0)} \mathbf{U}(0) P_1 \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$\triangleq \mathbf{U}(t) P_1 \mathbf{U}^{-1}(0) x_3,$$

$$\sum_{t_k \in [0, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

(19)

$$= \mathbf{U}(t) P_4 \mathbf{U}^{-1}(0) \sum_{t_k \in [0, +\infty)} \mathbf{U}(0) P_4 \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$\triangleq \mathbf{U}(t) P_4 \mathbf{U}^{-1}(0) x_4.$$

It follows from (17) that

$$h(t, (\tau, \xi)) = \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(0) x_0 + \mathbf{U}(t) P_1 \mathbf{U}^{-1}(0)$$

$$\times (x_0 + x_1 + x_3)$$

$$+ \mathbf{U}(t) P_4 \mathbf{U}^{-1}(0)(x_0 - x_2 - x_4)$$

$$- \int_{-\infty}^t \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$+ \int_t^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \int_0^t \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$+ \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi))).$$

(20)

We can assert that $P_1 \mathbf{U}^{-1}(0)(x_0 + x_1 + x_3) = 0$ and $P_4 \mathbf{U}^{-1}(0)(x_0 - x_2 - x_4) = 0$ hold. Otherwise, $h(t, (\tau, \xi))$

will be unbounded since it concludes unbounded part $\mathbf{U}(t) P_1 \mathbf{U}^{-1}(0)(x_0 + x_1 + x_3)$ or/and $\mathbf{U}(t) P_4 \mathbf{U}^{-1}(0)(x_0 - x_2 - x_4)$. Thus

$$h(t, (\tau, \xi)) = \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(0) x_0$$

$$- \int_{-\infty}^t \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$+ \int_t^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \int_0^t \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$+ \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi))).$$

(21)

Moreover, if $h(t, (\tau, \xi))$ satisfies the initial condition $(P_2 + P_3) \mathbf{U}^{-1}(0) h(0, (\tau, \xi)) = 0$, then

$$h(t, (\tau, \xi)) = - \int_{-\infty}^t \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$+ \int_t^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \int_0^t \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds$$

$$- \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$+ \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi)))$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (\tau, \xi))).$$

(22)

Now, we prove that $h(t, (\tau, \xi))$ is bounded. Due to the boundedness of $\mathbf{U}(t) P_2 = (0, \ldots, 0, \phi_{r-s+1}, \ldots, \phi_r, 0, \ldots, 0)$ and $\mathbf{U}(t) P_3 = (0, \ldots, 0, \phi_{r+1}, \ldots, \phi_{r+t-1}, 0, \ldots, 0)$, we assume that $|\mathbf{U}(t) P_2 \mathbf{U}^{-1}(0)| \leq C_1$, $|\mathbf{U}(t) P_3 \mathbf{U}^{-1}(0)| \leq C_2$, where C_1, C_2 are some positive constants. Together with (4) and (H_1), it follows that, if $t \geq 0$, then we have

$$\left| \int_0^t \mathbf{U}(t) P_2 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds \right|$$

$$\leq \left| \int_0^t \left\| \mathbf{U}(t) P_2 \mathbf{U}^{-1}(s) \right\| \cdot \left\| f(s, X(s, (\tau, \xi))) \right\| \, ds \right|$$ (23)

$$\leq K \int_0^t \psi(s) \, ds;$$

if $t < 0$, then we have

$$
\left| \int_0^t \mathbf{U}(t) P_2 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds \right|
$$

$$
\leq \left| \mathbf{U}(t) P_2 \mathbf{U}^{-1}(0) \int_0^t \left\| \mathbf{U}(0) P_2 \mathbf{U}^{-1}(s) \right\|
$$

$$
\cdot \left\| f(s, X(s, (\tau, \xi))) \right\| ds \right| \tag{24}
$$

$$
\leq K C_1 \int_t^0 \psi(s) \, ds.
$$

Similarly, if $t \geq 0$, then we have

$$
\left| \int_0^t \mathbf{U}(t) P_3 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds \right|
$$

$$
\leq \left| \mathbf{U}(t) P_3 \mathbf{U}^{-1}(0) \int_0^t \left\| \mathbf{U}(0) P_3 \mathbf{U}^{-1}(s) \right\|
$$

$$
\cdot \left\| f(s, X(s, (\tau, \xi))) \right\| ds \right| \tag{25}
$$

$$
\leq K C_2 \int_0^t \psi(s) \, ds;
$$

if $t < 0$, then we have

$$
\left| \int_0^t \mathbf{U}(t) P_3 \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds \right|
$$

$$
\leq \left| \int_0^t \left\| \mathbf{U}(t) P_3 \mathbf{U}^{-1}(s) \right\| \cdot \left\| f(s, X(s, (\tau, \xi))) \right\| ds \right| \tag{26}
$$

$$
\leq K \int_t^0 \psi(s) \, ds.
$$

On the other hand, it follows from (4) and (H_2) that, if $t \geq 0$, then we have

$$
\left| \sum_{t_k \in [0,t)} \mathbf{U}(t) P_2 \mathbf{U}^{-1}(t_k) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right| \leq K \sum_{t_k \in [0,t)} \widetilde{\psi}(t_k). \tag{27}
$$

