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Research Article

Robust H_∞ Control for a Class of Nonlinear Distributed Parameter Systems via Proportional-Spatial Derivative Control Approach

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This paper addresses the problem of robust H_∞ control design via the proportional-spatial derivative (P-sD) control approach for a class of nonlinear distributed parameter systems modeled by semilinear parabolic partial differential equations (PDEs). By using the Lyapunov direct method and the technique of integration by parts, a simple linear matrix inequality (LMI) based design method of the robust H_∞ P-sD controller is developed such that the closed-loop PDE system is exponentially stable with a given decay rate and a prescribed H_∞ performance of disturbance attenuation. Moreover, a suboptimal H_∞ controller is proposed to minimize the attenuation level for a given decay rate. The proposed method is successfully employed to address the control problem of the FitzHugh-Nagumo (FHN) equation, and the achieved simulation results show its effectiveness.

1. Introduction

A significant research area that has received a lot of attention over the past few decades is the control design for distributed parameter systems (DPSs) modeled by parabolic partial differential equations (PDEs). These DPSs can be applied to describe many industrial processes, such as thermal diffusion processes, fluid, and heat exchangers [1–4]. The key characteristic of DPSs is space distribution, which causes their outputs, inputs, process states, and parameters to be spatially varying. On the other hand, external disturbances and nonlinear phenomena appear in most real systems. In this situation, the study of robust H_∞ control design for nonlinear parabolic PDE systems is of theoretical and practical importance.

Significant research results have been reported in the past few decades for DPSs [1–3, 5–18]. The most interesting results within these research activities are those developed on the basis of PDE model [9–18]. For example, Krstic and Smyshlyaev have developed nonadaptive and adaptive kernel-based backstepping methods for linear boundary control PDE systems [9–11]. Fridman and Orlov [12] have presented exponential stabilization with H_∞ performance in terms of linear matrix inequalities (LMIs) for uncertain semilinear parabolic and hyperbolic systems via a robust collocated static output feedback boundary controller. These results [9–12] are only applicable for boundary control PDE systems. Motivated by significant recent advances in actuation and sensing technology, particularly the advances of microelectromechanical systems, it is possible to manufacture large arrays of microsensors and actuators with integrated control circuitry (for control applications of such devices, see [13] and the references therein). Hence, the problems on distributed control theory and design for PDE systems have received a great deal of attention [1–3, 14–18]. For example, Orlov et al. have developed state feedback tracking control design [3] for an uncertain heat diffusion process and exponential stabilization [14] for an uncertain wave equation via distributed dynamic input extension. Wang, Wu, and Li have established sufficient conditions of distributed exponential stabilization via simple fuzzy proportional state feedback controllers for first-order hyperbolic PDE systems [15–17] and via a fuzzy proportional-spatial derivative (P-sD) for semi-linear parabolic PDE system [18]. Wu, Wang, and Li [19] have proposed a Lyapunov-based distributed H_∞ fuzzy controller design with constraint for semi-linear first-order hyperbolic PDE systems. Notice that the results reported in [15–19] are presented in terms of spatial differential linear matrix inequalities (SDLMIs), which can be only approximately solved on the basis of standard finite difference method and the existing convex optimization techniques [20, 21]. Despite these efforts, to the best of the authors' knowledge, there are still few results on the robust H_∞ control design via the original PDE model of semi-linear parabolic PDE systems with external disturbances, which motivates this study.

In this study, we will deal with the problem of robust H_∞ control design for a class of semi-linear parabolic PDE systems with external disturbances via P-sD control approach. Based on the Lyapunov direct method and integration by parts, a sufficient condition for the exponential stabilization with a given decay rate and a prescribed H_∞ performance of disturbance attenuation is presented in terms of standard LMIs. Moreover, a suboptimal H_∞ controller is proposed to minimize the attenuation level for a given decay rate. Finally, the simulation study on the robust H_∞ control of a semi-linear parabolic PDE system represented by FitzHugh-Nagumo (FHN) equation is provided to show the effectiveness of the proposed method.

The remainder of this paper is organized as follows. The problem formulation and preliminaries are given in Section 2. The robust H_∞ P-sD control design is provided in Section 3. Section 4 presents an example to illustrate the effectiveness of the proposed method. Finally, Section 5 offers some conclusions.

Notations. The notations used throughout the paper are given as follows. $\mathfrak{R}$, $\mathfrak{R}^n$, and $\mathfrak{R}^{m\times n}$ denote the set of all real numbers, n-dimensional Euclidean space, and the set of all real $m \times n$ matrices, respectively. Identity matrix, of appropriate dimension, will be denoted by $\mathbf{I}$. For a symmetric matrix $\mathbf{M}$, $\mathbf{M} > (\geq, <, \leq)\ 0$ means that it is positive definite (semipositive definite, negative definite, and seminegative definite, resp.). $\mathscr{H}^n \triangleq \mathscr{L}_2([l_1, l_2]; \mathfrak{R}^n)$ is a Hilbert space of n-dimensional square integrable vector functions $\boldsymbol{\vartheta}(x,t) \in \mathfrak{R}^n$, $x \in [l_1, l_2] \subset \mathfrak{R}$, $\forall t \geq 0$ with the inner product and norm:

