Research Article

Numerical Algorithm for the Third-Order Partial Differential Equation with Three-Point Boundary Value Problem

Jing Niu and Ping Li

School of Mathematics and Sciences, Harbin Normal University, Harbin 150025, China

Correspondence should be addressed to Ping Li; njirwin@163.com

A numerical method based on the reproducing kernel theorem is presented for the numerical solution of a three-point boundary value problem with an integral condition. Using the reproducing property and the existence of orthogonal basis in a new reproducing kernel Hilbert space, we obtain a representation of exact solution in series form and its approximate solution by truncating the series. Moreover, the uniform convergency is proved and the effectiveness of the proposed method is illustrated with some examples.

1. Introduction

In this paper, we are concerned with the numerical solution of the following third-order partial differential equation with three-point boundary condition [1]:

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t),$$

$$\int_c^1 u(x,t)\,dx = 0, \quad t \in [0,T],\ 0 \le c < 1,$$

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \frac{\partial^2 u}{\partial t^2}(x,T) = 0,$$

$$x \in [0,1], \quad T > 0,$$

where $a(x,t)$ and its derivatives satisfy the conditions $0 < a_0 < a(x,t) < a_1$, $|(a(x,t))_x| \le b$, and $f(x,t)$ is given smooth function in $[0,1] \times [0,T]$.

Note that the third-order partial differential equations in (1) make a base of many mathematical models for dynamics of the soil moisture and subsoil waters [2], spreading of acoustic waves in a weakly heterogeneous environment [3]. Many physical phenomena and mechanical situations have been formulated into boundary value problems with integral boundary conditions [4, 5]. Later many works have appeared

such as Ashyralyev and Aggez [6], Ashyralyev and Tetikoglua [7], Pulkina [8], and Ashyralyev and Gercek [9]. It should be noted that there are so much work devoted to the existence of solution for this type of boundary value problems where parabolic equations, hyperbolic equations, and mixed-type equations are considered [10, 11]. The proof of existence and uniqueness of solution especially for (1) has been studied by Latrous and Memou [1]. Recently, the reproducing kernel space method (RKSM) plays a crucial role in numerical solutions of differential and integral equations [12–20]. The main ideas of RKSM are based on the construction of reproducing kernel space (RKS). The reproducing kernel function can absorb all definite conditions. Then the numerical solution of definite problem is approximated by the reproducing kernel function. It is obvious that constructing a suitable reproducing kernel space and effectively calculating the reproducing kernel function expression become the key to apply RKSM.

However, due to the complex three-point value conditions with an integral condition in (1), the RKSM has not constructed suitable RKS to deal with the numerical solution. More precisely, the establishment of traditional RKS relies heavily on the two endpoints. Hence it can not be extended to three-point nonlocal boundary value problem which is based on intermediate point, especially with the integral boundary condition. Moreover, to the best of the authors' knowledge, the numerical approximations of the problem equation (1)

have not been studied before. Motivated by all the works above, we describe an improvement of the RKSM to find the numerical solution for (1). A new RKS is successfully established by some techniques like (3) and (7). Furthermore, other partial differential equations with multipoint boundary value conditions may be numerically solved using a similar process.

The outline of this paper is as follows. In the next section, new reproducing kernel spaces for solving problem (1) are constructed. Section 3 establishes a bounded linear operator and an orthogonal basis to use the RKSM. As a result, the approximate solution of the considered problem is obtained. In Section 4, some numerical results are given to demonstrate the accuracy of the present method. Also a conclusion is given in Section 5. Note that we have computed the numerical results using mathematic programming.

2. Constructive Method for the Reproducing Kernel Space $W(\Omega)$

Definition 1 (see [12]). Let H be a Hilbert function space on a set X. H is called a reproducing kernel space if and only if, for any $x \in X$, there exists a unique function $K_x(y) \in H$, such that $\langle f, K_x \rangle = f(x)$ for any $f \in H$. Meanwhile, $K(x, y) \overset{\Delta}{=} K_x(y)$ is called a reproducing kernel.