If $t < 0$, then we have

$$
\left| \sum_{t_k \in (t,0]} \mathbf{U}(t) P_2 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right|
$$

$$
\leq \left| \mathbf{U}(t) P_2 \mathbf{U}^{-1}(0) \sum_{t_k \in (t,0]} \mathbf{U}(0) P_2 \mathbf{U}^{-1}
$$

$$
\times (t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right| \tag{28}
$$

$$
\leq K C_1 \sum_{t_k \in (t,0]} \widetilde{\psi}(t_k).
$$

Similarly, if $t \geq 0$, then we have

$$
\left| \sum_{t_k \in [0,t)} \mathbf{U}(t) P_3 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right|
$$

$$
\leq \left| \mathbf{U}(t) P_3 \mathbf{U}^{-1}(0) \sum_{t_k \in [0,t)} \mathbf{U}(0) P_3 \mathbf{U}^{-1}(t_k^+)
$$

$$
\times \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right| \tag{29}
$$

$$
\leq K C_2 \sum_{t_k \in [0,t)} \widetilde{\psi}(t_k).
$$

If $t < 0$, then we have

$$
\left| \sum_{t_k \in (t,0]} \mathbf{U}(t) P_3 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right|
$$

$$
\leq K \sum_{t_k \in (t,0]} \widetilde{\psi}(t_k). \tag{30}
$$

Therefore, we get

$$
\left| \int_0^t \mathbf{U}(t) (P_2 + P_3) \mathbf{U}^{-1}(s) f(s, X(s, (\tau, \xi))) \, ds \right|
$$

$$
\leq \max \{ C_1 + 1, C_2 + 1 \} K \int_{-\infty}^{+\infty} \psi(t) \, dt
$$

$$
\leq K (C_1 + C_2 + 1) \int_{-\infty}^{+\infty} \psi(t) \, dt,
$$

$$
\left| \sum_{t_k \in [0,t)} \mathbf{U}(t) (P_2 + P_3) \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (\tau, \xi))) \right| \tag{31}
$$

$$
\leq \max \{ C_1 + 1, C_2 + 1 \} K \sum_{t_k \in (-\infty, +\infty)} \widetilde{\psi}(t_k)
$$

$$
\leq K (C_1 + C_2 + 1) \sum_{t_k \in (-\infty, +\infty)} \widetilde{\psi}(t_k).
$$

Similarly, we get

$$
\left| \int_0^t \mathbf{U}(t) (P_2 + P_3) \mathbf{U}^{-1}(s) r(s) \, ds \right|
$$

$$
\leq K (C_1 + C_2 + 1) \int_{-\infty}^{+\infty} r(t) \, dt,
$$

$$
\left| \sum_{t_k \in [0,t)} \mathbf{U}(t) (P_2 + P_3) \mathbf{U}^{-1}(t_k^+) \widetilde{r}(t_k) \right| \tag{32}
$$

$$
\leq K (C_1 + C_2 + 1) \sum_{t_k \in (-\infty, +\infty)} \widetilde{r}(t_k).
$$

It follows from (4), (H_5), (22), and (31) that

$$
\begin{aligned}
\left|h\left(t,(\tau,\xi)\right)\right| &\leq K \int_{-\infty}^{t} \psi(s)\, ds + K \int_{t}^{+\infty} \psi(s)\, ds \\
&\quad + K \sum_{t_k \in (-\infty,t)} \widetilde{\psi}(t_k) \\
&\quad + K \sum_{t_k \in [t,+\infty)} \widetilde{\psi}(t_k) + K\left(C_1 + C_2 + 1\right) \int_{-\infty}^{+\infty} \psi(t)\, dt \\
&\quad + K\left(C_1 + C_2 + 1\right) \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \\
&= K\left(\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \right) \\
&\quad + K\left(C_1 + C_2 + 1\right)\left(\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \right) \\
&< KN + KN\left(C_1 + C_2 + 1\right) \\
&= KN\left(C_1 + C_2 + 2\right).
\end{aligned}
\tag{33}
$$

So $h(t,(\tau,\xi))$ is a bounded solution, and the bounded solution is unique with the initial condition $(P_2 + P_3)\mathbf{U}^{-1}(0)h(0,(\tau,\xi)) = 0$. The proof of Lemma 9 is complete. □