$$\left\langle \boldsymbol{\vartheta}_1(\cdot,t), \boldsymbol{\vartheta}_2(\cdot,t)\right\rangle = \int_{l_1}^{l_2} \boldsymbol{\vartheta}_1^T(x,t)\,\boldsymbol{\vartheta}_2(x,t)\,dx,$$
$$\left\|\boldsymbol{\vartheta}_1(\cdot,t)\right\|_2 = \left\langle \boldsymbol{\vartheta}_1(\cdot,t), \boldsymbol{\vartheta}_1(\cdot,t)\right\rangle^{1/2}, \tag{1}$$

where $\boldsymbol{\vartheta}_1(\cdot,t) \in \mathscr{H}^n$ and $\boldsymbol{\vartheta}_2(\cdot,t) \in \mathscr{H}^n$. The superscript "$T$" is used for the transpose of a vector or a matrix. The symbol "$*$" is used as an ellipsis in matrix expressions that are induced by symmetry; for example,

$$\begin{bmatrix} \mathbf{S} + [\mathbf{M} + \mathbf{N} + *] & \mathbf{X} \\ * & \mathbf{Y} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{S} + \left[\mathbf{M} + \mathbf{N} + \mathbf{M}^T + \mathbf{N}^T\right] & \mathbf{X} \\ \mathbf{X}^T & \mathbf{Y} \end{bmatrix}. \tag{2}$$

2. Preliminaries and Problem Formulation

Consider the following nonlinear DPSs modeled by semi-linear parabolic PDEs:

$$\mathbf{y}_t(x,t) = \boldsymbol{\Theta}_1 \mathbf{y}_{xx}(x,t) + \boldsymbol{\Theta}_2 \mathbf{y}_x(x,t) + \mathbf{A}\mathbf{y}(x,t) + \mathbf{f}\left(\mathbf{y}(x,t), x, t\right) + \mathbf{G}_u \mathbf{u}(x,t) + \mathbf{G}_w \mathbf{w}(x,t), \tag{3}$$

$$\mathbf{z}(x,t) = \mathbf{C}\mathbf{y}(x,t) + \mathbf{D}\mathbf{u}(x,t) \tag{4}$$

subject to the homogeneous Neumann boundary conditions:

$$\mathbf{y}_x(x,t)\big|_{x=l_1} = \mathbf{y}_x(x,t)\big|_{x=l_2} = 0 \tag{5}$$

and the initial condition:

$$\mathbf{y}(x,0) = \mathbf{y}_0(x), \tag{6}$$

where $\mathbf{y}(x,t) \triangleq [y_1(x,t)\ \cdots\ y_n(x,t)] \in \mathfrak{R}^n$ is the state, the subscripts x and t stand for the partial derivatives with respect to x, t, respectively, $x \in [l_1, l_2] \subset \mathfrak{R}$ and $t \in [0, \infty)$ denote the position and time, respectively, and $\mathbf{u}(x,t) \triangleq [u_1(x,t)\ \cdots\ u_m(x,t)] \in \mathfrak{R}^m$ is the control input. $\mathbf{z}(x,t) \in \mathfrak{R}^q$ is the controlled output. $\mathbf{w}(x,t) \in \mathfrak{R}^p$ is the exogenous disturbance satisfying $\int_0^\infty \|\mathbf{w}(\cdot,t)\|_2^2 dt < \infty$. $\mathbf{f}(\mathbf{y}(x,t), x, t) \in \mathfrak{R}^n$ is the nonlinear part in the system, which is a locally Lipschitz continuous function on $\mathbf{y}(x,t)$ and satisfies $\mathbf{f}(0, x, t) = 0$ for all $x \in [l_1, l_2]$ and $t \geq 0$. $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, $\mathbf{A}$, $\mathbf{G}_u$, $\mathbf{G}_w$, $\mathbf{C}$, and $\mathbf{D}$ are real-known matrices with appropriate dimensions.

This study considers the following P-sD state feedback controller:

$$\mathbf{u}(x,t) = \mathbf{K}_1 \mathbf{y}(x,t) + \mathbf{K}_2 \mathbf{y}_x(x,t), \tag{7}$$

where $\mathbf{K}_1 \in \mathfrak{R}^{m\times n}$ and $\mathbf{K}_2 \in \mathfrak{R}^{m\times n}$ are control gain matrices to be determined. The controller structure is shown in Figure 1, in which the notation "$\partial/\partial x$" means a first-order spatial differentiator.

Remark 1. It must be stressed that the implementation of the controller (7) requires distributed sensing and actuation. Although this is normally recognized as a critical drawback, with recent advances in technological developments of microelectromechanical systems, it becomes feasible to manufacture large arrays of microsensors and actuators with integrated control circuitry, which can be used for the implementation of distributed feedback control loops in some practical applications (see [13] and the references therein). The signal $\mathbf{y}_x(x,t)$ can be obtained using the finite difference method. In addition, it has been pointed out in [18] that the controller (7) can provide more convenient spatial performance.

Substituting (7) into (3) and (4) leads to the following PDE:

$$\mathbf{y}_t(x,t) = \boldsymbol{\Theta}_1 \mathbf{y}_{xx}(x,t) + \left[\boldsymbol{\Theta}_2 + \mathbf{G}_u \mathbf{K}_2\right] \mathbf{y}_x(x,t) + \mathbf{A}_c \mathbf{y}(x,t) + \mathbf{f}\left(\mathbf{y}(x,t), x, t\right) + \mathbf{G}_w \mathbf{w}(x,t),$$
$$\mathbf{z}(x,t) = \mathbf{C}_c \mathbf{y}(x,t) + \mathbf{D}\mathbf{K}_2 \mathbf{y}_x(x,t), \tag{8}$$

where $\mathbf{A}_c \triangleq \mathbf{A} + \mathbf{G}_u \mathbf{K}_1$ and $\mathbf{C}_c \triangleq \mathbf{C} + \mathbf{D}\mathbf{K}_1$.

