Research Article

Global Conservative Solutions of a Generalized Two-Component Camassa-Holm System

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The Cauchy problem for a generalized two-component Camassa-Holm system is investigated. Following the idea of fixed points and using new sets of independent and dependent variables, the existence of the global conservative solutions for the system is established.

1. Introduction

We consider a generalized two-component Camassa-Holm system

$$u_t - u_{txx} + [g(u)]_x = 2u_x u_{xx} + uu_{xxx} - \eta\overline{\eta}_x,$$
$$t > 0, \quad x \in R,$$
$$\eta_t + (u\eta)_x = 0, \quad t > 0, \quad x \in R, \tag{1}$$
$$u(0, x) = u_0(x), \quad \eta(0, x) = \eta_0(x), \quad x \in R,$$

which is a model for wave motion on shallow water; $u(t, x)$ describes the horizontal velocity of the fluid, $g(u) \in C^\infty(R)$, $g(0) = 0$, η is horizontal deviation of the free surface elevation from equilibrium (or depth, in the shallow-water interpretation), and $\eta = (1 - \partial_x^2)\overline{\eta}$, where $\overline{\eta}$ indicates locally averaged density. The classical Camassa-Holm equation in [1] has the form

$$u_t - u_{txx} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{2}$$

Combining its integrability property with free surface elevation dynamics in [2], the authors have extended the Camassa-Holm equation to two-component Camassa-Holm system (CH2):

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - \eta\eta_x,$$
$$\eta_t + (u\eta)_x = 0. \tag{3}$$

Moreover, the local well-posedness and blow-up solutions for CH2 have been established in [3, 4]. It was shown that the system has strong solutions which blow up in finite time [5]. Using the localization analysis in the transport equation, Gui and Liu [6] obtained the global solutions of (3) in the Sobolev space $H^s \times H^{s-1}$ with $s > 3/2$. Hu [7] considered the weakly dissipative CH2 which includes a nonlinearly dissipative term $L(u)$, where L is a differential operator or a quasidifferential operator. It has been shown that CH2 system possesses peakon solutions in u and cornerlike solutions in η, but singular solutions do not exist for its density variable [8]. By setting $\eta = (1 - \partial_x^2)\overline{\eta}$, Holm et al. [8] derived the modified CH2 (MCH2):

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - \eta\overline{\eta}_x,$$
$$\eta_t + (u\eta)_x = 0, \tag{4}$$

where the $\overline{\eta}$ is averaged or filtered density. Then, this slight modification of the system CH2 leads to the system MCH2. In [9], the global conservative and dissipative solutions of (4) have been testified.

The Cauchy problem for (2) has been studied extensively. It has been shown that this equation has global strong solutions [10–12]. On the other hand, it has global weak solutions in $H^1(R)$ [13–15]. Weak solutions for a weakly dissipative Camassa-Holm equation have been obtained in [16]. Moreover, the global and dissipative solutions have been

established in [17–20]. It is worth remarking that Bressan and Constantin [17] derived the existence of global conservative solutions to (2). The essential point in [17] is to introduce some independent and dependent variables.

For $g(u) = au^m + ku$ (a, k is constant and m is positive integer), the generalized Camassa-Holm equation

$$u_t - u_{txx} + [g(u)]_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in R,$$
$$u(0, x) = u_0(x), \quad x \in R, \tag{5}$$

has been discussed in [21, 22], and the existence of travelling waves solutions and peaked solitary wave solutions has been found. Mustafa [23] and Yin [24] obtained results of local well-posedness. The global conservative solutions and dissipative solutions of (5) have been established in [25]. However, the conservative solutions for the problem (1) have not been discussed yet. The aim of this paper is to establish the existence of global conservations for the problem (1) and prove uniqueness and continuous dependence on the initial data.

This paper is organized as follows. The preliminary is given in Section 2. In Section 3, we come up with an equivalent semilinear system for the problem (1) and the global solution of this system is constructed. In Section 4, we prove that the semilinear system yields global conservation solutions to the problem (1).

2. Preliminaries

2.1. The Basic Equations.
Firstly, we give some notations.

We define Banach space X with the norm $\|\cdot\|_X$ and let $p(x) = (1/2)e^{-|x|}$, $x \in R$; then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2$. We set $\varphi = \overline{\eta} - \overline{\eta}_0$ ($\overline{\eta}_0$ is a constant) and $\eta = \varphi - \varphi_{xx}$; then $\varphi_x = \overline{\eta}_x$, $p * \eta = \varphi$.

We rewrite the equivalent form of (1) as follows:

$$u_t + uu_x = -\Psi_{1x}, \quad t > 0, \ x \in R,$$
$$\varphi_t + u\varphi_x = -\Psi_{2x} - \Psi_3, \quad t > 0, \ x \in R, \tag{6}$$
$$u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x), \quad x \in R,$$

with initial data $(u_0(x), \varphi_0(x)) \in H^1 \times (H^1 \cap W^{1,\infty})$ and in which Ψ_i ($i = 1, 2, 3$) are defined by

$$\Psi_1 = p * \left(u^2 + \frac{1}{2}u_x^2 + g(u) + \frac{1}{2}\varphi^2 - \frac{1}{2}\varphi_x^2\right),$$
$$\Psi_2 = p * (u_x \varphi_x), \tag{7}$$
$$\Psi_3 = p * (u_x \varphi),$$

where $p(x) = (1/2)e^{-|x|}$, $g(u) \in C^\infty(R)$, and $g(0) = 0$. If u is in $H^1(R)$, due to $\|u\|_{L^\infty} \le \|u\|_{H^1}$, $g(u) \in C^\infty(R)$, and $g(0) = 0$, we deduce that

$$|g(u(x))| \le \sup_{|v| \le \|u\|_{L^\infty}} |g'(v)| |u(x)| \le c\left(\|u\|_{L^\infty}\right) |u(x)|$$
$$\le c\left(\|u\|_{H^1}\right) |u(x)|. \tag{8}$$

Since $u, \varphi \in H^1$, Young's inequality ensures that $\Psi_1, \Psi_2, \Psi_3 \in H^1$.

Definition 1. For a solution of the Cauchy problem (6) on $[t_1, t_2]$, it means a Hölder continuous function $(u(t, x), \eta(t, x))$ is defined on $[t_1, t_2] \times R$ with the following properties. At each fixed t it has $(u(t, \cdot), \varphi(t, \cdot)) \in H^1 \times (H^1 \cap W^{1,\infty})$. Moreover, the maps $u(t, \cdot)$ and $\varphi(t, \cdot)$ are Lipschitz continuous from $[t_1, t_2]$ to L^2, satisfying the initial condition and

$$u_t + uu_x = -\Psi_{1x}, \quad t > 0, \ x \in R,$$
$$\varphi_t + u\varphi_x = -\Psi_{2x} - \Psi_3, \quad t > 0, \ x \in R, \tag{9}$$

where (9) is understood as equalities between functions in L^2.

For smooth solutions, we have the conservation law

$$E(t) = \int_R \left(u^2 + u_x^2 + \varphi^2 + \varphi_x^2\right) dx$$
$$= \int_R \left(u_0^2 + u_{0x}^2 + \varphi_0^2 + \varphi_{0x}^2\right) dx \tag{10}$$
$$= E(0) = E_0.$$

Indeed, differentiating the first equation of (6) with respect to x and using the identity $\partial_x^2 p * f = p * f - f$, we have

$$u_{tx} + uu_{xx} + u_x^2 = f - p * f, \tag{11}$$

where $f = u^2 + (1/2)u_x^2 + g(u) + (1/2)\varphi^2 - (1/2)\varphi_x^2$. Using problem (6) and the above equality, we have

$$\int_R (uu_t + u_x u_{tx}) dx$$
$$= \int_R \left\{-u^2 u_x - \partial_x p * \left[u^2 + \frac{1}{2}u_x^2 + g(u) + \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi_x^2\right] u\right.$$
$$\left. - u_x^3 - uu_x u_{xxx} - \partial_x^2 p\right.$$
$$\left. * \left[u^2 + \frac{1}{2}u_x^2 + g(u) + \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi_x^2\right] u_x\right\}$$
$$= \frac{1}{2} \int_R \left(\varphi^2 u_x - \varphi_x^2 u_x\right) dx, \tag{12}$$

where $\int_R g(u)u_x dx = G'(u)|_R = 0$.