Lemma 10. *For each (τ,ξ), the system*

$$
\dot{Z} = A(t)\, Z + f\left(t, Y(t,\tau,\xi) + Z\right), \quad t \neq t_k,
$$

$$
\Delta Z(t_k) = \widetilde{A}(t_k)\, Z(t_k) + \widetilde{f}\left(t_k, Y(t_k,\tau,\xi) + Z(t_k)\right),
$$

$$
k \in \mathbb{Z},
\tag{34}
$$

has a unique bounded solution $g(t,(\tau,\xi))$ with $(P_2 + P_3)\mathbf{U}^{-1}(0)g(0,(\tau,\xi)) = 0$.

Proof. For a bounded continuous function $z(t)$ of $\mathbb{R}$ whose norm $\|z\| = \sup_{t \in \mathbb{R}}|z(t)|$, we define a map T as follows:

$$
\begin{aligned}
Tz(t) &= \int_{-\infty}^{t} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s)\, f\left(s, Y(s,\tau,\xi) + z(s)\right) ds \\
&\quad - \int_{t}^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s)\, f\left(s, Y(s,\tau,\xi) + z(s)\right) ds \\
&\quad + \int_{0}^{t} \mathbf{U}(t)\left(P_2 + P_3\right) \mathbf{U}^{-1}(s) \\
&\qquad \times f\left(s, Y(s,\tau,\xi) + z(s)\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{t_k \in (-\infty,t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}\left(t_k^+\right) \\
&\qquad \times \widetilde{f}\left(t_k, Y(t_k,\tau,\xi) + z(t_k)\right) \\
&\quad - \sum_{t_k \in [t,+\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}\left(t_k^+\right) \\
&\qquad \times \widetilde{f}\left(t_k, Y(t_k,\tau,\xi) + z(t_k)\right) \\
&\quad + \sum_{t_k \in [0,t)} \mathbf{U}(t)\left(P_2 + P_3\right) \mathbf{U}^{-1}\left(t_k^+\right) \\
&\qquad \times \widetilde{f}\left(t_k, Y(t_k,\tau,\xi) + z(t_k)\right).
\end{aligned}
\tag{35}
$$

It follows from (4), (H_5), and (31) that we can also obtain that

$$
\begin{aligned}
|Tz(t)| &\leq K \int_{-\infty}^{t} \psi(s)\, ds + K \int_{t}^{+\infty} \psi(s)\, ds + K \sum_{t_k \in (-\infty,t)} \widetilde{\psi}(t_k) \\
&\quad + K \sum_{t_k \in [t,+\infty)} \widetilde{\psi}(t_k) + K\left(C_1 + C_2 + 1\right) \int_{-\infty}^{+\infty} \psi(t)\, dt \\
&\quad + K\left(C_1 + C_2 + 1\right) \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \\
&= K\left(\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \right) \\
&\quad + K\left(C_1 + C_2 + 1\right) \\
&\quad \times \left(\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) \right) \\
&< KN + KN\left(C_1 + C_2 + 1\right) \\
&= KN\left(C_1 + C_2 + 2\right) \\
&\triangleq B.
\end{aligned}
\tag{36}
$$

So, we have $|Tz(t)| \leq B$. Therefore, T is a self-map of a sphere with radius B.

Moreover, it follows from (4), (H_3), (H_4), (H_6), and (32) that

$$
\begin{aligned}
\left|Tz_1(t) - Tz_2(t)\right| &\leq K \int_{-\infty}^{t} r(s)\left|z_1(s) - z_2(s)\right| ds \\
&\quad + K \int_{t}^{+\infty} r(s)\left|z_1(s) - z_2(s)\right| ds \\
&\quad + \int_{0}^{t} \left\| \mathbf{U}(t)\left(P_2 + P_3\right) \mathbf{U}^{-1}(s) \right\| \\
&\qquad \times r(s)\left|z_1(s) - z_2(s)\right| ds
\end{aligned}
$$

$$+ \sum_{t_k \in [0,t)} \left\| \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+) \right\| \tilde{r}(t_k)$$

$$\times \left| z_1(t_k) - z_2(t_k) \right|$$

$$+ K \sum_{t_k \in (-\infty, t)} \tilde{r}(t_k) \left| z_1(t_k) - z_2(t_k) \right|$$

$$+ K \sum_{t_k \in [t, +\infty)} \tilde{r}(t_k) \left| z_1(t_k) - z_2(t_k) \right|$$

$$\leq K (C_1 + C_2 + 1) \| z_1 - z_2 \|$$

$$\times \left(\int_{-\infty}^{+\infty} r(t) \, dt + \sum_{t_k \in (-\infty, +\infty)} \tilde{r}(t_k) \right)$$