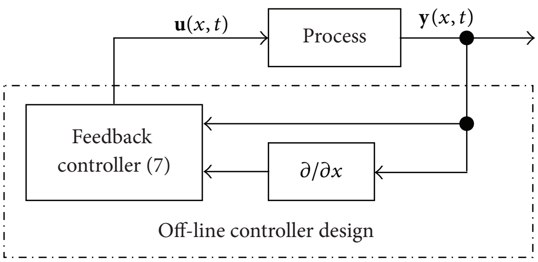

Figure 1: The structure of distributed P-sD state-feedback controller.

In order to attenuate the effect of $\mathbf{w}(x,t)$, robust H_∞ control will be employed in this paper to deal with the disturbance attenuation problem. Let us consider the following H_∞ control performance for the closed-loop PDE system of the form (5), (6), and (8):

$$\int_0^\infty \|\mathbf{z}(\cdot,t)\|_2^2 dt \le \langle \mathbf{y}_0(\cdot), \mathbf{P}\mathbf{y}_0(\cdot)\rangle + \gamma^2 \int_0^\infty \|\mathbf{w}(\cdot,t)\|_2^2 dt, \quad (9)$$

where $\mathbf{P} > 0$ is a real $n \times n$ matrix and $\gamma > 0$ is a prescribed level of disturbance attenuation. In general, it is desirable to make the attenuation level as small as possible to achieve the optimal disturbance attenuation performance.

For simplicity, when $\mathbf{u}(x,t) \equiv 0$, the PDE system (3)–(6) is referred to as an *unforced* PDE system, while when $\mathbf{w}(x,t) \equiv 0$, it is referred to as a *disturbance-free* PDE system. We introduce the following definitions.

Definition 2. Given a constant $\rho > 0$, the unforced disturbance-free PDE system of (5), (6), and (8) (i.e., $\mathbf{u}(x,t) \equiv 0$ and $\mathbf{w}(x,t) \equiv 0$) is said to be *exponentially stable with a given decay rate* ρ, if there exists a constant $\sigma > 0$ such that the following inequality holds:

$$\|\mathbf{y}(\cdot,t)\|_2^2 \le \sigma \exp(-2\rho t) \|\mathbf{y}_0(\cdot)\|_2^2, \quad \forall t \ge 0. \quad (10)$$

Definition 3. Given constants $\rho > 0$ and $\gamma > 0$, the unforced PDE system of (5), (6), and (8) is said to be *exponentially stable with a given decay rate* ρ *and* γ*-disturbance attenuation* if the response $\mathbf{z}(x,t)$ satisfies (9) and the disturbance-free system is *exponentially stable with a given decay rate* ρ.

Therefore, the objective of this study is to find a robust P-sD controller of the form (7) such that the resulting closed-loop system is exponentially stable and the H_∞ performance is ensured for a prescribed disturbance attenuation level $\gamma > 0$. To do this, the following assumption and lemma are useful for the development of the main results.

Assumption 4. There exists a scalar $\chi > 0$ such that the following inequality holds for any $\mathbf{y}(x,t) \in \Omega$:

$$\begin{aligned}&\int_{l_1}^{l_2} \mathbf{f}^T(\mathbf{y}(x,t),x,t)\,\mathbf{f}(\mathbf{y}(x,t),x,t)\,dx \\ &\quad \le \chi \int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{y}(x,t)\,dx,\end{aligned} \quad (11)$$

where $\Omega \triangleq \{\mathbf{y}(x,t) \mid \sigma_1(x) \le \mathbf{y}(x,t) \le \sigma_2(x),\ x \in [l_1,l_2],\ t \ge 0\}$.

Lemma 5. *For any two square integrable vector functions* $\mathbf{a}(x) \in \Re^n$, $\mathbf{b}(x) \in \Re^n$, $x \in [l_1,l_2]$, *the following inequality holds for any positive scalar* $\alpha \in \Re$, $x \in [l_1,l_2]$:

$$2\langle \mathbf{a},\mathbf{b}\rangle \le \langle \mathbf{a},\alpha\mathbf{a}\rangle + \left\langle \mathbf{b},\alpha^{-1}\mathbf{b}\right\rangle. \quad (12)$$

Proof. It is easily verified that $[\alpha\mathbf{a}(x)-\mathbf{b}(x)]^T[\alpha\mathbf{a}(x)-\mathbf{b}(x)] \ge 0$ holds for any $x \in [l_1,l_2]$ and any positive scalar α. Therefore,

$$\begin{aligned}0 &\le [\alpha\mathbf{a}(x)-\mathbf{b}(x)]^T[\alpha\mathbf{a}(x)-\mathbf{b}(x)] \\ &= \alpha^2\mathbf{a}^T(x)\mathbf{a}(x) - 2\alpha\mathbf{a}^T(x)\mathbf{b}(x) + \mathbf{b}^T(x)\mathbf{b}(x),\end{aligned} \quad (13)$$

which implies

$$\begin{aligned}2\mathbf{a}^{\mathbf{T}}(x)\mathbf{b}(x) &\le \alpha\mathbf{a}^T(x)\mathbf{a}(x) \\ &\quad + \alpha^{-1}\mathbf{b}^T(x)\mathbf{b}(x), \quad x \in [l_1,l_2].\end{aligned} \quad (14)$$