In the same way, for the second equation in (6), we have

$$\int_R (\varphi\varphi_t + \varphi_x \varphi_{tx}) dx$$
$$= \int_R \left(-u\varphi\varphi_x - \varphi_x(u\varphi_x)_x + u_x \varphi_x^2 - u_x \varphi^2\right) dx. \tag{13}$$

Combining (12) with (13), we deduce that

$$
\frac{d}{dt} \int_R \left(u^2 + u_x^2 + \varphi^2 + \varphi_x^2 \right) dx
$$

$$
= 2 \int_R \left(uu_t + u_x u_{tx} + \varphi \varphi_t + \varphi_x \varphi_{tx} \right) dx
$$

$$
= 2 \int_R \left[\frac{1}{2} \varphi^2 u_x - \frac{1}{2} \varphi_x^2 u_x - u \varphi \varphi_x - \varphi_x \left(u \varphi_x \right)_x \right. \quad (14)
$$

$$
\left. + u_x \varphi_x^2 - u_x \varphi^2 \right] dx
$$

$$
= \int_R \left(-u \varphi^2 \right)_x - \left(u \varphi_x^2 \right)_x dx = 0.
$$

Thus, (10) holds.

2.2. A New Set of Independent and Dependent Variables. Let $(\overline{u}, \overline{\varphi}) \in H^1 \times (H^1 \cap W^{1,\infty})$ be the initial data. For an energy variable $\xi \in R$, let the nondecreasing map $\xi \mapsto \overline{y}(\xi)$ be defined by

$$
\xi = \int_0^{\overline{y}(\xi)} \left(1 + \overline{u}_x^2 \right) dx. \quad (15)
$$

Assuming that the solution (u, η) to (6) remains Lipschitz continuous for $t \in [0, T]$, we derive an equivalent system of (6) by using the independent variables (t, ξ).

Let $\xi \mapsto y(\xi)$ be the characteristic starting at $\overline{y}(\xi)$ such that

$$
\frac{\partial}{\partial t} y(t, \xi) = u(t, y(t, \xi)), \quad y(0, \xi) = \overline{y}(\xi). \quad (16)
$$

Moreover, we write

$$
u(t, \xi) \doteq u(t, y(t, \xi)), \quad \varphi(t, \xi) \doteq \varphi(t, y(t, \xi)),
$$

$$
\varphi_x(t, \xi) \doteq \varphi_x(t, y(t, \xi)),
$$

$$
\Psi_i(t, \xi) \doteq \Psi_i(t, y(t, \xi)), \quad \Psi_{ix}(t, \xi) \doteq \Psi_{ix}(t, y(t, \xi)),
$$

$$
i = 1, 2, 3. \quad (17)
$$

The following further variables will be used: $v = v(t, \xi)$, $q = q(t, \xi)$, and $\psi = \psi(t, \xi)$, defined as

$$
v \doteq 2 \arctan u_x, \quad q \doteq \left(1 + u_x^2 \right) \cdot \frac{\partial y}{\partial \xi}, \quad \psi \doteq \varphi_x(t, \xi). \quad (18)
$$

Obviously, v is defined up to multiples of 2π. Notice that (15) implies

$$
q(0, \xi) = 1. \quad (19)
$$

For future use, we write the identities

$$
\frac{1}{1 + u_x^2} = \cos^2 \frac{v}{2}, \quad \frac{u_x}{1 + u_x^2} = \frac{1}{2} \sin v,
$$

$$
\frac{u_x^2}{1 + u_x^2} = \sin^2 \frac{v}{2}, \quad (20)
$$

$$
\frac{\partial y}{\partial \xi} = \frac{q}{1 + u_x^2} = \cos^2 \frac{v}{2} \cdot q. \quad (21)
$$

Using identity (21) yields

$$
y(t, \xi_2) - y(t, \xi_1) = \int_{\xi_1}^{\xi_2} \frac{\partial y}{\partial \xi} d\xi = \int_{\xi_1}^{\xi_2} \cos^2 \frac{v(t, \xi)}{2} q(t, \xi) d\xi. \quad (22)
$$

Furthermore, we have

$$
\Psi_1(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y(t,\xi)-x|}
$$

$$
\times \left(u^2 + \frac{1}{2} u_x^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \varphi_x^2 \right)
$$

$$
\times (t, x) dx,
$$

$$
\Psi_{1x}(t, \xi) = \frac{1}{2} \left(\int_{y(t,\xi)}^{\infty} - \int_{-\infty}^{y(t,\xi)} \right) e^{-|y(t,\xi)-x|}
$$

$$
\times \left(u^2 + \frac{1}{2} u_x^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \varphi_x^2 \right)
$$

$$
\times (t, x) dx,
$$

$$
\Psi_2(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y(t,\xi)-x|} u_x(t, x) \varphi_x(t, x) dx,
$$

$$
\Psi_{2x}(t, \xi) = \frac{1}{2} \left(\int_{y(t,\xi)}^{\infty} - \int_{-\infty}^{y(t,\xi)} \right) e^{-|y(t,\xi)-x|}
$$

$$
\times u_x(t, x) \varphi_x(t, x) dx,
$$

$$
\Psi_3(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y(t,\xi)-x|} u_x(t, x) \varphi(t, x) dx,
$$

$$
\Psi_{3x}(t, \xi) = \frac{1}{2} \left(\int_{y(t,\xi)}^{\infty} - \int_{-\infty}^{y(t,\xi)} \right) e^{-|y(t,\xi)-x|} u_x(t, x) \varphi(t, x) dx. \quad (23)
$$

In the above formulae, we use the change of variables $x = y(t, \xi')$, the validity of which will be checked in Section 4, and write the convolution as an integral over the variable ξ'. Using

identities (20)–(22), we obtain expressions for Ψ_i and Ψ_{ix} in terms of the new variable ξ, namely,

$$
\Psi_1(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\times\left[\left[u^2(\xi') + g(u(\xi'))\right]\cos^2\frac{v(\xi')}{2}\right.
$$
$$
+\frac{1}{2}\sin^2\frac{v(\xi')}{2} + \frac{1}{2}\varphi^2(\xi')\cos^2\frac{v(\xi')}{2}
$$
$$
\left.-\frac{1}{2}\psi^2(\xi')\cos^2\frac{v(\xi')}{2}\right]q(\xi')\,d\xi',
$$
(24)

$$
\Psi_{1x}(\xi) = \frac{1}{2}\left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\times\left[\left[u^2(\xi') + g(u(\xi'))\right]\cos^2\frac{v(\xi')}{2}\right.
$$
$$
+\frac{1}{2}\sin^2\frac{v(\xi')}{2} + \frac{1}{2}\varphi^2(\xi')\cos^2\frac{v(\xi')}{2}
$$
$$
\left.-\frac{1}{2}\psi^2(\xi')\cos^2\frac{v(\xi')}{2}\right]q(\xi')\,d\xi',
$$
(25)

$$
\Psi_2(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\times\frac{1}{2}\psi(\xi')\sin v(\xi')q(\xi')\,d\xi',
$$
(26)

$$
\Psi_{2x}(\xi) = \frac{1}{2}\left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\cdot\frac{1}{2}\psi(\xi')\sin v(\xi')q(\xi')\,d\xi',
$$
(27)

$$
\Psi_3(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\times\frac{1}{2}\varphi(\xi')\sin v(\xi')q(\xi')\,d\xi',
$$
(28)

$$
\Psi_{3x}(\xi) = \frac{1}{2}\left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_{\xi}^{\xi'}\cos^2\frac{v(s)}{2}q(s)\,ds\right|\right\}
$$
$$
\cdot\frac{1}{2}\varphi(\xi')\sin v(\xi')q(\xi')\,d\xi'.
$$
(29)

From (6) and (16), the evolution equations for u and η in the new variables (t,ξ) take the form

$$
\frac{\partial}{\partial t}u(t,\xi) = u_t + uu_x = -\Psi_{1x}(t,\xi),
$$
$$
\frac{\partial}{\partial t}\varphi(t,\xi) = \varphi_t + u\varphi_x = -\Psi_{2x}(t,\xi) - \Psi_3(t,\xi),
$$
(30)

where Ψ_i and Ψ_{ix} $(i = 1, 2, 3)$ are given in (24)–(29).