Since $K(x, y) = K_x(y) = \langle K_x, K_y \rangle = \overline{\langle K_y, K_x \rangle} = \overline{K_y(x)} = \overline{K(y, x)}$, one has the following property.

Lemma 2. *A reproducing kernel function of real reproducing kernel space is symmetric.*

Definition 3. $W_1[0, 1] = \{u(x) \mid u''(x)$ is an absolutely continuous real value function in $[0, 1]$, $u^{(3)}(x) \in L^2[0, 1]$, $u(0) = 0$, and $\int_c^1 u(x)dx = 0, 0 < c < 1\}$. The inner product is given by

$$\langle u(x), v(x) \rangle_1 = u'(0) v'(0) + u''(0) v''(0)$$
$$+ \int_0^1 u^{(3)}(x) v^{(3)}(x) dx. \tag{2}$$

Theorem 4. $W_1[0, 1]$ *is a reproducing kernel space. Moreover the reproducing kernel $R(x, y)$ can be denoted by*

$R(x, y)$

$$= \begin{cases} r_1(x, y) = a_1 + a_2 x + a_3 x^2 + \cdots + a_6 x^5, & x < y < c, \\ r_2(x, y) = b_1 + b_2 x + b_3 x^2 \\ \quad + \cdots + b_6 x^5 + \lambda_1 \dfrac{x^6}{6!}, & c < x < y, \\ r_3(x, y) = c_1 + c_2 x + c_3 x^2 + \cdots + c_6 x^5, & y < c < x, \\ r_4(x, y) = d_1 + d_2 x + d_3 x^2 + \cdots + d_6 x^5, & y < x < c, \\ r_5(x, y) = e_1 + e_2 x + e_3 x^2 \\ \quad + \cdots + e_6 x^5 + \lambda_1 \dfrac{x^6}{6!}, & c < y < x, \\ r_6(x, y) = f_1 + f_2 x + f_3 x^2 \\ \quad + \cdots + f_6 x^5 + \lambda_2 \dfrac{x^6}{6!}, & x < c < y. \end{cases} \tag{3}$$

Proof. $W_1[0, 1]$ is a RKS which is a generalization of [12, Theorem 1.3.1, 1.3.2] with essentially the same proofs. Let $R(x, y)$ be the reproducing kernel function of $W_1[0, 1]$. In view of (2), $u(0) = 0$, and $\int_c^1 u(y)dy = 0$, we have the following equality using the integration by parts:

$$\langle u(x), R(x, y) \rangle_1$$
$$= u'(0) \partial_x R(0, y) + u''(0) \partial_x^2 R(0, y)$$
$$+ \int_0^1 u^{(3)}(x) \partial_x^3 R(x, y) dx$$
$$= u'(0) \partial_x R(0, y) + u''(0) \partial_x^2 R(0, y)$$
$$+ \int_0^1 u^3(x) \partial_x^3 R(x, y) dx + \lambda \int_c^1 u(x) dx$$
$$= u'(0) \left[\partial_x R(0, y) + \partial_x^4 R(0, y) \right]$$
$$+ u''(0) \left[\partial_x^2 R(0, y) - \partial_x^3 R(0, y) \right]$$
$$+ u(1) \partial_x^5 R(1, y) - u'(1) \partial_x^4 R(1, y)$$
$$+ u''(1) \partial_x^3 R(1, y) - \int_0^c u(x) \partial_x^6 R(x, y) dx$$
$$- \int_c^1 u(x) \left(\partial_x^6 R(x, y) - \lambda \right) dx. \tag{4}$$

Here λ is an arbitrary function of y. In order to obtain reproducing property, namely,

$$\langle u(x), R(x, y) \rangle_1 = u(y), \tag{5}$$

let

$$\partial_x R(0, y) + \partial_x^4 R(0, y) = 0,$$
$$\partial_x^2 R(0, y) - \partial_x^3 R(0, y) = 0, \tag{6}$$
$$\partial_x^i R(1, y) = 0, \quad (i = 3, 4, 5),$$
$$\partial_x^6 R(x, y) = -\delta(x - y) \quad (y < c, x < c),$$
$$\partial_x^6 R(x, y) - \lambda_2 = 0 \quad (y > c, x < c),$$
$$\partial_x^6 R(x, y) - \lambda_1 = -\delta(x - y) \quad (y > c, x > c),$$
$$\partial_x^6 R(x, y) = 0 \quad (y < c, x > c). \tag{7}$$