$$+ K \| z_1 - z_2 \|$$

$$\times \left(\int_{-\infty}^{+\infty} r(t) \, dt + \sum_{t_k \in (-\infty, +\infty)} \tilde{r}(t_k) \right)$$

$$\leq KC (C_1 + C_2 + 2) \| z_1 - z_2 \|. \tag{37}$$

Let C be a positive constant such that $C < 1/K(C_1 + C_2 + 2)$; then $L \triangleq KC(C_1 + C_2 + 2) < 1$. By the contraction principle map T has a unique fixed point $z_0(t)$; that is, $z_0(t)$ satisfies

$$z_0(t) = \int_{-\infty}^{t} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s) f(s, Y(s, \tau, \xi) + z_0(s)) \, ds$$

$$- \int_{t}^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s) f(s, Y(s, \tau, \xi) + z_0(s)) \, ds$$

$$+ \int_{0}^{t} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s)$$

$$\times f(s, Y(s, \tau, \xi) + z_0(s)) \, ds$$

$$+ \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_0(t_k))$$

$$- \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_0(t_k))$$

$$+ \sum_{t_k \in [0,t)} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_0(t_k)). \tag{38}$$

By direct differentiation, we can verify that $z_0(t)$ is a solution of (34). Furthermore, the solution is bounded with $|z_0(t)| \leq B$ and

$$(P_2 + P_3) \mathbf{U}^{-1}(0) z(0) = 0. \tag{39}$$

Now, we are going to prove that the bounded solution with initial condition (39) is unique. For this purpose, we assume that $z_1(t)$ is another solution of (34). Following steps similar to (17)–(22), it is not hard to show that any bounded solution of (34) with initial value condition (39) can be written as follows:

$$z_1(t) = \int_{-\infty}^{t} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(s)$$

$$\times f(s, Y(s, \tau, \xi) + z_1(s)) \, ds$$

$$- \int_{t}^{+\infty} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(s)$$

$$\times f(s, Y(s, \tau, \xi) + z_1(s)) \, ds$$

$$+ \int_{0}^{t} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s)$$

$$\times f(s, Y(s, \tau, \xi) + z_1(s)) \, ds$$

$$+ \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_1(t_k))$$

$$- \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_1(t_k))$$

$$+ \sum_{t_k \in [0,t)} \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \tilde{f}(t_k, Y(t_k, \tau, \xi) + z_1(t_k)). \tag{40}$$

Calculate $z_1(t) - z_0(t)$, by (4), (H_3), (H_4), (H_6), and (32) that

$$\left| z_1(t) - z_0(t) \right| \leq K \int_{-\infty}^{t} r(s) \left| z_1(s) - z_0(s) \right| \, ds$$

$$+ K \int_{t}^{+\infty} r(s) \left| z_1(s) - z_0(s) \right| \, ds$$

$$+ \int_{0}^{t} \left\| \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(s) \right\| r(s)$$

$$\times \left| z_1(s) - z_0(s) \right| \, ds$$

$$+ \sum_{t_k \in [0,t)} \left\| \mathbf{U}(t)(P_2 + P_3) \mathbf{U}^{-1}(t_k^+) \right\| \tilde{r}(t_k)$$

$$\times \left| z_1(t_k) - z_0(t_k) \right|$$

$$+ K \sum_{t_k \in (-\infty, t)} \tilde{r}(t_k) \left| z_1(t_k) - z_0(t_k) \right|$$

$$+ K \sum_{t_k \in [t, +\infty)} \tilde{r}(t_k) \left| z_1(t_k) - z_0(t_k) \right|$$

$$\leq K (C_1 + C_2 + 1) \| z_1 - z_0 \|$$

$$\times \left(\int_{-\infty}^{+\infty} r(t)\,dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k) \right)$$
$$+ K \|z_1 - z_0\|$$
$$\times \left(\int_{-\infty}^{+\infty} r(t)\,dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k) \right)$$
$$\leq KC(C_1 + C_2 + 2)\|z_1 - z_0\|$$
$$= L\|z_1 - z_0\|.$$

$$(41)$$

Hence $\|z_1 - z_0\| \leq L\|z_1 - z_0\|$, and $L < 1$, so we have $z_1(t) = z_0(t)$. This implies that the bounded solution of (34) with initial condition (39) is unique. The proof of Lemma 10 is complete. $\square$

Lemma 11. *Let $x(t)$ be any solution of the system (7); then the system*

$$\dot{Z} = A(t)Z + f(t, x(t) + Z) - f(t, x(t)), \quad t \neq t_k,$$
$$\Delta Z(t_k) = \widetilde{A}(t_k)Z(t_k) + \widetilde{f}(t_k, x(t_k) + Z(t_k)) \quad (42)$$
$$- \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z},$$

has a unique bounded solution $z(t) = 0$ with $(P_2 + P_3)\mathbf{U}^{-1}(0)z(0) = 0$.