Next, to derive an evolution equation for the variable q, we observe from (16) that

$$
\int_{\xi_1}^{\xi_2}q(t,\xi)\,d\xi = \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left(1 + u_x^2(t,x)\right)dx
$$
(31)

holds. We have the equality

$$
\frac{d}{dt}\int_{\xi_1}^{\xi_2}q(t,\xi)\,d\xi = \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left\{\left(1+u_x^2\right)_t + \left[u\left(1+u_x^2\right)\right]_x\right\}dx
$$
$$
= \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left[2u_xu_{xt} + u_x + \left(uu_x^2\right)_x\right]dx
$$
$$
= \int_{y(t,\xi_1)}^{y(t,\xi_2)}\left[2u^2 + 2g(u) + \varphi^2 - \varphi_x^2\right.
$$
$$
\left.-2\Psi_1 + 1\right]u_x\,dx.
$$
(32)

Therefore, it has

$$
\frac{\partial}{\partial t}q(t,\xi) = \left(u^2 + g(u) + \frac{1}{2}\varphi^2 - \frac{1}{2}\psi^2 - \Psi_1 + \frac{1}{2}\right)\sin v\cdot q.
$$
(33)

Applying (6), (9), and (16)–(18) yields

$$
\frac{\partial}{\partial t}v(t,\xi) = \left(2u^2 + 2g(u) + \varphi^2 - \psi^2 - 2\Psi_1\right)\cos^2\frac{v}{2} - \sin^2\frac{v}{2},
$$
$$
\frac{\partial}{\partial t}\psi(t,\xi) = -\Psi_2 - \Psi_{3x}.
$$
(34)

3. Global Solutions of the Semilinear System

Let initial data $(\overline{u},\overline{\varphi}) \in H^1 \times (H^1 \cap W^{1,\infty})$ be given. From Section 2, we rewrite the corresponding Cauchy problem (6) for the variables (u, φ, ψ, v, q) in the form

$$
\frac{\partial u}{\partial t} = -\Psi_{1x},
$$
$$
\frac{\partial \varphi}{\partial t} = -\Psi_{2x} - \Psi_3,
$$
$$
\frac{\partial \psi}{\partial t} = -\Psi_2 - \Psi_{3x},
$$
(35)
$$
\frac{\partial v}{\partial t} = \left(2u^2 + 2g(u) + \varphi^2 - \psi^2 - 2\Psi_1\right)\cos^2\frac{v}{2} - \sin^2\frac{v}{2},
$$
$$
\frac{\partial q}{\partial t} = \left(u^2 + g(u) + \frac{1}{2}\varphi^2 - \frac{1}{2}\psi^2 - \Psi_1 + \frac{1}{2}\right)\sin v\cdot q,
$$

with initial data

$$u(0, \xi) = \overline{u}(\overline{y}(\xi)),$$

$$\varphi(0, \xi) = \overline{\varphi}(\overline{y}(\xi)),$$

$$\psi(0, \xi) = \overline{\varphi}_x(\overline{y}(\xi)), \tag{36}$$

$$v(0, \xi) = 2 \arctan \overline{u}_x(\overline{y}(\xi)),$$

$$q(0, \xi) = 1.$$

We regard (35) as an ordinary differential equation in the Banach space:

$$X \doteq H^1 \times H^1 \times (L^2 \cap L^\infty) \times (L^2 \cap L^\infty) \times L^\infty, \tag{37}$$

with the norm

$$\|(u, \varphi, \psi, v, q)\|_X \doteq \|u\|_{H^1} + \|\varphi\|_{H^1} + \|\psi\|_{L^2}$$
$$+ \|\psi\|_{L^\infty} + \|v\|_{L^2} + \|v\|_{L^\infty} + \|q\|_{L^\infty}. \tag{38}$$

Then the solution of the Cauchy problem means a fixed point of the integral transformation:

$$T(u, \varphi, \psi, v, q) = (\widetilde{u}, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{v}, \widetilde{q}), \tag{39}$$

where

$$\widetilde{u}(t, \xi) = \overline{u}(\overline{y}(\xi)) - \int_0^t \Psi_{1x} dt,$$

$$\widetilde{\varphi}(t, \xi) = \overline{\varphi}(\overline{y}(\xi)) - \int_0^t (\Psi_{2x} + \Psi_3) dt,$$

$$\widetilde{\psi}(t, \xi) = \overline{\varphi}_x(\overline{y}(\xi)) - \int_0^t (\Psi_2 + \Psi_{3x}) dt,$$

$$\widetilde{v}(t, \xi) = 2 \arctan \overline{u}_x(\overline{y}(\xi))$$
$$+ \int_0^t \left[\left(2u^2 + 2g(u) + \varphi^2 - \psi^2 - 2\Psi_1 \right) \right.$$
$$\left. \times \cos^2 \frac{v}{2} - \sin^2 \frac{v}{2} \right] dt, \tag{40}$$

$$\widetilde{q}(t, \xi) = 1 + \int_0^t \left(u^2 + g(u) + \frac{1}{2}\varphi^2 - \frac{1}{2}\psi^2 \right.$$
$$\left. -\Psi_1 + \frac{1}{2} \right) \sin v \cdot q \, dt.$$

Proving the local Lipschitz continuity of the right-hand side of (35), the local existence of solution follows from the standard theorem for ordinary differential equations in Banach spaces. Then, we show the conservation of energy property expressed by (10). Moreover, we prove that this local solution can be extended globally in time.

Theorem 2. *If* $(\overline{u}, \overline{\varphi}) \in H^1 \times (H^1 \cap W^{1,\infty})$, *then the Cauchy problem* (35)-(36) *has a unique solution for all* $t \geq 0$ *in the sense of Definition 1.*

Proof.

Step 1 (local solution). In order to establish the local existence of solution, it suffices to show that the operator determined by the right-hand side of (35) mapping (u, φ, ψ, v, q) to

$$\left(-\Psi_{1x}, -\Psi_{2x} - \Psi_3, -\Psi_2 - \Psi_{3x}, \right.$$
$$\left(2u^2 + 2g(u) + \varphi^2 - \psi^2 - 2\Psi_1 \right)$$
$$\times \cos^2 \frac{v}{2} - \sin^2 \frac{v}{2},$$
$$\left. \left(u^2 + g(u) + \frac{1}{2}\varphi^2 - \frac{1}{2}\psi^2 - \Psi_1 + \frac{1}{2} \right) \sin v \cdot q \right) \tag{41}$$

is Lipschitz continuous on every bounded domain $\Omega \subset X$ in the form of

$$\Omega = \left\{ (u, \varphi, \psi, v, q) : \|u\|_{H^1} \leq \alpha, \|\varphi\|_{H^1} \leq \beta, \right.$$
$$\|\psi\|_{L^2} \leq m, \|\psi\|_{L^\infty} \leq \gamma, \tag{42}$$
$$\left. \|v\|_{L^2} \leq n, \|v\|_{L^\infty} \leq \frac{3\pi}{2}, q(x) \in [q^-, q^+] \right\},$$

for any constants $\alpha, \beta, m, n, \gamma, q^-, q^+ > 0$.