Since the eigenvalues of (7) are all zero and sixfold, the general solutions of (7) have the form of (3). Next, we need to establish 38 equations for calculating the coefficients which are functions on y. It is obvious that 4 equations can be obtained from boundary value conditions and (6) give 10 equations. According to (7), we have 12 equations. Finally, 12 equations follow from the continuity at c. □

For $c = 1/2$, the concrete expression of $R(x, y)$ is given by Lemma 2

$$r_1(x, y)$$
$$= \frac{y^5}{120} + \frac{x^2 y^2}{12}(y + 3) + xy - \frac{xy^4}{24}$$
$$- \frac{7xy}{9594180}\left(\left(1080 + 210x + 70x^2 - 45x^3 + 12x^4\right)\right.$$
$$\left.\times \left(1080 + 210y + 70y^2 - 45y^3 + 12y^4\right)\right);$$

$$r_2(x, y)$$
$$= \frac{y^5}{120} + \frac{x^2 y^2}{12}(y + 3) + xy - \frac{xy^4}{24}$$
$$+ \frac{7y}{153506880}\left(\left(1 - 17292x - 3300x^2 - 1280x^3\right.\right.$$
$$\left. + 960x^4 - 384x^5 + 64x^6\right)$$
$$\left.\times \left(1080 + 210y + 70y^2 - 45y^3 + 12y^4\right)\right);$$

$$r_3(x, y)$$
$$= \frac{y^5}{120} + \frac{x^2 y^2}{12}(y + 3) + xy - \frac{xy^4}{24}$$
$$- \frac{7}{2456110080}\left(\left(1 - 17292x - 3300x^2 - 1280x^3\right.\right.$$
$$\left. + 960x^4 - 384x^5 + 64x^6\right)$$
$$\times \left(1 - 17292y - 3300y^2 - 1280y^3\right.$$
$$\left.\left. + 960y^4 - 384y^5 + 64y^6\right)\right);$$

$$r_4(x, y) = r_1(y, x); \qquad r_5(x, y) = r_2(y, x);$$
$$r_6(x, y) = r_3(y, x).$$

$$(8)$$

Similar to Theorem 4, we show another RKS: $W_2[0, T] = \{u(t) \mid u^{(3)}(t)$ is an absolutely continuous real value function in $[0, T]$, $u^{(4)}(t) \in L^2[0, T]$, and $u(0) = u'(0) = u''(T) = 0\}$. The inner product is given by

$$\langle u(t), v(t)\rangle_2 = u^{(3)}(0) v^{(3)}(0) + \int_0^T u^{(4)}(t) v^{(4)}(t) dt \quad (9)$$

and we calculate the reproducing kernel function is $Q(t, s)$ as the following form:

$$q_1(t, s)$$
$$= \frac{1}{5040}\left(t^2\left(7st^4 - t^5 + 140s^2 t(s - 3T)\right.\right.$$
$$+ 35s^2 t^2(s - 3T) + 21s^2\left(s^3 - 5s(4 + s)T\right.$$
$$\left.\left.\left. + 60T^2 + 20T^3\right)\right)\right),$$
$$t \le s,$$

$$q_2(t, s)$$
$$= \frac{1}{5040}\left(s^2\left(7s^4 t - s^5 + 21t^5\right.\right.$$
$$+ 35t^3\left(s(s + 4) - 12T\right)$$
$$\left.\left. - 105t^4 T + 105t^2 T\left(4T^2 + 12T - s(s + 4)\right)\right)\right),$$
$$s \le t.$$
$$(10)$$