Proof. Obviously, $z \equiv 0$ is a bounded solution of system (42) with the initial condition $(P_2 + P_3)\mathbf{U}^{-1}(0)z(0) = 0$. Now we show that the bounded solution is unique. If not, there is another bounded solution $z_1(t)$, by Lemma 10, which can be written as follows:

$$z_1(t) = \int_{-\infty}^{t} \mathbf{U}(t)P_1\mathbf{U}^{-1}(s)$$
$$\times [f(s, x(s) + z_1(s)) - f(s, x(s))]\,ds$$
$$- \int_{t}^{+\infty} \mathbf{U}(t)P_4\mathbf{U}^{-1}(s)$$
$$\times [f(s, x(s) + z_1(s)) - f(s, x(s))]\,ds$$
$$+ \int_{0}^{t} \mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(s)$$
$$\times [f(s, x(s) + z_1(s)) - f(s, x(s))]\,ds$$
$$+ \sum_{t_k \in (-\infty,t)} \mathbf{U}(t)P_1\mathbf{U}^{-1}(t_k^+)$$
$$\times [\widetilde{f}(t_k, x(t_k) + z_1(t_k)) - \widetilde{f}(t_k, x(t_k))]$$

$$- \sum_{t_k \in [t,+\infty)} \mathbf{U}(t)P_4\mathbf{U}^{-1}(t_k^+)$$
$$\times [\widetilde{f}(t_k, x(t_k) + z_1(t_k)) - \widetilde{f}(t_k, x(t_k))]$$
$$+ \sum_{t_k \in [0,t)} \mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(t_k^+)$$
$$\times [\widetilde{f}(t_k, x(t_k) + z_1(t_k)) - \widetilde{f}(t_k, x(t_k))].$$

$$(43)$$

Then it follows from (4), (H_3), (H_4), (H_6), and (32) that

$$|z_1(t)| \leq K\int_{-\infty}^{t} r(s)|z_1(s)|\,ds + K\int_{t}^{+\infty} r(s)|z_1(s)|\,ds$$
$$+ \int_{0}^{t} \|\mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(s)\| r(s)|z_1(s)|\,ds$$
$$+ \sum_{t_k \in [0,t)} \|\mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(t_k^+)\| \widetilde{r}(t_k)|z_1(t_k)|$$
$$+ K\sum_{t_k \in (-\infty,t)} \widetilde{r}(t_k)|z_1(t_k)|$$
$$+ K\sum_{t_k \in [t,+\infty)} \widetilde{r}(t_k)|z_1(t_k)|$$
$$\leq K(C_1 + C_2 + 1)\|z_1\|$$
$$\times \left(\int_{-\infty}^{+\infty} r(t)\,dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k) \right)$$
$$+ K\|z_1\| \left(\int_{-\infty}^{+\infty} r(t)\,dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k) \right)$$
$$\leq KC(C_1 + C_2 + 2)\|z_1\|$$
$$= L\|z_1\|.$$

$$(44)$$

Since $L < 1$, so $z_1(t) \equiv 0$ with the initial condition $(P_2 + P_3)\mathbf{U}^{-1}(0)z(0) = 0$. The proof of Lemma 11 is complete. $\square$

Let

$$H(t, x) = x + h(t, (t, x)),$$
$$G(t, y) = y + g(t, (t, y)),$$

$$(45)$$

where $h(t, (t, x))$ is given by (22), and

$$h(t, (t, x)) = -\int_{-\infty}^{t} \mathbf{U}(t)P_1\mathbf{U}^{-1}(s)f(s, X(s, (t, x)))\,ds$$
$$+ \int_{t}^{+\infty} \mathbf{U}(t)P_4\mathbf{U}^{-1}(s)f(s, X(s, (t, x)))\,ds$$
$$- \int_{0}^{t} \mathbf{U}(t)(P_2 + P_3)\mathbf{U}^{-1}(s)f(s, X(s, (t, x)))\,ds$$

$$- \sum_{t_k \in (-\infty, t)} \mathbf{U}(t) P_1 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (t, x)))$$

$$+ \sum_{t_k \in [t, +\infty)} \mathbf{U}(t) P_4 \mathbf{U}^{-1}(t_k^+) \widetilde{f}(t_k, X(t_k, (t, x)))$$

$$- \sum_{t_k \in [0, t)} \mathbf{U}(t) (P_2 + P_3) \mathbf{U}^{-1}(t_k^+)$$

$$\times \widetilde{f}(t_k, X(t_k, (t, x))).$$

$$(46)$$

Lemma 12. *Let $x(t)$ be any solution of system (7); then $H(t, x(t))$ is a solution of system (6).*

Proof. If $x(t)$ is any solution of system (7), then $H(t, x(t)) = x(t) + h(t, (t, x(t))) = x(t) + h(t, (0, x(0)))$ since, by (46), $h(t, (t, x(t))) = h(t, (0, x(0)))$.