Integrating both sides of (14) from l_1 to l_2, we can obtain that

$$\begin{aligned}2\int_{l_1}^{l_2} \mathbf{a}^T(x)\mathbf{b}(x)\,dx &\le \int_{l_1}^{l_2} \alpha\mathbf{a}^T(x)\mathbf{a}(x)\,dx \\ &\quad + \int_{l_1}^{l_2} \alpha^{-1}\mathbf{b}^T(x)\mathbf{b}(x)\,dx,\end{aligned} \quad (15)$$

which implies that the inequality (12) holds. The proof is complete. □

3. Robust H_∞ P-sD Control Design

The aim of this section is to develop a robust H_∞ P-sD state feedback controller to not only exponentially stabilize the semi-linear PDE system (3)–(6) but also achieve the H_∞ performance with a prescribed disturbance attenuation level $\gamma > 0$.

Consider the following Lyapunov functional for the system (5), (6), and (8):

$$V(t) = \int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{y}(x,t)\,dx, \quad (16)$$

where $\mathbf{P} > 0$ is a real $n \times n$ gain matrix to be determined. The time derivative of $V(t)$ along the solution of the system (5), (6), and (8) is given by

$$\begin{aligned}\dot{V}(t) &= 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{y}_t(x,t)\,dx \\ &= 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{\Theta}_1\mathbf{y}_{xx}(x,t)\,dx \\ &\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\left[\mathbf{\Theta}_2 + \mathbf{G}_u\mathbf{K}_2\right]\mathbf{y}_x(x,t)\,dx \\ &\quad + \int_{l_1}^{l_2} \mathbf{y}^T(x,t)\left[\mathbf{P}\mathbf{A}_c + *\right]\mathbf{y}(x,t)\,dx \\ &\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{f}\left(\mathbf{y}(x,t),x,t\right)dx \\ &\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{G}_w\mathbf{w}(x,t)\,dx.\end{aligned} \tag{17}$$

Integrating by parts and taking into account (5) yield

$$\begin{aligned}\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{\Theta}_1\mathbf{y}_{xx}(x,t)\,dx &= \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{\Theta}_1\mathbf{y}_x(x,t)\Big|_{x=l_1}^{x=l_2} \\ &\quad - \int_{l_1}^{l_2} \mathbf{y}_x^T(x,t)\,\mathbf{P}\mathbf{\Theta}_1\mathbf{y}_x(x,t)\,dx \\ &= -\int_{l_1}^{l_2} \mathbf{y}_x^T(x,t)\,\mathbf{P}\mathbf{\Theta}_1\mathbf{y}_x(x,t)\,dx.\end{aligned} \tag{18}$$

Applying Assumption 4, for any scalar $\alpha > 0$,

$$\begin{aligned}&2\int_{z_1}^{z_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{f}\left(\mathbf{y}(x,t),x,t\right)dx \\ &\le \alpha\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{P}\mathbf{y}(x,t)\,dx \\ &\quad + \alpha^{-1}\int_{l_1}^{l_2} \mathbf{f}^T\left(\mathbf{y}(x,t),x,t\right)\mathbf{f}\left(\mathbf{y}(x,t),x,t\right)dx \\ &\le \alpha\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{P}\mathbf{y}(x,t)\,dx \\ &\quad + \alpha^{-1}\chi\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{y}(x,t)\,dx \\ &= \int_{l_1}^{l_2} \mathbf{y}^T(x,t)\left[\alpha\mathbf{P}\mathbf{P} + \alpha^{-1}\chi\mathbf{I}\right]\mathbf{y}(x,t)\,dx.\end{aligned} \tag{19}$$

Substitution of (18) and (19) into (17) implies

$$\begin{aligned}&\dot{V}(t) + 2\rho V(t) \\ &\le \int_{l_1}^{l_2} \mathbf{y}^T(x,t)\left[\left[\mathbf{P}\mathbf{A}_c + *\right] + \alpha\mathbf{P}\mathbf{P} + \alpha^{-1}\chi\mathbf{I} + 2\rho\mathbf{P}\right]\mathbf{y}(x,t)\,dx \\ &\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\left[\mathbf{\Theta}_2 + \mathbf{G}_u\mathbf{K}_2\right]\mathbf{y}_x(x,t)\,dx \\ &\quad - \int_{l_1}^{l_2} \mathbf{y}_x^T(x,t)\left[\mathbf{P}\mathbf{\Theta}_1 + *\right]\mathbf{y}_x(x,t)\,dx \\ &\le \int_{l_1}^{l_2} \tilde{\mathbf{y}}^T(x,t)\,\overline{\Psi}\tilde{\mathbf{y}}(x,t)\,dx \\ &\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{G}_w\mathbf{w}(x,t)\,dx,\end{aligned} \tag{20}$$

where $\tilde{\mathbf{y}}(x,t) \triangleq [\mathbf{y}^T(x,t)\ \ \mathbf{y}_x^T(x,t)]^T$ and

$$\overline{\Psi} \triangleq \begin{bmatrix}\left[\mathbf{P}\mathbf{A}_c + *\right] + \alpha\mathbf{P}\mathbf{P} + \alpha^{-1}\chi\mathbf{I} + 2\rho\mathbf{P} & \mathbf{P}\left[\mathbf{\Theta}_2 + \mathbf{G}_u\mathbf{K}_2\right] \\ * & -\left[\mathbf{P}\mathbf{\Theta}_1 + *\right]\end{bmatrix}. \tag{21}$$