Definition 5. Let $\Omega = [0, 1] \times [0, T]$, and $W(\Omega) = \{u(x, t) \mid (\partial^5 u/\partial x^2 \partial t^3)$ is a completely continuous real value function in Ω, $u(0, t) = u(x, 0) = (\partial u/\partial t)(x, 0) = (\partial^2 u/\partial t^2)(x, T) = 0$, $\int_c^1 u(x, t) dx = 0$, and $(\partial^7 u/\partial x^3 \partial t^4) \in L^2(\Omega)\}$. The inner product in $W(\Omega)$ is given by

$$\langle u(x, t), v(x, t)\rangle_w$$
$$= \sum_{i=1}^2 \int_0^T \left[\frac{\partial^4}{\partial t^4}\frac{\partial^i}{\partial x^i}u(0, t)\frac{\partial^4}{\partial t^4}\frac{\partial^i}{\partial x^i}v(0, t)\right] dt$$
$$+ \left\langle \frac{\partial^3}{\partial t^3}u(x, 0), \frac{\partial^3}{\partial t^3}v(x, 0)\right\rangle_1$$
$$+ \int\int_\Omega \left[\frac{\partial^3}{\partial x^3}\frac{\partial^4}{\partial t^4}u(x, t)\frac{\partial^3}{\partial x^3}\frac{\partial^4}{\partial t^4}v(x, t)\right] dx\, dt.$$
$$(11)$$

Theorem 6. *The space $W(\Omega)$ is a reproducing kernel space and its reproducing kernel is*

$$K((x, t), (y, s)) = R(x, y) Q(t, s), \quad (12)$$

where $R(x, y)$ and $Q(t, s)$ are reproducing kernel functions of $W_1[0, 1]$ and $W_2[0, T]$, respectively.

Proof. We need to prove that $K((x, t, y, s))$ satisfies the reproducing property; namely,

$$\langle u(x, t), K((x, t, y, s))\rangle_w$$
$$= \langle u(x, t), R(x, y) Q(t, s)\rangle_w$$
$$= \sum_{i=1}^2 \int_0^T \left[\frac{\partial^4}{\partial t^4}\frac{\partial^i}{\partial x^i}u(0, t)\frac{\partial^4}{\partial t^4}Q(t, s)\frac{\partial^i}{\partial x^i}R(0, y)\right] dt$$
$$+ \left\langle \frac{\partial^3}{\partial t^3}u(x, 0), R(x, y)\frac{\partial^3}{\partial t^3}Q(0, s)\right\rangle_1$$
$$+ \int\int_\Omega \left[\frac{\partial^3}{\partial x^3}\frac{\partial^4}{\partial t^4}u(x, t)\frac{\partial^3}{\partial x^3}R(x, y)\frac{\partial^4}{\partial t^4}Q(t, s)\right] dx\, dt$$
$$= \int_0^T \sum_{i=1}^2 \left[\frac{\partial^4}{\partial t^4}\frac{\partial^i}{\partial x^i}u(0, t)\frac{\partial^4}{\partial t^4}Q(t, s)\frac{\partial^i}{\partial x^i}R(0, y)\right] dt$$
$$+ \frac{\partial^3}{\partial t^3}Q(0, s)\frac{\partial^3}{\partial t^3}u(y, 0)$$

$$+ \int_0^T \int_0^1 \left[\frac{\partial^3}{\partial x^3} \frac{\partial^4}{\partial t^4} u(x,t) \frac{\partial^3}{\partial x^3} R(x,y) \frac{\partial^4}{\partial t^4} Q(t,s) \right] dx\, dt$$

$$= \int_0^T \left\{ \sum_{i=1}^2 \left[\frac{\partial^4}{\partial t^4} \frac{\partial^i}{\partial x^i} u(0,t) \frac{\partial^4}{\partial t^4} Q(t,s) \frac{\partial^i}{\partial x^i} R(0,y) \right] \right.$$

$$+ \int_0^1 \left[\frac{\partial^3}{\partial x^3} \frac{\partial^4}{\partial t^4} u(x,t) \frac{\partial^3}{\partial x^3} R(x,y) \right.$$

$$\left. \left. \times \frac{\partial^4}{\partial t^4} Q(t,s) \right] dx \right\} dt$$

$$+ \frac{\partial^3}{\partial t^3} Q(0,s) \frac{\partial^3}{\partial t^3} u(y,0)$$