We assume that $H(t) = H(t, x(t))$; then we have

$$\dot{H}(t) = A(t) x(t) + f(t, x(t)) + A(t) h(t, (t, x(t)))$$

$$- f(t, x(t))$$

$$= A(t) (x(t) + h(t, (t, x(t))))$$

$$= A(t) H(t),$$

$$\Delta H(t_k) = \widetilde{A}(t_k) x(t_k) + \widetilde{f}(t_k, x(t_k))$$

$$+ \widetilde{A}(t_k) h(t_k, (t_k, x(t_k))) - \widetilde{f}(t_k, x(t_k))$$

$$= \widetilde{A}(t_k) (x(t_k) + h(t_k, (t_k, x(t_k))))$$

$$= \widetilde{A}(t_k) H(t_k).$$

$$(47)$$

So, $H(t, x(t))$ is the solution of system (6). □

Lemma 13. *Let $y(t)$ be any solution of system (6); then $G(t, y(t))$ is a solution of system (7).*

Proof. The proof is similar to Lemma 12. □

Lemma 14. *For any $t \in \mathbb{R}$, $x \in \mathbb{R}^n$,*

$$G(t, H(t, x)) = x. \qquad (48)$$

Proof. According to the above arguments, if $x(t)$ is a solution of system (7), from Lemma 12, $H(t, x(t))$ is a solution of (6). On the other hand, in view of Lemma 13, it is easy to see that $x_1(t) = G(t, H(t, x(t)))$ is another solution of (7). Let $J(t) = x_1(t) - x(t)$; we have

$$\dot{J}(t) = A(t) x_1(t) + f(t, x_1(t)) - A(t) x(t) + f(t, x(t))$$

$$= A(t) J(t) + f(t, x(t) + J(t)) - f(t, x(t)),$$

$$\Delta J(t_k) = \widetilde{A}(t_k) x_1(t_k) + \widetilde{f}(t_k, x_1(t_k))$$

$$- \widetilde{A}(t_k) x(t_k) + \widetilde{f}(t_k, x(t_k))$$

$$= \widetilde{A}(t_k) J(t_k) + \widetilde{f}(t_k, x(t_k) + J(t_k)) - f(t_k, x(t_k)).$$

$$(49)$$

Thus $J(t)$ is a solution of the system (42). On the other hand, following the definition of H, G, we can obtain

$$|J(t)| = |G(t, H(t, x(t))) - x(t)|$$

$$\leq |G(t, H(t, x(t))) - H(t, x(t))|$$

$$+ |H(t, x(t)) - x(t)|$$

$$= |g(t, (t, H(t, x(t))))| + |h(t, (t, x(t)))|.$$

$$(50)$$

It follows from Lemmas 9 and 10 that $J(t)$ is a bounded solution of the system (42); by Lemma 11, system (42) has only one zero bounded solution with the initial condition $(P_2+P_3)\mathbf{U}^{-1}(0)z(0) = 0$. Hence $J(t) = 0$ and thus $x_1(t) = x(t)$. That is, $G(t, H(t, x(t))) = x(t)$. Since $x(t)$ is arbitrary, we have

$$G(t, H(t, x)) = x. \qquad (51)$$

□

Lemma 15. *For any $t \in \mathbb{R}$, $y \in \mathbb{R}^n$,*

$$H(t, G(t, y)) = y. \qquad (52)$$

Proof. If $y(t)$ is any solution of system (6), from Lemma 13, $G(t, y(t))$ is a solution of (7). On the other hand, in view of Lemma 12, it is easy to see that $y_1(t) = H(t, G(t, y(t)))$ is another solution of (6). Let $J(t) = y_1(t) - y(t)$; we have

$$\dot{J}(t) = A(t) y_1(t) - A(t) y(t) = A(t) J(t),$$

$$\Delta J(t_k) = \widetilde{A}(t_k) y_1(t_k) - \widetilde{A}(t_k) y(t_k) = \widetilde{A}(t_k) J(t_k).$$

$$(53)$$

Thus $J(t)$ is a solution of the system (6). On the other hand, following the definition of H, G, we can obtain

