

Research Article

Impulsive Stabilization of Dynamic Equations on Time Scales

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In this paper we study the impulsive stabilization of dynamic equations on time scales via the Lyapunov's direct method. Our results show that dynamic equations on time scales may be ψ -exponentially stabilized by impulses. Furthermore, we give some examples to illustrate our results.

1. Introduction

Differential equations with impulse effect provide an adequate mathematical description of various real-world phenomena in physics, engineering, biology, economics, neural network, social sciences, and so forth. Since the 1960s, the theory of impulsive differential or difference equations has been studied by many authors [1–3].

Aulbach and Hilger [4, 5] introduced the theory of time scales (measure chains) in order to create a theory that can unify continuous and discrete analysis. The theory of dynamic systems on time scales has been developed as a generalization of both continuous and discrete dynamic systems simultaneously and applied to many different fields of mathematics [6, 7].

It is widely known that the various types of stability of nonlinear impulsive differential equations or impulsive difference equations can be characterized by using Lyapunov's second method and inequalities [8–11]. In recent years, some authors studied the stability of impulsive dynamic systems on time scales [12–16]. Furthermore, Hatipoğlu et al. [12] studied the ψ -exponential stability of nonlinear impulsive dynamic equations on time scales. Liu [17] investigated the impulsive stabilization of nonlinear systems by employing Lyapunov's direct method and obtained sufficient conditions for both stabilization and destabilization.

In this paper we study the impulsive stabilization of dynamic equations on time scales via the Lyapunov's direct method. Our results show that dynamic equations on time

scales may be ψ -exponentially stabilized by impulses. We give some examples to illustrate our results.

2. Preliminaries

We refer the reader to [6] for all the basic definitions and results from time scales calculus that we will use in the sequel (e.g., delta differentiability, rd-continuity, and exponential function and its properties).

It is assumed throughout that a time scale \mathbb{T} will be unbounded above and $\mu(t)$ is bounded. Let \mathbb{R}^n be the n -dimensional real Euclidean space. $C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ denotes the set of all rd-continuous functions from $\mathbb{T} \times \mathbb{R}^n$ to \mathbb{R}^n and $\mathbb{R}_+ = [0, \infty)$. Also, for any $t_0 \in \mathbb{T}$, let $\mathbb{T}_{t_0} := [t_0, \infty) \cap \mathbb{T}$. We denote by \mathcal{R} (resp., \mathcal{R}^+) the set of all regressive (resp., positively regressive) functions from \mathbb{T} to \mathbb{R} . The set of all rd-continuous and regressive functions from \mathbb{T} to \mathbb{R} is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Also, let

$$C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \{p \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)p(t) > 0 \forall t \in \mathbb{T}\}. \quad (1)$$

We consider the impulsive dynamic system with impulses at constant times

$$\begin{aligned} x^\Delta(t) &= f(t, x), \quad t \neq t_k, \quad t \in \mathbb{T}_{t_0}; \\ \Delta x(t) &:= x(t^+) - x(t) = I_k(x(t)), \quad t = t_k; \\ x(t_0) &= x_0, \end{aligned} \quad (2)$$

with the following conditions:

- (i) $t_0 < t_1 < t_2 < \dots < t_k < \dots$, with $\lim_{k \rightarrow \infty} t_k = \infty, t_k \in \mathbb{T}$ for $k \in \mathbb{N}$.
- (ii) The function $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is rd-continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ and $f(t, 0) = 0$ for $t \in \mathbb{T}$.
- (iii) The function $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and $I_k(0) = 0$ for $k \in \mathbb{N}$;
- (iv) $x(t_k^+)$ represents the right limit of $x(t)$ at $t = t_k$.

The solution of the impulsive dynamic equation with impulse effect (2) depends not only on the initial condition (t_0, x_0) but also on the moments of impulses t_k for each $k \in \mathbb{N}$. Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (2) satisfying the initial condition $x(t_0, t_0, x_0) = x_0$. For the existence and continuation of solutions of impulsive dynamic equations, see [3, 18].

We need the following well-known impulsive inequality of Gronwall's type to prove our main results.