Combining (4) and (20) gives

$$\begin{aligned}&\dot{V}(t) + 2\rho V(t) + \|\mathbf{z}(\cdot,t)\|_2^2 - \gamma^2\|\mathbf{w}(\cdot,t)\|_2^2 \\ &\quad \le \int_{l_1}^{l_2} \tilde{\mathbf{y}}^T(x,t)\left[\overline{\Psi} + \left[\mathbf{C}_c\ \ \mathbf{D}\mathbf{K}_2\right]^T\left[\mathbf{C}_c\ \ \mathbf{D}\mathbf{K}_2\right]\right]\tilde{\mathbf{y}}(x,t)\,dx \\ &\quad\quad + 2\int_{l_1}^{l_2} \mathbf{y}^T(x,t)\,\mathbf{P}\mathbf{G}_w\mathbf{w}(x,t)\,dx \\ &\quad\quad - \gamma^2\int_{l_1}^{l_2} \mathbf{w}^T(x,t)\,\mathbf{w}(x,t)\,dx \\ &\quad = \int_{l_1}^{l_2} \hat{\mathbf{y}}^T(x,t)\,\widehat{\Psi}\hat{\mathbf{y}}(x,t)\,dx,\end{aligned} \tag{22}$$

where $\hat{\mathbf{y}}(x,t) \triangleq [\tilde{\mathbf{y}}^T(x,t)\ \ \mathbf{w}^T(x,t)]^T$ and

$$\begin{aligned}\widehat{\Psi} &\triangleq \begin{bmatrix}\left[\mathbf{P}\mathbf{A}_c + *\right] + \alpha\mathbf{P}\mathbf{P} + \alpha^{-1}\chi\mathbf{I} + 2\rho\mathbf{P} & \mathbf{P}\left[\mathbf{\Theta}_2 + \mathbf{G}_u\mathbf{K}_2\right] & \mathbf{P}\mathbf{G}_w \\ * & -\left[\mathbf{P}\mathbf{\Theta}_1 + *\right] & 0 \\ * & * & -\gamma^2\mathbf{I}\end{bmatrix} \\ &\quad + \begin{bmatrix}\mathbf{C}_c^T \\ \mathbf{K}_2^T\mathbf{D}^T \\ 0\end{bmatrix}\begin{bmatrix}\mathbf{C}_c^T \\ \mathbf{K}_2^T\mathbf{D}^T \\ 0\end{bmatrix}^T.\end{aligned} \tag{23}$$

From the above analysis, we have the following theorem.

Theorem 6. *Consider the semi-linear PDE system* (3)–(6) *with the P-sD controller* (7). *For some given scalar* $\rho > 0$ *and* $\gamma > 0$, *the closed-loop PDE system is exponentially stable with a decay rate* ρ *and the* γ*-disturbance attenuation, if there exist a* $n \times n$ *matrix* $\mathbf{Q} > 0$, $m \times n$ *matrices* $\mathbf{Z}_i$, $i \in \{1, 2\}$, *and a positive scalar* α *satisfying the following LMI:*

$$\Xi \triangleq \begin{bmatrix} \Xi_{11} & \mathbf{\Theta}_2\mathbf{Q} + \mathbf{G}_u\mathbf{Z}_2 & \mathbf{G}_w & \mathbf{Q} & \widetilde{\mathbf{C}}^T \\ * & -[\mathbf{\Theta}_1\mathbf{Q} + *] & 0 & 0 & \mathbf{Z}_2^T\mathbf{D}^T \\ * & * & -\gamma^2\mathbf{I} & 0 & 0 \\ * & * & * & -\alpha\chi^{-1}\mathbf{I} & 0 \\ * & * & * & * & -\mathbf{I} \end{bmatrix} < 0, \quad (24)$$

where $\Xi_{11} \triangleq [\mathbf{A}\mathbf{Q} + \mathbf{G}_u\mathbf{Z}_1 + *] + 2\rho\mathbf{Q} + \alpha\mathbf{I}$ *and* $\widetilde{\mathbf{C}} \triangleq \mathbf{C}\mathbf{Q} + \mathbf{D}\mathbf{Z}_1$. *In this case, the gain matrices* $\mathbf{K}_i$, $i \in \{1, 2\}$ *can be constructed as*

$$\mathbf{K}_i = \mathbf{Z}_i\mathbf{Q}^{-1}, \quad i \in \{1, 2\}. \quad (25)$$

Proof. Set

$$\mathbf{Q} = \mathbf{P}^{-1} > 0, \qquad \mathbf{Z}_i = \mathbf{K}_i\mathbf{Q}, \quad i \in \{1, 2\}. \quad (26)$$

By pre- and post-multiplying the matrix $\widehat{\Psi}$ by the matrix $\operatorname{diag}\{\mathbf{Q}, \mathbf{Q}, \mathbf{I}\}$, respectively, we get

$$\overline{\Xi} \triangleq \begin{bmatrix} \Xi_{11} + \alpha^{-1}\chi\mathbf{Q}\mathbf{Q} & \mathbf{\Theta}_2\mathbf{Q} + \mathbf{G}_u\mathbf{Z}_2 & \mathbf{G}_w \\ * & -[\mathbf{\Theta}_1\mathbf{Q} + *] & 0 \\ * & * & -\gamma^2\mathbf{I} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{C}}^T \\ \mathbf{Z}_2^T\mathbf{D}^T \\ 0 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{C}}^T \\ \mathbf{Z}_2^T\mathbf{D}^T \\ 0 \end{bmatrix}^T. \quad (27)$$

Using the Schur complement two times, LMI (24) is equivalent to the inequality $\overline{\Xi} < 0$. Since $\operatorname{diag}\{\mathbf{Q}, \mathbf{Q}, \mathbf{I}\} > 0$ and $\overline{\Xi} < 0$, we can get the inequality $\widehat{\Psi} < 0$.