Applying Sobolev's inequality $\|u\|_{L^\infty} \leq \|u\|_{H^1}$, $\|\varphi\|_{L^\infty} \leq \|\varphi\|_{H^1}$, we know that the maps

$$u^2, \quad g(u), \quad g(u)\cos v, \quad u^2 \cos^2 \frac{v}{2}, \quad \varphi^2, \quad \varphi^2 \cos^2 \frac{v}{2},$$
$$\sin v, \quad \sin^2 \frac{v}{2}, \quad \sin v \cdot q, \quad \varphi^2, \quad \varphi^2 \cos^2 \frac{v}{2} \tag{43}$$

are all Lipschitz continuous from $\Omega \mapsto L^2 \cap L^\infty$. Now, we only need to prove the Lipschitz continuity of the maps

$$(u, \varphi, \psi, v, q) \longmapsto \Psi_i, \quad (u, \varphi, \psi, v, q) \longmapsto \Psi_{ix}, \quad i = 1, 2, 3, \tag{44}$$

defined in (24)–(29) from Ω to H^1. This will also imply the Lipschitz continuity of these maps from Ω to $L^2 \cap L^\infty$. To get this goal, we first observe that, as long as $|v| \leq 3\pi/2$, it holds that

$$\sin^2 \frac{v}{2} \leq \frac{v^2}{4} \leq \frac{9\pi^2}{8} \sin^2 \frac{v}{2},$$

$$\text{measure} \left\{ \xi \in R; \left| \frac{v(\xi)}{2} \right| \geq \frac{\pi}{4} \right\}$$

$$\leq \text{measure} \left\{ \xi \in R; \sin^2 \frac{v(\xi)}{2} \geq \frac{1}{18} \right\} \tag{45}$$

$$\leq 18 \int_{\{\xi \in R; \sin^2(v(\xi)/2) \geq 1/18\}} \sin^2 \frac{v(\xi)}{2} d\xi$$

$$\leq \frac{9}{2} n^2.$$

Therefore, for any $\xi_1 < \xi_2$, it holds that

$$\int_{\xi_1}^{\xi_2} \cos^2 \frac{\nu(\xi)}{2} q(\xi') \, d\xi \geq \int_{\{\xi \in [\xi_1, \xi_2]: |\nu(\xi)/2| \leq \pi/4\}} \frac{q^-}{2} \, d\xi$$
$$\geq q^- \left(\frac{\xi_1 - \xi_2}{2} - \frac{9}{4} n^2 \right). \tag{46}$$

Introducing the exponentially decaying function

$$F(\zeta) \doteq \min \left\{ 1, \exp \left(\frac{9}{4} n^2 q^- - \frac{|\zeta|}{2} q^- \right) \right\}, \tag{47}$$

we show that

$$\|F\|_{L^1} = \left(\int_{|\zeta| \leq (9/2)n^2} + \int_{|\zeta| \geq (9/2)n^2} \right) F(\zeta) \, d\zeta = 9n^2 + \frac{4}{q^-}. \tag{48}$$

Next, we begin to show $\Psi_i, \Psi_{ix} \in H^1$, $i = 1, 2, 3$. We only give the estimates for Ψ_1, since the estimates for others are similar. It follows from definition (24) that

$$|\Psi_1(\xi)| \leq \frac{q^+}{2} \left| F * \left((u^2 + g(u)) \cos^2 \frac{\nu}{2} \right. \right.$$
$$+ \frac{1}{2} \sin^2 \frac{\nu}{2} + \frac{1}{2} \varphi^2 \cos^2 \frac{\nu}{2} \tag{49}$$
$$\left. \left. - \frac{1}{2} \psi^2 \cos^2 \frac{\nu}{2} \right)(\xi) \right|.$$

Using standard properties of convolutions, we obtain

$$\|\Psi_1(\xi)\|_{L^2} \leq \frac{q^+}{2} \|F\|_{L^1}$$
$$\cdot \left(\|u^2 + g(u)\|_{L^2} + \frac{1}{8} \|v^2\|_{L^2} + \frac{1}{2} \|\varphi^2\|_{L^2} \right.$$
$$\left. + \frac{1}{2} \|\psi^2\|_{L^2} \right)$$
$$\leq \frac{q^+}{2} \|F\|_{L^1} \left(C \|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^\infty} \|v\|_{L^2} \right.$$
$$\left. + \frac{1}{2} \|\varphi\|_{L^\infty} \|\varphi\|_{L^2} + \frac{1}{2} \|\psi\|_{L^\infty} \|\psi\|_{L^2} \right)$$
$$< \infty, \tag{50}$$

where we have used $H^1 \subset L^\infty$ and (8). Differentiating (24), we get

$$\partial_\xi \Psi_1(\xi) = \frac{1}{2} \int_{-\infty}^\infty \exp \left\{ - \left| \int_\xi^{\xi'} \cos^2 \frac{\nu(s)}{2} q(s) \, ds \right| \right\}$$
$$\cdot \cos^2 \frac{\nu(\xi)}{2} q(\xi) \operatorname{sign}(\xi' - \xi)$$
$$\cdot \left[[u^2(\xi') + g(u(\xi'))] \cos^2 \frac{\nu(\xi')}{2} \right.$$
$$+ \frac{1}{2} \sin^2 \frac{\nu(\xi')}{2} \tag{51}$$
$$+ \frac{1}{2} \varphi^2(\xi') \cos^2 \frac{\nu(\xi')}{2}$$
$$\left. - \frac{1}{2} \psi^2(\xi') \cos^2 \frac{\nu(\xi')}{2} \right]$$
$$\cdot q(\xi') \, d\xi',$$

thus

$$|\partial_\xi \Psi_1(\xi)| \leq \frac{1}{2} (q^+)^2 \left| \cos^2 \frac{\nu}{2} \right|$$
$$\cdot \left| F * \left([u^2 + g(u)] \cos^2 \frac{\nu}{2} + \frac{1}{2} \sin^2 \frac{\nu}{2} \right. \right.$$
$$\left. + \frac{1}{2} \varphi^2 \cos^2 \frac{\nu}{2} - \frac{1}{2} \psi^2 \cos^2 \frac{\nu}{2} \right) \bigg| \tag{52}$$
$$\leq \frac{1}{2} (q^+)^2 \cdot \left| F * \left[\left(u^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \psi^2 \right) \right. \right.$$
$$\left. \left. \times \cos^2 \frac{\nu}{2} + \frac{1}{2} \sin^2 \frac{\nu}{2} \right] \right|.$$

Therefore, we have

$$\|\partial_\xi \Psi_1\|_{L^2} \leq \frac{1}{2} (q^+)^2 \|F\|_{L^1}$$
$$\cdot \left(C \|u\|_{L^\infty} \|u\|_{L^2} + \frac{1}{8} \|v\|_{L^\infty} \|v\|_{L^2} \right.$$
$$+ \frac{1}{2} \|\varphi\|_{L^\infty} \|\varphi\|_{L^2} \tag{53}$$
$$\left. + \frac{1}{2} \|\psi\|_{L^\infty} \|\psi\|_{L^2} \right)$$
$$< \infty,$$

and $\Psi_1 \in H^1(R)$.

To establish the Lipschitz continuity of (44), it suffices to show that their partial derivatives

$$\frac{\partial \Psi_i}{\partial u}, \frac{\partial \Psi_i}{\partial \varphi}, \frac{\partial \Psi_i}{\partial \psi}, \frac{\partial \Psi_i}{\partial v}, \frac{\partial \Psi_i}{\partial q},$$
$$\frac{\partial \Psi_{ix}}{\partial u}, \frac{\partial \Psi_{ix}}{\partial \varphi}, \frac{\partial \Psi_{ix}}{\partial \psi}, \frac{\partial \Psi_{ix}}{\partial v}, \frac{\partial \Psi_{ix}}{\partial q}, \quad i = 1, 2, 3 \tag{54}$$

are uniformly bounded as (u, φ, ψ, v, q) ranges inside the domain Ω. We observe that these derivatives are bounded operators from the appropriate space into H^1. For sake of illustration, we will work out the detailed estimate for $\partial \Psi_1 / \partial u$. All other derivatives can be estimated by the same way.