$$= \int_0^T \frac{\partial^4}{\partial t^4} Q(t,s) \frac{\partial^4}{\partial t^4} \left\{ \sum_{i=1}^2 \left[\frac{\partial^i}{\partial x^i} u(0,t) \frac{\partial^i}{\partial x^i} R(0,y) \right] \right.$$

$$+ \int_0^1 \left[\frac{\partial^3}{\partial x^3} u(x,t) \right.$$

$$\left. \left. \times \frac{\partial^3}{\partial x^3} R(x,y) \right] dx \right\} dt$$

$$+ \frac{\partial^3}{\partial t^3} Q(0,s) \frac{\partial^3}{\partial t^3} u(y,0)$$

$$= \frac{\partial^3}{\partial t^3} Q(0,s) \frac{\partial^3}{\partial t^3} u(y,0)$$

$$+ \int_0^T \frac{\partial^4}{\partial t^4} Q(t,s) \frac{\partial^4}{\partial t^4} \langle u(x,t), R(x,y) \rangle_1 dt$$

$$= \frac{\partial^3}{\partial t^3} Q(0,s) \frac{\partial^3}{\partial t^3} u(y,0) + \int_0^T \frac{\partial^4}{\partial t^4} Q(t,s) \frac{\partial^4}{\partial t^4} u(y,t)\, dt$$

$$= \langle u(y,t), Q(t,s) \rangle_2 = u(y,s).$$

$$(13)$$
□

3. The Numerical Method

A subspace in $L^2(\Omega)$ is defined by

$$L_0(\Omega) = \{ u(x,t) \mid u(x,t) \in L^2(\Omega),$$

$$u(x,t) \text{ is continuous in } \Omega \}. \qquad (14)$$

Then we define a linear operator $\mathscr{L} : W(\Omega) \to L_0(\Omega)$:

$$\mathscr{L}u(x,t) \triangleq \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial u}{\partial x} \right). \qquad (15)$$

Lemma 7. $\mathscr{L}$ *is an invertible bounded linear operator.*

Proof. Consider the following

$$\|\mathscr{L}u\|_{L_0}^2$$

$$= \int \int_\Omega ((\mathscr{L}u)(x,t))^2 dx\, dt$$

$$= \int \int_\Omega \left[\frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial u}{\partial x} \right) \right]^2 dx\, dt$$

$$\leq C \left(\int \int_\Omega \left[\left(\frac{\partial^3 u}{\partial t^3} \right)^2 + 2\left| \frac{\partial u}{\partial x} \right| \left| \frac{\partial^3 u}{\partial t^3} \right| + \left(\frac{\partial^2 u}{\partial t^2} \right)^2 \right] dx\, dt \right).$$

$$(16)$$

Due to the definition of $W(\Omega)$, we get $\|\partial_x^i R(x,y) \partial_t^j Q(t,s)\|_w \leq M$, $i = 0, 1, 2$ and $j = 0, 1, 2, 3$; then

$$\left| \partial_{x^i t^j}^{i+j} u(x,t) \right| = \left| \left\langle u(y,s), \partial_x^i R(x,y) \partial_t^j Q(t,s) \right\rangle_w \right|$$

$$\leq \left\| \partial_x^i R(x,y) \partial_t^j Q(t,s) \right\|_w \|u\|_w \leq M \|u\|_w;$$

$$(17)$$

it follows that

$$\|\mathscr{L}u\|_{L_0}^2 \leq M^2 \|u\|_w^2. \qquad (18)$$

Therefore (1) is turned into the following operator equation:

$$(\mathscr{L}u)(x,t) = f(x,t). \qquad (19)$$

Since (19) has a unique solution [1], it indicates $\mathscr{L}$ is invertible. The proof is complete. □

For the reproducing kernel function of $W(\Omega)$, we choose a countable dense subset

$$S = \{ (x_1, t_1), (x_2, t_2), \ldots \} \subset \Omega \qquad (20)$$

and define

$$\Psi_i(x,t) = \mathscr{L}_{(y,s)} K((x,t),(y,s)) \big|_{(y,s)=(x_i,t_i)}, \quad i = 1, 2, \ldots, \qquad (21)$$

where $\mathscr{L}_{(y,s)}$ denotes operator $\mathscr{L}$ which acts on (y,s).