From the inequality $\widehat{\Psi} < 0$ and (22), we can drive

$$\dot{V}(t) + 2\rho V(t) + \|\mathbf{z}(\cdot, t)\|_2^2 - \gamma^2\|\mathbf{w}(\cdot, t)\|_2^2 \leq 0. \quad (28)$$

Since $\rho V(t) \geq 0$, we can obtain from (28) that

$$\dot{V}(t) + \|\mathbf{z}(\cdot, t)\|_2^2 - \gamma^2\|\mathbf{w}(\cdot, t)\|_2^2 \leq 0. \quad (29)$$

Integrating (29) from $t = 0$ to $t = \infty$ yields

$$\int_0^\infty \dot{V}(t)\,dt + \int_0^\infty \left(\|\mathbf{z}(\cdot, t)\|_2^2 - \gamma^2\|\mathbf{w}(\cdot, t)\|_2^2\right) dt \leq 0, \quad (30)$$

which implies

$$V(\infty) - V(0) + \int_0^\infty \left(\|\mathbf{z}(\cdot, t)\|_2^2 - \gamma^2\|\mathbf{w}(\cdot, t)\|_2^2\right) dt \leq 0. \quad (31)$$

Since $V(\infty) \geq 0$, we obtain (9) from (31).

Next, we will show the exponential stability with a given decay rate ρ of the disturbance-free system of (5), (6), and (8). When $\mathbf{w}(x, t) = 0$, inequality (20) can be rewritten as

$$\dot{V}(t) + 2\rho V(t) \leq \int_{l_1}^{l_2} \widetilde{\mathbf{y}}^T(x, t)\,\overline{\Psi}\widetilde{\mathbf{y}}(x, t)\,dx. \quad (32)$$

We can easily derive $\overline{\Psi} < 0$ from $\widehat{\Psi} < 0$. Hence, the inequality (32) can be further written as

$$\dot{V}(t) + 2\rho V(t) \leq 0. \quad (33)$$

Integration of (33) from 0 to t yields

$$V(t) \leq V(0)\exp(-2\rho t). \quad (34)$$

Since $\mathbf{P} > 0$, it is easily observed that $V(t)$ given by (16) satisfies the following inequality:

$$p_m\|\mathbf{y}(\cdot, t)\|_2^2 \leq V(t) \leq p_M\|\mathbf{y}(\cdot, t)\|_2^2, \quad (35)$$

where $p_m \triangleq \lambda_{\min}(\mathbf{P})$ and $p_M \triangleq \lambda_{\max}(\mathbf{P})$ are two positive scalars. Inequalities (34) and (35) imply

$$\|\mathbf{y}(\cdot, t)\|_2^2 \leq p_m^{-1}p_M\|\mathbf{y}(\cdot, 0)\|_2^2\exp(-2\rho t). \quad (36)$$

Thus, from Definition 2 and (36), the disturbance-free system of (5), (6), and (8) is exponentially stable with a given decay rate ρ. From Definition 3, the closed-loop system (5), (6), and (8) is exponentially stable with a given decay rate ρ and γ-disturbance attenuation. Moreover, from (26), we have (25). The proof is complete. □

From Theorem 6, since the controller (7) has been shown to be an effective control which can attenuate the effect of uncertain external disturbances, it is appealing to eliminate the influence brought by external disturbances as possible, that is, making the attenuation level as small as possible. To achieve this goal, for a given decay rate ρ, setting $\vartheta = \gamma^2$, we consider the following minimization optimization problem:

$$\min_{\{\vartheta, \mathbf{Q} > 0, \mathbf{Z}_1, \mathbf{Z}_2, \alpha > 0\}} \vartheta \quad (37)$$

subject to the following LMI

$$\begin{bmatrix} \Xi_{11} & \mathbf{\Theta}_2\mathbf{Q} + \mathbf{G}_u\mathbf{Z}_2 & \mathbf{G}_w & \mathbf{Q} & \widetilde{\mathbf{C}}^T \\ * & -[\mathbf{\Theta}_1\mathbf{Q} + *] & 0 & 0 & \mathbf{Z}_2^T\mathbf{D}^T \\ * & * & -\vartheta\mathbf{I} & 0 & 0 \\ * & * & * & -\alpha\chi^{-1}\mathbf{I} & 0 \\ * & * & * & * & -\mathbf{I} \end{bmatrix} < 0. \quad (38)$$

Remark 7. Notice that the control design proposed in this paper is different from the results reported in [18, 19]. The result in [18] only considers simple exponential stabilization for a class of semi-linear parabolic PDE systems. The main difference between the result in this study and [19] is that the system under consideration in the latter one is a class of semi-linear first-order hyperbolic PDE systems, whereas the system addressed in this study is a class of semi-linear parabolic PDE systems. On the other hand, different from the SDLMI-based control designs in [18, 19], the main result of this study is presented in terms of standard LMI, which can be directly verified via the existing convex optimization techniques [20, 21].