For a given point $(u, \varphi, \psi, v, q) \in \Omega$, the partial derivative $(\partial \Psi_1 / \partial u) : H^1 \to L^2$ is the linear operator defined by

$$
\begin{aligned}
&\left[\frac{\partial \Psi_1 (u, \varphi, \psi, v, q)}{\partial u} \cdot \tilde{u} \right] (\xi) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{v(s)}{2} q(s) \, ds \right| \right\} \\
&\quad \cdot \left[2u(\xi') + g'(u(\xi')) \right] \cos^2 \frac{v(\xi')}{2} q(\xi') \\
&\quad \cdot \tilde{u}(\xi') \, d\xi'.
\end{aligned}
\tag{55}
$$

Thus,

$$
\left\| \frac{\partial \Psi_1}{\partial u} \cdot \tilde{u} \right\|_{L^2} \leq \frac{q^+}{2} \left\| F * \left[2u + g'(u) \right] \right\|_{L^2} \cdot \| \tilde{u} \|_{L^\infty}.
\tag{56}
$$

By use of $\| \tilde{u} \|_{L^\infty} \leq \| \tilde{u} \|_{H^1}$ and $|g'(u)| \leq C(\| u \|_{H^1}) |u(x)|$, the above operator norm satisfies

$$
\left\| \frac{\partial \Psi_1}{\partial u} \right\| \leq C q^+ \| F \|_{L^1} \| u \|_{L^2} < \infty.
\tag{57}
$$

From (51), $(\partial (\partial_\xi \Psi_1) / \partial u) : H^1 \to L^2$ is the linear operator defined by

$$
\begin{aligned}
&\left[\frac{\partial (\partial_\xi \Psi_1)(u, \varphi, \psi, v, q)}{\partial u} \cdot \tilde{u} \right] (\xi) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{v(s)}{2} q(s) \, ds \right| \right\} \\
&\quad \cdot \cos^2 \frac{v(\xi)}{2} q(\xi) \operatorname{sign}(\xi' - \xi) \\
&\quad \cdot \left[2u(\xi') + g'(u(\xi')) \right] \\
&\quad \cdot \cos^2 \frac{v(\xi')}{2} q(\xi') \cdot \tilde{u}(\xi') \, d\xi',
\end{aligned}
\tag{58}
$$

whose norm is bounded by

$$
\left\| \frac{\partial (\partial_\xi \Psi_1)}{\partial u} \right\| \leq C(q^+)^2 \| F \|_{L^1} \| u \|_{L^2} < \infty.
\tag{59}
$$

Combining (57) and (59) yields the boundedness of $\partial \Psi_1 / \partial u$ as a linear operator from H^1 to H^1. The bounds on the other partial derivatives in (54) are obtained by a similar way.

The local existence of a solution to the Cauchy problem (35)-(36) on some small time interval $[0, T]$ with $T > 0$

now follows from the standard theory of ordinary differential equations in Banach space. Thus, we have showed that the right-hand side of (35) is Lipschitz continuity on a neighbourhood of the initial data in the space X.

Step 2 (extension to a global solution). To ensure that the local solution of (35) constructed above can be extended to a global solution defined for all $t \geq 0$, it suffices to show that the quantity

$$
\| u \|_{H^1} + \| \varphi \|_{H^1} + \| \psi \|_{L^2} + \| \psi \|_{L^\infty} + \| v \|_{L^2} + \| v \|_{L^\infty} + \| q \|_{L^\infty}
\tag{60}
$$

remains uniformly bounded on any bounded time interval. The a priori bounds on (60) will actually follow from the conservation of the total energy (10). In the following, we rederive this energy conservation property in terms of the new variables (u, φ, ψ, v, q) and ξ.

As long as the local solution of (35) is defined, we claim that

$$
u_\xi = \frac{q}{2} \sin v,
\tag{61}
$$

$$
\varphi_\xi = \psi \cos^2 \frac{v}{2} q,
\tag{62}
$$

$$
\frac{d}{dt} \int_R \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2} + \varphi^2 \cos^2 \frac{v}{2} + \psi^2 \cos^2 \frac{v}{2} \right) \cdot q \, d\xi = 0.
\tag{63}
$$

Using (35) and (25) yields

$$
\begin{aligned}
u_{\xi t} &= \left[\left(u^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \psi^2 \right) \cos^2 \frac{v}{2} + \frac{1}{2} \sin^2 \frac{v}{2} \right] \\
&\quad \cdot q - \cos^2 \frac{v}{2} q \cdot \Psi_1.
\end{aligned}
\tag{64}
$$

From (35), we get

$$
\begin{aligned}
\left(\frac{q}{2} \sin v \right)_t &= \frac{q_t}{2} \sin v + \frac{q}{2} \cos v \cdot v_t \\
&= \frac{\sin v}{2} \left(u^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \psi^2 \right. \\
&\qquad \left. - \Psi_1 + \frac{1}{2} \right) \sin v \cdot q \\
&\quad + \frac{q}{2} \cos v \left(2u^2 + 2g(u) + \varphi^2 \right. \\
&\qquad \left. - \psi^2 - 2\Psi_1 \right) \cos^2 \frac{v}{2} - \sin^2 \frac{v}{2} \\
&= \left[\left(u^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \psi^2 \right) \cos^2 \frac{v}{2} \right. \\
&\qquad \left. + \frac{1}{2} \sin^2 \frac{v}{2} \right] \cdot q - \cos^2 \frac{v}{2} q \cdot \Psi_1.
\end{aligned}
\tag{65}
$$

Thus, we have

$$
u_{\xi t} = \left(\frac{q}{2} \sin v \right)_t.
\tag{66}
$$

Moreover, at the initial time $t = 0$, using (20) and (36), we get

$$\frac{\partial u}{\partial \xi} = \frac{u_x}{1 + u_x^2} = \frac{\sin v}{2}, \quad q \equiv 1. \tag{67}$$

Therefore, we obtain that (61) remains valid for all times $t \geq 0$ as long as the solution is defined. The (62) holds true by the same way.

To prove (63), we proceed as follows. From (35), we deduce that

$$\frac{d}{dt} \int_R \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2} + \varphi^2 \cos^2 \frac{v}{2} + \psi^2 \cos^2 \frac{v}{2} \right) \cdot q \, d\xi$$
$$= \int_R \left(\frac{3}{2} u^2 + g(u) + \varphi^2 - \Psi_1 \right) \sin v \cdot q \, d\xi$$
$$- 2 \int_R \left[u \Psi_{1x} + \varphi \left(\Psi_{2x} + \Psi_3 \right) + \psi \left(\Psi_2 + \Psi_{3x} \right) \right]$$
$$\times \cos^2 \frac{v}{2} \cdot q \, d\xi. \tag{68}$$

On the other hand, from (24)–(29), we have

$$\Psi_{i\xi} = \Psi_{ix} \cos^2 \frac{v}{2} \cdot q, \quad i = 1, 2, 3. \tag{69}$$

Applying (61)-(62) and (68)-(69), we have

$$\frac{d}{dt} \int_R \left(u^2 \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2} + \varphi^2 \cos^2 \frac{v}{2} + \psi^2 \cos^2 \frac{v}{2} \right) \cdot q \, d\xi$$
$$= \int_R \left(3u^2 + 2g(u) + 2\varphi^2 - 2\Psi_1 \right) u_\xi \, d\xi$$
$$- 2 \int_R \left(u \Psi_{1\xi} + \varphi \Psi_{2\xi} + \varphi \Psi_3 \cos^2 \frac{v}{2} \cdot q \right.$$
$$\left. + \varphi_\xi \Psi_2 + \varphi_\xi \Psi_{3\xi} \right) d\xi$$
$$= 2 \int_R \left[\varphi^2 u_\xi - \varphi \Psi_3 \cos^2 \frac{v}{2} \cdot q \right.$$
$$\left. + \varphi \left(\Psi_3 \cos^2 \frac{v}{2} \cdot q - \varphi \sin v \cdot \frac{v}{2} \right) \right]$$
$$= 0, \tag{70}$$

which implies (63).

We can now rewrite the total energy (10) in terms of the new variables. According to (70), we have that the energy remains constant in time,

$$E(t) = \int_R \left(u^2(t, \xi) \cos^2 \frac{v(t, \xi)}{2} + \sin^2 \frac{v(t, \xi)}{2} \right.$$
$$+ \varphi^2(t, \xi) \cos^2 \frac{v(t, \xi)}{2}$$
$$\left. + \psi^2(t, \xi) \cos^2 \frac{v(t, \xi)}{2} \right) \cdot q(t, \xi) \, d\xi \tag{71}$$
$$= E(0) = E_0,$$

along any solution of (35)-(36).