$$|J(t)| = |H(t, G(t, y(t))) - y(t)|$$

$$\leq |H(t, G(t, y(t))) - G(t, y(t))|$$

$$+ |G(t, y(t)) - y(t)|$$

$$= |h(t, (t, G(t, y(t))))| + |g(t, (t, y(t)))|.$$

$$(54)$$

It follows from Lemmas 9 and 10 that $J(t)$ is a bounded solution of the system (6), and it is easy to see that system (6) has only one zero bounded solution with the initial condition $(P_2 + P_3)\mathbf{U}^{-1}(0)x(0) = 0$; therefore, $J(t) = 0$ and thus $y_1(t) = y(t)$. That is, $H(t, G(t, y(t))) = y(t)$. Since $y(t)$ is arbitrary, we have

$$H(t, G(t, y)) = y. \qquad (55)$$

So H and G are inverses of each other for each fixed t and they are both homeomorphisms for each fixed t. □

Now we are in a position to prove the main results.

Proof of Theorem 4. We are going to show that $H(t, \cdot)$ satisfies the four conditions of Definition 3.

Proof of Condition (i). For any fixed t, it follows from Lemmas 14 and 15 that $H(t, \cdot)$ is homeomorphism and $G(t, \cdot) = H^{-1}(t, \cdot)$.

Proof of Condition (ii). It follows from $H(t, x) = x + h(t, (t, x))$ and Lemma 9 that $|H(t, x) - x|$ is bounded, uniformly with respect to t.

Proof of Condition (iii). It follows from $G(t, y) = y + g(t, (t, y))$ and Lemma 10 that $|G(t, y) - y|$ is bounded, uniformly with respect to t.

Proof of Condition (iv). Following from Lemma 12 and Lemma 13, we easily prove that condition (iv) is true.

　　Therefore, system (7) is topologically conjugated to system (6). This completes the proof of Theorem 4. □

4. Examples

Now we present two examples to show the feasibility of our results. Consider the following impulsive systems:

$$x'(t) = A(t)x, \quad t \neq t_k,$$
$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k), \quad k \in \mathbb{Z}, \tag{56}$$

$$x'(t) = A(t)x + f(t, x), \quad t \neq t_k,$$
$$\Delta x(t_k) = \widetilde{A}(t_k)x(t_k) + \widetilde{f}(t_k, x(t_k)), \quad k \in \mathbb{Z}, \tag{57}$$

where

$$A(t) = \text{diag}\left\{-\frac{1}{1+|t|}, -\frac{1}{1+t^2}, \frac{1}{1+\sqrt{|t|}}, \arctan t\right\},$$

$$f(t, x) = \frac{1}{3\pi(1+t^2)}(\sin x, 1, \cos x, 1)^T, \tag{58}$$

$$\widetilde{f}(t, x) = \frac{1}{(1+t)^2}(\cos x, \sin x, 1, 1)^T.$$

From system (57), we can easily see that $f(t, x)$ satisfies

$$\|f(t, x)\| < \frac{1}{1+t^2} \triangleq \psi(t),$$

$$\|\widetilde{f}(t, x)\| < \frac{1}{(1+t)^2} \triangleq \widetilde{\psi}(t),$$

$$\|f(t, x_1) - f(t, x_2)\| < \frac{1}{3\pi(1+t^2)}\|x_1 - x_2\| \tag{59}$$
$$\triangleq r(t)\|x_1 - x_2\|,$$

$$\|\widetilde{f}(t, x_1) - \widetilde{f}(t, x_2)\| < \frac{1}{(1+t)^2}\|x_1 - x_2\|$$
$$\triangleq \widetilde{r}(t)\|x_1 - x_2\|.$$

It is easy to see that the fundamental matrix of $x' = A(t)x$ is

$$\Phi(t) = \text{diag}\left\{e^{\int_0^t -(1/(1+|u|))du}, e^{-\arctan t},\right.$$
$$\left. e^{\int_0^t (1/(1+\sqrt{|u|}))du}, e^{\int_0^t \arctan u du}\right\}. \tag{60}$$

Then the fundamental matrix of the impulsive linear system (56) is

$$\mathbf{U}(t) = \Phi(t)\prod_{t_k \in [0,t)}\Phi^{-1}(t_k)\left(I + \widetilde{A}(t_k)\right)\Phi(t_k)\Phi^{-1}(0). \tag{61}$$

In what follows, we give two examples of $\widetilde{A}(t_k)$.