Lemma 1 (see [14]). *Let $t_0 \in \mathbb{T}$, let $u \in C_{rd} \mathcal{R}(\mathbb{T}_{t_0}, \mathbb{R}_+)$, let $p \in C_{rd} \mathcal{R}^+(\mathbb{T}_{t_0}, \mathbb{R})$, and let $c, b_k \in \mathbb{R}_+$ for each $k \in \mathbb{N}$. Then,*

$$u(t) \leq c + \int_{t_0}^t p(s) u(s) \Delta s + \sum_{t_0 < t_k < t} b_k u(t_k), \quad t \in \mathbb{T}_{t_0} \quad (3)$$

implies

$$u(t) \leq c \prod_{t_0 < t_k < t} (1 + b_k) e_p(t, t_0), \quad t \in \mathbb{T}_{t_0}. \quad (4)$$

Lemma 2 (see [19]). *For every positive constant λ with $-\lambda \in \mathcal{R}^+$, the following inequalities hold:*

$$e_{-\lambda}(t, t_0) \leq e^{-\lambda(t-t_0)} \leq e_{\ominus\lambda}(t, t_0), \quad t \in \mathbb{T}_{t_0}, \quad (5)$$

where $\ominus\lambda = -\lambda/(1 + \mu(t)\lambda)$.

Lemma 3 (see [6]). *If $p, q \in \mathcal{R}$, then we have, for all $t, s, r \in \mathbb{T}$,*

- (i) $e_0(t, s) = 1$ and $e_p(t, t) = 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $1/e_p(t, s) = e_{\ominus p}(t, s)$ and $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
- (iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (v) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ and $e_p(t, s)/e_q(t, s) = e_{p \ominus q}(t, s)$.

Akiyele [20] introduced the notion of ψ -stability of degree k with respect to a function $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$, increasing and differentiable on \mathbb{R}_+ and such that $\psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t \rightarrow \infty} \psi(t) = b, b \in [1, \infty)$.

Now, we give notions of ψ -exponential, ψ -uniformly exponential, and ψ -globally exponential stability for solutions of nonlinear impulsive dynamic equations on time scales.

Definition 4 (see [12]). Let $\psi \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$. System (2) is called ψ -exponentially stable if any solution $x(t, t_0, x_0)$ of (2) satisfies

$$\|\psi(t) x(t, t_0, x_0)\| \leq \beta(\|x_0\|, t_0) [e_{-\lambda}(t, t_0)]^d, \quad t \in \mathbb{T}_{t_0}, \quad (6)$$

where the function $\beta(h, t) : \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{R}_+$ is increasing in $h \in \mathbb{R}_+, \lambda > 0, -\lambda \in \mathcal{R}^+$, and d is a positive constant.

Moreover, system (2) is said to be ψ -uniformly exponentially stable if β is independent of t_0 .

System (2) is said to be ψ -globally exponentially stable if system (2) is ψ -exponentially stable for each $(t_0, x_0) \in \mathbb{T}_{t_0} \times \mathbb{R}^n$ and the function β is independent on each t_0 and x_0 in the definition of ψ -exponential stability; that is, there exist constants $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ and $M \geq 1$ such that for any initial value $(t_0, x_0) \in \mathbb{T}_0 \times \mathbb{R}^n$,

$$\|\psi(t) x(t, t_0, x_0)\| \leq M \|\psi(t_0) x_0\| [e_{-\lambda}(t, t_0)]^d, \quad t \in \mathbb{T}_{t_0}, \quad (7)$$

where $x(t, t_0, x_0)$ is any solution of system (2).

Remark 5. System (2) is exponentially stable if we set $\psi(t) = 1$ in the definition of ψ -exponential stability.

Moreover, system (2) is uniformly exponentially stable if we set $\psi(t) = 1$ in the definition of ψ -uniformly exponential stability.

For the Lyapunov-like function $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+)$, we recall the following definition.