From (61) and (71), we have

$$\sup_{\xi \in R} \left| u^2(t, \xi) \right| \leq 2 \int_R \left| u u_\xi \right| d\xi$$
$$\leq 2 \int_R |u| \left| \sin \frac{v}{2} \cos \frac{v}{2} \right| q \, d\xi \leq E_0, \tag{72}$$

$$\sup_{\xi \in R} \left| \varphi^2(t, \xi) \right| \leq 2 \int_R \left| \varphi \varphi_\xi \right| d\xi \leq 2 \int_R \left| \varphi \psi \cos^2 \frac{v}{2} \right| q \, d\xi \leq E_0. \tag{73}$$

Applying (8) and (72), we get $|g(u)| \leq CE_0$. Moreover, it holds that

$$\left\| \Psi_1 \right\|_{L^\infty}, \left\| \Psi_{1x} \right\|_{L^\infty}$$
$$\leq \left\| \frac{1}{2} e^{|\xi|} \right\|_{L^\infty}$$
$$\cdot \left\| \left[\left(u^2 + g(u) + \varphi^2 + \psi^2 \right) \cos^2 \frac{v}{2} + \sin^2 \frac{v}{2} \right] \cdot q \right\|_{L^1} \tag{74}$$
$$\leq CE_0.$$

Notice that

$$\varphi(\xi') \sin v(\xi') \leq \varphi(\xi)^2 \cos^2 \frac{v(\xi')}{2} + \sin^2 \frac{v(\xi')}{2},$$
$$\psi(\xi') \sin v(\xi') \leq \psi(\xi)^2 \cos^2 \frac{v(\xi')}{2} + \sin^2 \frac{v(\xi')}{2}. \tag{75}$$

Similar calculations yield

$$\left\| \Psi_2 \right\|_{L^\infty}, \quad \left\| \Psi_{2x} \right\|_{L^\infty} \leq CE_0,$$
$$\left\| \Psi_3 \right\|_{L^\infty}, \quad \left\| \Psi_{3x} \right\|_{L^\infty} \leq CE_0. \tag{76}$$

Noting (35) and (76), we deduce that, as long as the solution is defined,

$$\left| \psi_t \right| \leq E_0,$$
$$\left\| \psi \right\|_{L^\infty} \leq E_0 t + \left\| \overline{\varphi}_x \right\|_{L^\infty}. \tag{77}$$

Thus, we have

$$\left| q_t \right| \leq CE_0 \|q\|_{L^\infty}, \quad C \geq 0. \tag{78}$$

Since $q(0, \xi) = 1$, using Gronwall's inequality, it holds that

$$e^{-CE_0 t} \leq \|q(t)\|_{L^\infty} \leq e^{CE_0 t}. \tag{79}$$

From (35), it derives that

$$\|v(t)\|_{L^\infty} \leq e^{Mt}, \quad M = M\left(E_0, \|\overline{\varphi}\|_{W^{1,\infty}} \right). \tag{80}$$

Moreover, (35) implies

$$\left| \frac{d}{dt} \left(\int_R u^2(t,\xi)\, d\xi \right) \right| \leq 2 \|u(t)\|_{L^\infty} \|\Psi_{1x}(t)\|_{L^1},$$

$$\left| \frac{d}{dt} \left(\int_R u_\xi^2(t,\xi)\, d\xi \right) \right| \leq 2 \|u_\xi(t)\|_{L^\infty} \|\partial_\xi \Psi_{1x}(t)\|_{L^1},$$

$$\left| \frac{d}{dt} \left(\int_R \varphi^2(t,\xi)\, d\xi \right) \right| \leq 2 \|\varphi(t)\|_{L^\infty} \left(\|\Psi_{2x}(t)\|_{L^1} + \|\Psi_3\|_{L^\infty} \right),$$

$$\left| \frac{d}{dt} \left(\int_R \varphi_\xi^2(t,\xi)\, d\xi \right) \right|$$

$$\leq 2 \|\varphi_\xi(t)\|_{L^\infty} \left(\|\partial_\xi \Psi_{2x}(t)\|_{L^1} + \|\partial_\xi \Psi_3\|_{L^\infty} \right). \tag{81}$$

From (61)–(63) and (72)–(73), we get uniform bounds of u, u_ξ, φ, φ_ξ on bounded time interval, respectively. The estimates on $\|u\|_{H^1}$ and $\|\varphi\|_{H^1}$ will follow from bounds on the L^1-norms of $\Psi_{ix}(t)$ and $\partial_\xi \Psi_{ix}(t)$, $i = 1, 2, 3$. To get this goal, letting r be the right-hand side of (79), we have $r^{-1} \leq \|q(t)\|_{L^\infty} \leq r$. Indeed, from (8) and (35), we have

$$\left| \left[\left(g(u) + u^2 \right) \cos^2 \frac{v}{2} + \frac{1}{2} \sin^2 \frac{v}{2} \right] q(t) \right| \leq C E_0. \tag{82}$$

It follows from (51) that

$$\|\partial_\xi \Psi_1(t)\|_{L^1} \leq \frac{C}{2} E_0 \int_R \exp \left\{ -\left| \int_\xi^{\xi'} r^{-1} \cos^2 \frac{v}{2}\, ds \right| \right\} d\xi \tag{83}$$

$$\leq C E_0 \|F\|_{L^1},$$

where

$$F(\zeta) \doteq \min \left\{ 1, \exp \left(9 E_0 r^{-1} - r^{-1} \frac{|\zeta|}{2} \right) \right\}. \tag{84}$$

We deduce

$$\|F\|_{L^1} \leq 36 E_0 + 4r. \tag{85}$$

The estimate for $\|\partial_\xi \Psi_{1x}\|_{L^1}$ is similar. Note that $\varphi \sin v \leq \varphi^2 \cos^2(v/2) + \sin^2(v/2)$, $\psi \sin v \leq \psi^2 \cos^2(v/2) + \sin^2(v/2)$; similar calculations show that the L^1-norms of Ψ_{2x}, $\partial_\xi \Psi_{2x}$, Ψ_{3x}, $\partial_\xi \Psi_{3x}$ are uniformly bounded. This proves the boundedness of $\|u\|_{H^1}$ and $\|\varphi\|_{H^1}$ for t in bounded interval.

Finally, the equations in (35) imply that

$$\frac{d}{dt} \|\psi\|_{L^2}^2 \leq \|\psi\|_{L^\infty} \left(\|\Psi_2\|_{L^1} + \|\Psi_{3x}\|_{L^1} \right). \tag{86}$$

Combining the bounds $\|\Psi_2\|_{L^1}$, $\|\Psi_{3x}\|_{L^1}$ with (77), we have that $\|\psi\|_{L^\infty}$ is uniformly bounded for t in bounded interval. Thus,

$$\frac{d}{dt} \|v\|_{L^2}^2 \leq C \left(\|u\|_{L^\infty} \|u\|_{L^2} + \|\varphi\|_{L^\infty} \|\varphi\|_{L^2} \right.$$

$$\left. + \|\psi\|_{L^\infty} \|\psi\|_{L^2} + \|v\|_{L^\infty} \|u\|_{L^2} \right). \tag{87}$$

By the previous bounds, it is clear that $\|v\|_{L^2}$ remains bounded on bounded time interval. This shows that the solution of (35) can be extended globally in time. □

For future use, we record here a property of the solution. Namely, consider the set of times

$$\mathfrak{R} \doteq \{ t \geq 0 : \text{measure} \{ \xi \in R : v(t,\xi) = -\pi \} > 0 \}. \tag{88}$$

Then, we claim

$$\text{measure} (\mathfrak{R}) = 0. \tag{89}$$

Its validity will be proved in the next section.

4. Solutions to the Original Equation

We now show that the global solution of the system (35) yields a global conservative solution to system (6) in the original variables (t, x).