Example 1. Taking

$$\widetilde{A}(t_k) = \text{diag}\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \tag{62}$$

denote that N is a positive constant such that the interval $[0, t)$ contains no more than $N\{t_k\}_{k \in \mathbb{Z}}$. Then we have

$$\mathbf{U}(t) = \text{diag}\left\{\frac{1}{2^N}e^{\int_0^t -(1/(1+|u|))du}, \frac{1}{2^N}e^{-\arctan t}, \right.$$
$$\left. \frac{1}{2^N}e^{\int_0^t (1/(1+\sqrt{|u|}))du}, \frac{1}{2^N}e^{\int_0^t \arctan u du}\right\}. \tag{63}$$

Let

$$P = \text{diag}\{1, 1, 0, 0\}. \tag{64}$$

Consequently,

$$\mathbf{U}(t)P\mathbf{U}^{-1}(s)$$
$$= \text{diag}\left\{\frac{1}{4^N}e^{\int_s^t -(1/(1+|u|))du}, \frac{1}{4^N}e^{-\arctan t + \arctan s}, 0, 0\right\},$$

$$\mathbf{U}(t)(I - P)\mathbf{U}^{-1}(s)$$
$$= \text{diag}\left\{0, 0, \frac{1}{4^N}e^{\int_s^t (1/(1+\sqrt{|u|}))du}, \frac{1}{4^N}e^{\int_s^t \arctan u du}\right\}. \tag{65}$$

Obviously,

$$\|\mathbf{U}(t)P\mathbf{U}^{-1}(s)\| \leq \frac{1}{4^N} < 1,$$
$$\|\mathbf{U}(t)(I - P)\mathbf{U}^{-1}(s)\| \leq \frac{1}{4^N} < 1. \tag{66}$$

That is, system (56) possesses an ordinary dichotomy. On the other hand, we see that

$$
\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k)
$$
$$
= \int_{-\infty}^{+\infty} \frac{1}{1+t^2}\, dt + \sum_{t_k \in (-\infty,+\infty)} \frac{1}{(1+t_k)^2}
$$
$$
= N\left(\pi + \frac{\pi^2}{6}\right),
$$
$$
\int_{-\infty}^{+\infty} r(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k)
$$
$$
= \int_{-\infty}^{+\infty} \frac{1}{3\pi(1+t^2)}\, dt + \sum_{t_k \in (-\infty,+\infty)} \frac{1}{(1+t_k)^2}
$$
$$
= N\left(\frac{1}{3} + \frac{\pi^2}{6}\right).
$$

(67)

From Theorem 4, we conclude that (56) and (57) are topologically conjugated.

Example 2. Taking

$$
\widetilde{A}(t_k) = \mathrm{diag}\left\{-\frac{1}{t_k+2}, -\frac{1}{t_k+2}, -\frac{1}{t_k+2}, -\frac{1}{t_k+2}\right\},
$$
$$
P = \mathrm{diag}\{1,1,0,0\}.
$$

(68)

Taking $t_k = k = 0,1,2,\dots$, we have

$$
\mathbf{U}(t) = \mathrm{diag}\left\{\frac{1}{k+2} e^{\int_0^t -(1/(1+|u|))du}, \frac{1}{k+2} e^{-\arctan t}, \right.
$$
$$
\left. \frac{1}{k+2} e^{\int_0^t (1/(1+\sqrt{|u|}))du}, \frac{1}{k+2} e^{\int_0^t \arctan u\, du}\right\}.
$$

(69)

That is,

$$
\mathbf{U}(t) P \mathbf{U}^{-1}(s) = \mathrm{diag}\left\{\frac{1}{(k+2)^2} e^{\int_s^t -(1/(1+|u|))du},\right.
$$
$$
\left. \frac{1}{(k+2)^2} e^{-\arctan t + \arctan s}, 0, 0\right\},
$$
$$
\mathbf{U}(t)(I-P)\mathbf{U}^{-1}(s) = \mathrm{diag}\left\{0, 0, \frac{1}{(k+2)^2} e^{\int_s^t (1/(1+\sqrt{|u|}))du},\right.
$$
$$
\left. \frac{1}{(k+2)^2} e^{\int_s^t \arctan u\, du}\right\}.
$$

(70)

Obviously,

$$
\left\|\mathbf{U}(t) P \mathbf{U}^{-1}(s)\right\| \le \frac{1}{(k+2)^2} < 1,
$$
$$
\left\|\mathbf{U}(t)(I-P)\mathbf{U}^{-1}(s)\right\| \le \frac{1}{(k+2)^2} < 1.
$$

(71)

In this case, system (56) also possesses an ordinary dichotomy. On the other hand, we have

$$
\int_{-\infty}^{+\infty} \psi(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{\psi}(t_k) = \pi + \frac{\pi^2}{6},
$$
$$
\int_{-\infty}^{+\infty} r(t)\, dt + \sum_{t_k \in (-\infty,+\infty)} \widetilde{r}(t_k) = \frac{1}{3} + \frac{\pi^2}{6}.
$$

(72)

From Theorem 4, we conclude that (56) and (57) are topologically conjugated.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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