4. Simulation Study on the FHN Equation

To illustrate the effectiveness of the proposed methods, the control problem of the FHN equation is considered, which is a wavy behavior model extensively applied to excitable media in biology [22] and chemistry [23]. The FHN equation has the following closed-form description:

$$\begin{aligned} y_{1,t}(x,t) = {} & y_{1,xx}(x,t) - 0.5y_{1,x}(x,t) + y_1(x,t) \\ & - 0.9y_2(x,t) - 0.1y_1^3(x,t) \\ & + u(x,t) + 0.5w(x,t), \\ y_{2,t}(x,t) = {} & 4y_{2,xx}(x,t) - 0.5y_{2,x}(x,t) + 0.2y_1(x,t) \\ & - y_2(x,t) + 0.2w(x,t) \end{aligned} \tag{39}$$

subject to the boundary conditions

$$\begin{aligned} y_{1,x}(x,t)\big|_{x=0} = y_{1,x}(x,t)\big|_{x=L} = 0, \\ y_{2,x}(x,t)\big|_{x=0} = y_{2,x}(x,t)\big|_{x=L} = 0 \end{aligned} \tag{40}$$

and the initial conditions

$$y_1(x,0) = y_{1,0}(x), \qquad y_2(x,0) = y_{2,0}(x), \tag{41}$$

where $y_1(x,t)$ and $y_2(x,t)$ are the state variables and $u(x,t)$ is the manipulated input. t, x, and L denote the independent time, space variables, and the length of the spatial domain, respectively. $y_{1,0}(x)$ and $y_{2,0}(x)$ are the initial conditions.

To more intuitively illustrate the effectiveness of the proposed design method, for the above values, we first verify through simulation that the operating steady states $y_1(x,t) = 0$ and $y_2(x,t) = 0$ of the system (39)–(41) are unstable ones. The initial conditions in (41) are assumed to be $y_{1,0}(x) = 0.6\cos(\pi x/L)$ and $y_{2,0}(x) = 0.1$. The length of spatial domain is set to be 20; that is, $L = 20$. The disturbance input is chosen as $w(x,t) = \cos(x)\exp(-0.5t)$. Figure 2 shows the open-loop profiles of the evolution of $y_1(x,t)$ and $y_2(x,t)$ starting from the initial conditions. It is easily observed from Figure 2 that the equilibria $y_1(x,t) = 0$ and $y_2(x,t) = 0$ of the system (39)–(41) are unstable ones and $-1 \le y_i(x,t) \le 1$, $x \in [0,L]$, $t \ge 0$, $i \in \{1,2\}$.

Equations (39) can be rewritten as the form of PDE (3) with the following parameters:

$$\begin{gathered} \mathbf{\Theta}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad \mathbf{\Theta}_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0.2 & -1 \end{bmatrix}, \\ \mathbf{G_u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{G_w} = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \\ \mathbf{f}(\mathbf{y}(x,t), x, t) = \begin{bmatrix} -0.3y_1^3(x,t) & 0 \end{bmatrix}^T, \end{gathered} \tag{42}$$

where $\mathbf{y}(x,t) \triangleq [y_1(x,t) \;\; y_2(x,t)]^T$ and $y_1^3(x,t)$ is a nonlinear term. The controlled output $\mathbf{z}(x,t)$ is chosen as $\mathbf{z}(x,t) = y_1(x,t)$. Hence, the parameter matrices in (4) are chosen as $\mathbf{C} = [1 \;\; 0]$ and $\mathbf{D} = 0$.

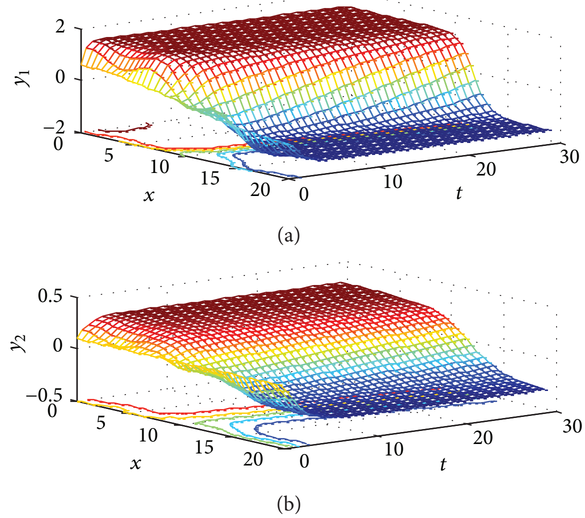

Figure 2: Open-loop profiles of evolution of $y_1(x,t)$ and $y_2(x,t)$.