Let us start with a global solution (u, φ, ψ, v, q) to (35). Define

$$y(t, \xi) \doteq \overline{y}(\xi) + \int_0^t u(\tau, \xi)\, d\tau. \tag{90}$$

For each fixed ξ, the function $t \mapsto y(t, \xi)$ provides a solution to the Cauchy problem:

$$\frac{\partial}{\partial t} y(t, \xi) = u(t, \xi), \qquad y(0, \xi) = \overline{y}(\xi). \tag{91}$$

We claim that a solution of (6) can be obtained by setting

$$u(t, x) \doteq u(t, \xi); \qquad \varphi(t, x) \doteq \varphi(t, \xi), \quad \text{if } y(t, \xi) = x. \tag{92}$$

Theorem 3. *Let (u, φ, ψ, v, q) provide a global solution to (35)-(36). Then the pair of functions $(u(t, x), \varphi(t, x))$ defined by (90)–(92) provide the global solution to the problem (6)-(7).*

The solution (u, φ) constructed in this way has the following properties. The energy is almost always conserved; namely,

$$\|u(t)\|_{H^1} + \|\varphi(t)\|_{H^1} = \|\overline{u}(t)\|_{H^1} + \|\overline{\varphi}(t)\|_{H^1}, \quad a.e. \ t \geq 0. \tag{93}$$

Furthermore, consider the sequence of initial data $(\overline{u}_n, \overline{\varphi}_n)$, such that

$$\|\overline{u}_n - \overline{u}\|_{H^1} \longrightarrow 0, \qquad \|\overline{\varphi}_n - \overline{\varphi}\|_{W^{1,\infty}} \longrightarrow 0,$$

$$\|\overline{\varphi}_n - \overline{\varphi}\|_{H^1} \longrightarrow 0. \tag{94}$$

Then the corresponding solutions $(u_n(t, x), \varphi_n(t, x))$ converge to $(u(t, x), \varphi(t, x))$ uniformly for (t, x) in any bounded sets.

Proof. Using the uniform bound $|u(t, \xi)| \leq E_0^{1/2}$ in (72) and combining with (90), we have the estimate

$$\overline{y}(\xi) - E_0^{1/2} t \leq y(t, \xi) \leq \overline{y}(\xi) + E_0^{1/2} t, \quad t \geq 0. \tag{95}$$

Applying the definition of ξ in (15), we obtain

$$\lim_{\xi \to \pm\infty} \overline{y}(t, \xi) = \pm\infty. \tag{96}$$

Therefore, the image of the continuous map $(t, \xi) \mapsto (t, y(t, \xi))$ is the half-plane $R^+ \times R$. Now we claim

$$y_\xi = q \cos^2 \frac{v}{2}, \tag{97}$$

for all $t \geq 0$ and a.e. $\xi \in R$. Indeed, from (35) and (61) we have

$$\frac{\partial}{\partial t} \left(q \cos^2 \frac{v}{2} \right) (t, \xi) = -q v_t \cos \frac{v}{2} \sin \frac{v}{2} + q_t \cos^2 \frac{v}{2} = u_\xi (t, \xi). \tag{98}$$

On the other hand, (90) implies

$$\frac{\partial}{\partial t} y_\xi (t, \xi) = u_\xi (t, \xi). \tag{99}$$

Since the function $x \mapsto 2 \arctan \overline{u}_x(x)$ is measurable, identity (97) holds true for almost every $\xi \in R$ at $t = 0$. By the above calculation it remains true for all $t \geq 0$.

Next, we prove the set $\mathfrak{R}$ that defined in (88) has measure zero. Indeed, if $v(t_0, \xi) = -\pi$, then $y_\xi(t_0, \xi) = \cos^2(v(t_0, \xi)/2)q(t_0, \xi) = 0$. Using (61), we get

$$\frac{\partial}{\partial t} y_\xi (t_0, \xi) = u_\xi (t_0, \xi) = \frac{q}{2} \sin v (t_0, \xi) = 0. \tag{100}$$

By (64), we have

$$\frac{\partial^2}{\partial t^2} y_\xi (t_0, \xi) = u_{\xi t} (t_0, \xi) = \frac{1}{2} q (t_0, \xi) > 0, \tag{101}$$

which implies that t satisfying $y_\xi(t, \xi) = 0$ is isolated. Thus, t satisfying $v(t, \xi) = -\pi$ is also isolated. Since $v \in L^2$, it infers

$$\text{measure} (\mathfrak{R}) = 0. \tag{102}$$

From (97), we get $y(t, \xi)$ is nondeceasing. Moreover, if $\xi < \xi'$ and $y(t, \xi) = y(t, \xi')$, then

$$\int_\xi^{\xi'} y_\xi (t, s) \, ds = \int_\xi^{\xi'} q (t, s) \cos^2 \frac{v (t, s)}{2} \, ds = 0. \tag{103}$$

Hence, $\cos(v/2) \equiv 0$ throughout the interval of integration. Therefore, from (61)-(62), we have

$$u (t, \xi') - u (t, \xi) = \int_\xi^{\xi'} \frac{q (t, s)}{2} \sin v (t, s) \, ds = 0,$$

$$\varphi (t, \xi') - \varphi (t, \xi) = \int_\xi^{\xi'} \psi (t, s) q (t, s) \cos^2 \frac{v (t, s)}{2} ds = 0, \tag{104}$$

which proves that the map $(u(t, x), \varphi(t, x))$ in (92) is well defined for all $t \geq 0$ and $x \in R$.

If $x = y(t, \xi)$ holds, using the fact $\varphi_\xi = \psi q \cos^2(v/2)$, we have $\varphi_x(t, x) = \psi(t, \xi)$. This implies $\varphi_x \in L^\infty$. For every fixed t, we have

$$\int_R \left(u^2 (t, x) + u_x^2 (t, x) + \varphi^2 (t, x) + \varphi_x^2 (t, x) \right) dx$$

$$= \int_{\{\cos v > -1\}} \left(u^2 (t, \xi) \cos^2 \frac{v (t, \xi)}{2} + \sin^2 \frac{v (t, \xi)}{2} \right.$$

$$+ \varphi^2 (t, \xi) \cos^2 \frac{v (t, \xi)}{2}$$

$$\left. + \psi^2 (t, \xi) \cos^2 \frac{v (t, \xi)}{2} \right) \cdot q (t, \xi) \, d\xi \tag{105}$$

$$\leq E_0.$$

Since the measure of $\mathfrak{R}$ is zero, the equality holds true for almost all t.

By Sobolev's inequality, this implies the uniform Hölder continuity with the exponent $1/2$ of u and φ as functions of x. By (35) and the bounds $\|\Psi_{1x}\|_{L^\infty} \leq E_0/2$, $\|\Psi_{2x}\|_{L^\infty} \leq E_0/2$, and $\|\Psi_3\|_{L^\infty} \leq E_0/2$, we get that $(u(t, y(t)), \varphi(t, y(t)))$ is Hölder continuous with the exponent $1/2$. Indeed

$$|u (t, y) - u (s, x)| \leq |u (t, y) - u (t, x)|$$

$$+ |u (t, x) - u (t, y (t, \xi))|$$

$$+ |u (t, y (t, \xi)) - u (s, y (s, \xi))|$$

$$\leq E_0^{1/2} |y - x|^{1/2}$$

$$+ E_0^{1/2} |y (t, \xi) - y (s, \xi)|$$

$$+ \int_s^t |\Psi_{1x} (\tau, \xi)| \, d\tau$$

$$\leq C \left(|y - x|^{1/2} + |t - s|^{1/2} + |t - s| \right), \tag{106}$$

where we choose $\xi \in R$ such that the characteristic $t \mapsto y(t, \xi)$ passes through the point (s, x). Notice that $u(t, x) \leq E_0^{1/2}$. This implies that $u(t, x)$ is uniform Hölder continuous with the exponent $1/2$. The same computation shows that $\varphi(t, x)$ is uniform Hölder continuous with the exponent $1/2$.