Lemma 8. *The function system $\{\Psi_i(x,t)\}_{i=1}^\infty$ is a complete system in $W(\Omega)$.*

Proof. It follows that $\Psi_i(x,t) \in W(\Omega)$ from the definition of $W(\Omega)$ and $\Psi_i(x,t)$. Next we will see it is complete; that is, if $\langle u(x,t), \Psi_i(x,t) \rangle_w = 0$, then we can get $u(x,t) \equiv 0$. For every i, it holds

$$0 = \langle u(x,t), \Psi_i(x,t) \rangle_w$$

$$= \left\langle u(x,t), \mathscr{L}_{(y,s)} K((x,t),(x_i,t_i)) \right\rangle_w \qquad (22)$$

$$= \mathscr{L}_{(y,s)} \langle u(x,t), K((x,t),(x_i,t_i)) \rangle_w = \mathscr{L}u(x_i,t_i).$$

Note that $(x_i, t_i) \in S$ is a countable dense subset in Ω; hence, $\mathscr{L}u(x,t) = 0$. It follows that $u(x,t) \equiv 0$ from the existence of $\mathscr{L}^{-1}$. □

Applying Gram-Schmidt process, we obtain an orthogonal basis $\{\widetilde{\Psi}_i(x,t)\}_{i=1}^{\infty}$ in $W(\Omega)$:

$$\widetilde{\Psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik} \Psi_k(x,t), \quad i = 1, 2, \ldots. \tag{23}$$

Theorem 9. *If $u(x,t)$ is the solution of (19), then the approximate solution can be formed by*

$$u_m(x,t) = \sum_{i=1}^{m}\sum_{k=1}^{i} \beta_{ik} f(x_k, t_k) \widetilde{\Psi}_i(x,t). \tag{24}$$

Proof. From $u(x,t) \in W(\Omega)$ and (21), it holds that

$$u(x,t) = \sum_{i=1}^{\infty} \left\langle u(x,t), \widetilde{\Psi}_i(x,t) \right\rangle_w \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty} \left\langle u(x,t), \sum_{k=1}^{i} \beta_{ik} \Psi_k(x,t) \right\rangle_w \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x,t), \Psi_k(x,t) \right\rangle_w \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x,t), \right.$$

$$\left. \mathscr{L}_{(y,s)} K((x,t),(y,s))\big|_{(y,s)=(x_k,t_k)} \right\rangle_w$$

$$\times \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left[\mathscr{L}_{(y,s)} \left\langle u(x,t), \right.\right.$$

$$\left.\left. K((x,t),(y,s)) \right\rangle_w \right]\big|_{(y,s)=(x_k,t_k)}$$

$$\times \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \mathscr{L}_{(y,s)} u(x_k, t_k) \widetilde{\Psi}_i(x,t)$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k, t_k) \widetilde{\Psi}_i(x,t). \tag{25}$$

The approximate solution of (19) is m-truncation of Fourier series about the exact solution $u(x,t)$ in (28), so $u_m(x,t) \to u(x,t)$ in $W(\Omega)$ as $m \to \infty$. □

Theorem 10. *The approximate solution $u_m(x,t)$ and its derivatives uniformly converge to exact solution $u(x,t)$ and its derivatives, respectively.*

Proof. By the properties of $K((x,t),(y,s))$, we know that there exist positive real numbers $M = M_{(i,j)}$ ($i = 0, 1, 2$ and $j = 0, 1, 2, 3$), such that

$$\left\| \partial_{x^i t^j}^{i+j} K((x,t),(y,s)) \right\|_w \leq M. \tag{26}$$

TABLE 1: Numerical results for Example 11.