From Figure 2, we can easily observe that $-1 \le y_i(x,t) \le 1$, $x \in [0,L]$, $t \ge 0$, $i \in \{1,2\}$. Let $\Omega \triangleq \{\mathbf{y}(x,t) \mid -1.0 \le y_i(x,t) \le 1.0,\ x \in [0,L],\ t \ge 0,\ i \in \{1,2\}\}$. The parameter χ satisfying Assumption 4 is chosen as

$$\begin{aligned} \chi &\triangleq \max_{\mathbf{y}(x,t)} \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial f_i(\mathbf{y}(x,t))}{\partial y_j(x,t)} \right\} \\ &= 0.09 \max_{\mathbf{y}(x,t)} \left\{ 9y_1^4(x,t) \right\} = 0.81. \end{aligned} \tag{43}$$

We first show the effectiveness of the proposed design method. Set $\rho = 0.04$. Solving the optimization problem (37), we can get the optimized level of attenuation γ as $\gamma^* = \sqrt{\vartheta} = 4.3435 \times 10^{-5}$. Setting $\gamma = 0.8$ and solving LMI (24), the control gain matrices in (7) can be derived as follows:

$$\begin{aligned} \mathbf{K}_1 &= \begin{bmatrix} -5.2143 & -5.2365 \end{bmatrix}^T, \\ \mathbf{K}_2 &= \begin{bmatrix} 0.0999 & -0.0002 \end{bmatrix}^T. \end{aligned} \tag{44}$$

Applying the P-sD controller (7) with the control gain matrices given in (44) to the semi-linear PDE system (39)–(41), the closed-loop profiles of evolution of $y_1(x,t)$ and $y_2(x,t)$ are shown in Figure 3, which implies that the proposed P-sD controller (7) with the control gain matrices given in (44) can stabilize the semi-linear PDE system (39)–(41). Moreover, profile of evolution of $\mathbf{u}(x,t)$ is shown in Figure 4.

Define the function $\eta(t)$ as

$$\begin{aligned} \eta(t) \triangleq {} & \int_0^t \|\mathbf{z}(\cdot,\tau)\|_2^2 d\tau - \langle \mathbf{y}_0(\cdot), \mathbf{P}\mathbf{y}_0(\cdot) \rangle \\ & - 0.64 \int_0^t \|\mathbf{w}(\cdot,\tau)\|_2^2 d\tau. \end{aligned} \tag{45}$$

Figure 5 shows the value of $\eta(t)$. From this figure, we can see that $\eta(t) < 0$ for all time $t \ge 0$, which implies that the H_∞ control performance in (9) is ensured.

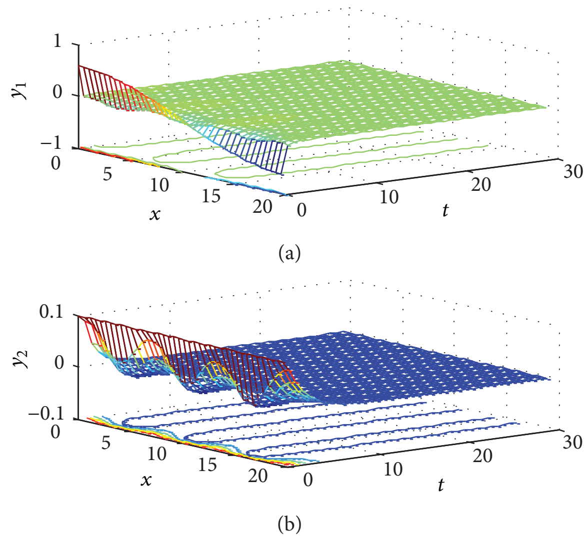

Figure 3: Closed-loop profiles of evolution of $y_1(x,t)$ and $y_2(x,t)$.

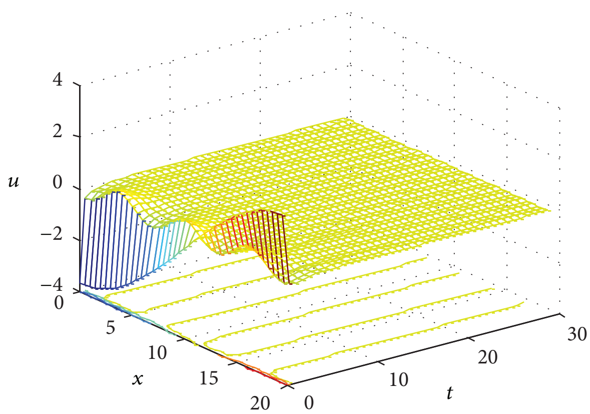

Figure 4: Profile of evolution of $u(x,t)$.

5. Conclusions

In this paper, we have addressed the problem of robust H_∞ P-sD state-feedback controller design for a class of semi-linear parabolic PDE systems with external disturbances. Based on the Lyapunov technique, the robust H_∞ P-sD state-feedback controller design is formulated as a standard LMI optimization problem. The proposed controller can not only exponentially stabilize the semi-linear PDE system but also satisfy the H_∞ performance in (9). The influence caused by external disturbances is eliminated as possible by the minimization optimization problem. Finally, the developed design method is successfully applied to the control of the FHN equation, and the achieved simulation results illustrate its effectiveness. Compared to one node in the paper, it is interesting to study the collective control in a coupled network with multiple nodes described by nonlinear parabolic PDEs in the future work.

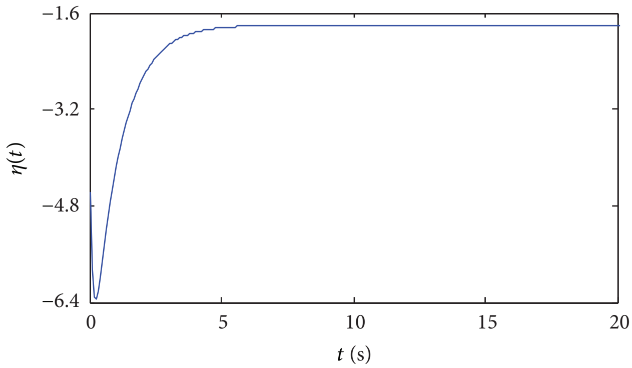

Figure 5: Trajectory of $\eta(t)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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