We now prove the Lipschitz continuity of $(u(t, x), \varphi(t, x))$ with values in L^2. Consider any interval $[\tau, \tau + h]$. For a given point x, we choose ξ such that the characteristic $t \mapsto y(t, \xi)$ passes through the point (τ, x). By (35) and (72), it follows that

$$|u (\tau + h, x) - u (\tau, x)|$$

$$\leq |u (\tau + h, x) - u (\tau + h, y (\tau + h, \xi))|$$

$$+ |u (\tau + h, y (\tau + h, \xi)) - u (\tau, x)|$$

$$\leq \sup_{|y - x| \leq E_0^{1/2} h} |u (\tau + h, y) - u (\tau + h, x)|$$

$$+ \int_\tau^{\tau + h} |\Psi_{1x} (t, \xi)| \, dt. \tag{107}$$

Integrating over the whole real line, in view of the boundedness of Ψ_{1x}, (8), (79), and (105), we obtain

$$
\int_R |u(\tau + h, x) - u(\tau, x)|^2 dx
$$

$$
\leq 2 \int_R \left(\int_{x - E^{1/2}h}^{x + E^{1/2}h} |u_x(\tau + h, y)| \, dy \right)^2 dx
$$

$$
+ 2 \int_R \left(\int_\tau^{\tau + h} |\Psi_{1x}(t, \xi)| \, dt \right)^2 q(\tau, \xi) \cos^2 \frac{v(\tau, \xi)}{2} d\xi
$$

$$
\leq 2 \int_R \left(2 E_0^{1/2} h \int_{x - E_0^{1/2}h}^{x + E_0^{1/2}h} |u_x(\tau + h, y)|^2 \, dy \right) dx
$$

$$
+ 2 \int_R \left(h \int_\tau^{\tau + h} |\Psi_{1x}(t, \xi)|^2 dt \right) \|q(\tau, \xi)\|_{L^\infty} d\xi
$$

$$
= 4 E_0^{1/2} h \int_R \int_{y - E_0^{1/2}h}^{y + E_0^{1/2}h} |u_x(\tau + h, y)|^2 dx \, dy
$$

$$
+ 2h \|q(\tau)\|_{L^\infty} \int_R \int_\tau^{\tau + h} |\Psi_{1x}(t, \xi)|^2 dt \, d\xi
$$

$$
\leq 8 E_0 h^2 \|u_x(\tau + h)\|_{L^2}^2 + 2h \|q(\tau)\|_{L^\infty} \int_\tau^{\tau + h} \|\Psi_{1x}(t)\|_{L^2}^2 dt
$$

$$
\leq C h^2,
$$

$$
(108)
$$

where constant C depends only on T. This clearly implies the Lipschitz continuity of the map $t \mapsto u(t)$ in terms of the x variable. Similar calculation shows the Lipschitz continuity of $\varphi(t, x)$ as a map $t \mapsto L^2$.

Since L^2 is a reflexive space, the left-hand side of (6) is well defined for almost all $t \in R$. Note that we have proved that the right-hand side of (6) also lies in L^2 for almost all $t \in R$. To establish the equality between these two sides, we observe that

$$
\frac{d}{dt} u(t, \xi) = -\Psi_{1x}(t, \xi),
$$

$$
\frac{d}{dt} \varphi(t, \xi) = -\Psi_{2x}(t, \xi) - \Psi_3(t, \xi).
$$

$$
(109)
$$

On the other hand, recalling (88), for every $t \notin \mathfrak{R}$ the map $\xi \mapsto x(t, \xi)$ is one-to-one. Then, the change of variable formulae (24)–(29) yields

$$
\Psi_{1x}(t, \xi) = \Psi_{1x}(t, y(t, \xi))
$$

$$
= \frac{1}{2} \left(\int_{y(t,\xi)}^\infty - \int_{-\infty}^{y(t,\xi)} \right) e^{-|y(t,\xi) - x|}
$$

$$
\times \left(u^2 + \frac{1}{2} u_x^2 + g(u) + \frac{1}{2} \varphi^2 - \frac{1}{2} \varphi_x^2 \right)(t, x) \, dx,
$$

$$
\Psi_{2x}(t, \xi) = \Psi_{2x}(t, y(t, \xi))
$$

$$
= \frac{1}{2} \left(\int_{y(t,\xi)}^\infty - \int_{-\infty}^{y(t,\xi)} \right) e^{-|y(t,\xi) - x|} u_x(t, x) \varphi_x(t, x) dx,
$$

$$
\Psi_3(t, \xi) = \Psi_3(t, y(t, \xi))
$$

$$
= \frac{1}{2} \int_{-\infty}^\infty e^{-|y(t,\xi) - x|} u_x(t, x) \varphi(t, x) \, dx.
$$

$$
(110)
$$

Hence, identity (6) is valid for almost all $t \geq 0$. This implies that (u, φ) is a global solution of the generalized two-component Camassa-Holm system in the sense of Definition 1.

From (71) and (105), we obtain identity (93) for almost all $t \geq 0$.

Finally, let $(\overline{u}_n, \overline{\varphi}_n)$ be a sequence of initial data converging to $(\overline{u}, \overline{\varphi})$ in $H^1 \times (H^1 \cap W^{1,\infty})$. In this case, from (35) and the boundedness of u, φ, ψ, we can infer $\|q_n(t, \xi) - q(t, \xi)\|_{L^2} < \infty$. Recalling (15) and (36) at time $t = 0$, we have

$$
\sup_{\xi \in R} |y_n(0, \xi) - y(0, \xi)| \longrightarrow 0,
$$

$$
\|u_n(0, \xi) - u(0, \xi)\|_{H^1} \longrightarrow 0,
$$

$$
\|\varphi_n(0, \xi) - \varphi(0, \xi)\|_{H^1} \longrightarrow 0,
$$

$$
\|\psi_n(0, \xi) - \psi(0, \xi)\|_{L^\infty} \longrightarrow 0,
$$

$$
\|\psi_n(0, \xi) - \psi(0, \xi)\|_{L^2} \longrightarrow 0,
$$

$$
\|v_n(0, \xi) - v(0, \xi)\|_{L^2} \longrightarrow 0,
$$

$$
\|q_n(0, \xi) - q(0, \xi)\|_{L^2} \longrightarrow 0.
$$

$$
(111)
$$

Now from (35) and the bounds of u, φ, ψ, v, q, we obtain

$$
\frac{d}{dt} \left(\|u_n(t, \xi) - u(t, \xi)\|_{L^\infty} + \|\varphi_n(t, \xi) - \varphi(t, \xi)\|_{L^\infty} \right.
$$

$$
+ \|\psi_n(t, \xi) - \psi(t, \xi)\|_{L^\infty}
$$

$$
+ \|v_n(t, \xi) - v(t, \xi)\|_{L^2}
$$

$$
\left. + \|q_n(t, \xi) - q(t, \xi)\|_{L^2} \right)
$$

$$
\leq C \left(\|u_n(t, \xi) - u(t, \xi)\|_{L^\infty} + \|\varphi_n(t, \xi) - \varphi(t, \xi)\|_{L^\infty} \right.
$$

$$
+ \|\psi_n(t, \xi) - \psi(t, \xi)\|_{L^\infty}
$$

$$
\left. + \|v_n(t, \xi) - v(t, \xi)\|_{L^2} + \|q_n(t, \xi) - q(t, \xi)\|_{L^2} \right).
$$

$$
(112)
$$

Thus, Gronwall's inequality implies that $u_n(t, \xi) \rightarrow u(t, \xi)$, $\varphi_n(t, \xi) \rightarrow \varphi(t, \xi)$, uniformly for (t, ξ) in bounded sets. Returning to the original coordinates, it yields that

$$
y_n(t, \xi) \longrightarrow y(t, \xi),
$$

$$
u_n(t, \xi) \longrightarrow u(t, \xi),
$$

$$
\varphi_n(t, \xi) \longrightarrow \varphi(t, \xi),
$$

$$
(113)
$$

uniformly on bounded sets since all functions $u, u_n, \varphi, \varphi_n$ are uniformly Hölder continuous. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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