(x,t)	$u(x,t)$	$u_{20}(x,t)$	Relative error
$(0.65, 0.1)$	-0.062127	-0.0621255	$2.33726E-5$
$(0.85, 0.3)$	0.340119	0.340117	$6.44956E-6$
$(0.45, 0.5)$	-2.09138	-2.09151	$6.47869E-6$
$(0.5, 0.7)$	-3.8955	-3.89524	$6.66563E-5$
$(0.4, 0.9)$	-6.14693	-6.14721	$4.66011E-5$
$(0.35, 1.1)$	-8.3421	-8.34287	$9.21041E-5$
$(0.25, 1.3)$	-8.93588	-8.93797	$2.34743E-4$
$(0.4, 1.5)$	-15.066	-15.0659	$6.33660E-6$
$(0.45, 1.7)$	-18.9015	-18.9	$8.02067E-5$
$(1, 2)$	48.0	47.9863	$2.84837E-4$

Therefore, as $m \to \infty$ we have

$$\left| \partial_{x^i t^j}^{i+j} u_m(x,t) - \partial_{x^i t^j}^{i+j} u(x,t) \right|$$

$$= \left| \partial_{x^i t^j}^{i+j} (u_m(x,t) - u(x,t)) \right|$$

$$= \left| \partial_{x^i t^j}^{i+j} \left\langle u_m(y,s) - u(y,s), K((x,t),(y,s)) \right\rangle_w \right|$$

$$= \left| \left\langle u_m(y,s) - u(y,s), \partial_{x^i t^j}^{i+j} K((x,t),(y,s)) \right\rangle_w \right| \tag{27}$$

$$\leq \left\| u_m - u \right\|_w \left\| \partial_{x^i t^j}^{i+j} K((x,t),(y,s)) \right\|_w$$

$$\leq M \left\| u_m - u \right\|_w.$$

□

4. Numerical Simulation and Comparison

In this section we will give some numerical examples of multipoint boundary value problem that show the exactness and usefulness of our presented process.

Example 11. This problem corresponds to (1) with $a(x,t) = (t+1)\sin x$ and $f(x,t) = 6x(5-8x^2) - t^2(t-6)(t+1)((24x^2 - 5)\cos x + 48x \sin x)$. The exact solution is $u(x) = (8x^3 - 5x)(-t^3 + 6t^2)$. The numerical results are given in Table 1 for $m = 20$. Here we take $T = 2$ and $c = 1/2$.

Example 12. Consider the following three-point nonlocal elliptic-parabolic problem in [9]:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left((1+x)\frac{\partial u}{\partial x} \right)$$

$$= -t\sin x + \left(e^{-t} + t \right)(\cos x - x\sin x)$$

$$0 < t < 1, \quad 0 < x < \pi,$$

Table 2: Maximum absolute error for Example 12.

Method	$m = 30$	$m = 60$	$m = 90$
Finite difference in [9]	0.015167	0.007318	0.004822
Finite difference in [9]	0.000908	0.000227	0.000101
RKSM	0.000441195	0.0000895156	0.0000527183

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left((1+x)\frac{\partial u}{\partial x}\right) = \left(-2e^{-t} + 1 - t\right)\sin x$$

$$+ \left(e^{-t} + t\right)(\cos x - x\sin x),$$

$$-1 < t < 0, \quad 0 < x < \pi,$$

$$u(1, x) = \frac{1}{2}u(-1, x) + \frac{1}{2}u\left(-\frac{1}{2}, x\right)$$

$$+ \left(e^{-1} - \frac{e}{2} - \frac{1}{2}e^{1/2} + \frac{7}{4}\right)\sin x,$$

$$0 \leq x \leq \pi,$$

$$u(t, 0) = u(t, \pi) = 0, \quad 1 \leq t \leq -1. \tag{28}$$

The exact solution of this problem is $u(t, x) = (e^{-t} + t)\sin x$. In terms of (24), we calculate the approximate solution $u_m(x, t)$ for $m = 30, 60, 90$. Comparing the maximum absolute error by our method with finite difference methods, Table 2 shows that our method has better accuracy.

5. Conclusion

In summary, a new numerical algorithm is provided to solve three-point boundary value problems in a very favorable reproducing kernel space. Using the good properties of reproducing kernel space such as reproducing property and existence of orthogonal basis, we obtain the series pattern approximate solution through operator equation. Numerical results show that the present method is an accurate and reliable analytical technique.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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