Definition 6 (see [7, Definition 3.1.1]). We define the *generalized derivative* $D^+ V_{(2)}^\Delta(t, x(t))$ of $V(t, x)$ relative to system (2) as follows: given $\varepsilon > 0$, there exists a neighborhood U of $t \in \mathbb{T}$ such that

$$\begin{aligned} & \frac{1}{\sigma(t) - s} [V(\sigma(t), x(\sigma(t))) \\ & \quad - V(s, x(\sigma(t)) - (\sigma(t) - s)f(t, x(t)))] \\ & < D^+ V_{(2)}^\Delta(t, x(t)) + \varepsilon, \quad s \in U, s > t, \end{aligned} \quad (8)$$

where $x(t)$ is any solution of system (2) and the *upper right Dini derivative* $V_*^\Delta(t)$ of $V_*(t)$ is given by

$$V_*^\Delta(t) = \begin{cases} \overline{\lim}_{\eta \rightarrow 0^+, \eta+t \in \mathbb{T}} \frac{V_*(t+\eta) - V_*(t)}{\eta}, & \text{if } t = \sigma(t), \\ \frac{V_*(\sigma(t)) - V_*(t)}{\mu(t)}, & \text{if } t < \sigma(t), \end{cases} \quad (9)$$

where $V_*(t) = V(t, x(t))$.

Then it is well-known that

$$D^+ V_{(2)}^\Delta(t, x(t)) = V_*^\Delta(t) \quad (10)$$

if $V(t, x)$ is Lipschitzian in x for each $t \in \mathbb{T}$ [21].

In case $t \in \mathbb{T}$ is right-dense, we have

$$\begin{aligned}
 & D^+V_{(2)}^\Delta(t, x(t)) \\
 &= D^+V_{(2)}(t, x(t)) \\
 &= \overline{\lim}_{\eta \rightarrow 0^+} \frac{1}{\eta} [V(t + \eta, x(t) + \eta f(t, x(t))) - V(t, x(t))] \\
 &= \overline{\lim}_{\eta \rightarrow 0^+, \eta+t \in \mathbb{T}} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \\
 &= D^+V_*(t).
 \end{aligned} \tag{11}$$

In case $t \in \mathbb{T}$ is right-scattered and $V(t, x(t))$ is continuous at t , we have

$$D^+V_{(2)}^\Delta(t, x(t)) = \frac{1}{\mu(t)} [V(\sigma(t), x(\sigma(t))) - V(t, x(t))]. \tag{12}$$

In fact, if $x(t)$ is a solution of system (2), then we have

$$\begin{aligned}
 & V^\Delta(t, x(t)) \\
 &= V^{\Delta t}(t, x(t)) \\
 &+ \left[\int_0^1 D_2V(\sigma(t), x(t) + \eta\mu(t)x^\Delta(t)) d\eta \right] x^\Delta(t) \\
 &= V^{\Delta t}(t, x(\sigma(t))) \\
 &+ \left[\int_0^1 D_2V(t, x(t) + \eta\mu(t)x^\Delta(t)) d\eta \right] x^\Delta(t)
 \end{aligned} \tag{13}$$

by the chain rule of a differentiable function $V(t, x(t))$ [22, Theorem 1].

Definition 7 (see [23]). $V: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to belong to the class ν_0 if

- (i) V is rd-continuous in $((t_{k-1}, t_k] \cap \mathbb{T}) \times \mathbb{R}^n$ and for each $x \in \mathbb{R}^n, t \in (t_{k-1}, t_k] \cap \mathbb{T}, k \in \mathbb{N}, \lim_{(t,y) \rightarrow (t_k^+, x)} V(t, y) = V(t_k^+, x)$ exists.
- (ii) $V(t, x)$ is locally Lipschitzian in $x \in \mathbb{R}^n$ and $V(t, 0) = 0$ for $t \in \mathbb{T}$.

3. Main Results

In this section we investigate ψ -exponential stability for impulsive dynamic equations on time scales via Lyapunov's direct method.

The following result shows that dynamic equations on time scales may be ψ -exponentially stabilized by impulses. It is adapted from Theorem 3.1 in [11].

Theorem 8. Assume that there exists a function $V \in \nu_0$ and constants $p, q, c, c_1, c_2 > 0$ and $\alpha > 0, \lambda > c$ with $-\lambda \in \mathcal{R}^+$ such that the following conditions hold:

- (i) $c_1 \|\psi(t)x\|^p \leq V(t, x) \leq c_2 \|\psi(t)x\|^q$ for $(t, x) \in \mathbb{T}_{t_0} \times \mathbb{R}^n$;
- (ii) $V^\Delta(t, x) \leq cV(t, x)$ for all $t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0}$;
- (iii) $V(t_k^+, x(t_k) + I_k(x(t_k))) \leq d_k V(t_k, x(t_k))$, where each d_k is a positive constant;
- (iv) $0 < t_k - t_{k-1} < \alpha$ and $d_k < e_{-\lambda}(t_{k+1}, t_k)e^{-\alpha\lambda}$ for each $k \in \mathbb{N}$.

Then the zero solution of system (2) is ψ -exponentially stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2) with initial value $x(t_0) = x_0$, and $V_*(t) = V(t, x(t))$.

We will show that

$$\begin{aligned}
 & V_*(t) \\
 &\leq M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t, t_0), \quad t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0}, k \in \mathbb{N}.
 \end{aligned} \tag{14}$$

We can choose $M_1 \geq 1$ such that

$$\begin{aligned}
 & c_2 \|\psi(t_0)x_0\|^q < M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0) e^{-\alpha c} \\
 &< M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0).
 \end{aligned} \tag{15}$$

We first show that

$$V_*(t) \leq M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0), \quad t \in [t_0, t_1] \cap \mathbb{T}_{t_0}. \tag{16}$$

In view of conditions (i) and (15), we have

$$\begin{aligned}
 & V_*(t) \leq V_*(t_0) e_c(t, t_0) \\
 &\leq c_2 \|\psi(t_0)x_0\|^q e_c(t_1, t_0) \\
 &\leq M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0) e^{-\alpha c} e_c(t_1, t_0) \\
 &< M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0), \quad t \in [t_0, t_1] \cap \mathbb{T}_{t_0}.
 \end{aligned} \tag{17}$$

Next, we show that

$$V_*(t) < M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_2, t_0), \quad t \in (t_1, t_2] \cap \mathbb{T}_{t_0}. \tag{18}$$

From conditions (i)–(iv), (17) and Lemma 2, we have

$$\begin{aligned}
 & V_*(t) \leq V_*(t_1^+) e_c(t, t_1) \\
 &\leq d_1 V_*(t_1) e_c(t, t_1) \\
 &\leq e_{-\lambda}(t_2, t_1) e^{-\lambda\alpha} M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_1, t_0) e_c(t_2, t_1) \\
 &\leq M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_2, t_0), \quad t \in (t_1, t_2] \cap \mathbb{T}_{t_0}.
 \end{aligned} \tag{19}$$

Now we assume that (14) holds for $k = 1, 2, \dots, m$ ($m \in \mathbb{N}$); that is,

$$\begin{aligned}
 & V_*(t) < M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_k, t_0), \\
 &t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0}, \quad k = 1, 2, \dots, m.
 \end{aligned} \tag{20}$$

From conditions (iii) and (20), we have

$$\begin{aligned}
V_*(t) &\leq V_*(t_m^+) e_c(t, t_m) \\
&\leq d_m V_*(t_m) e_c(t_{m+1}, t_m) \\
&\leq e_{-\lambda}(t_{m+1}, t_m) e^{-\lambda\alpha} M \|\psi(t_0)x_0\|^q e_{-\lambda}(t_m, t_0) \\
&\quad \times e_c(t_{m+1}, t_m) \\
&\leq M_1 \|\psi(t_0)x_0\|^q e_{-\lambda}(t_{m+1}, t_0), \quad t \in (t_m, t_{m+1}] \cap \mathbb{T}_{t_0}.
\end{aligned} \tag{21}$$

Thus (14) holds for each $k = m + 1$. Then it follows from mathematical induction that (14) holds for each $k \in \mathbb{N}$.

In view of conditions (i) and (14), we get

$$\begin{aligned}
&\|\psi(t)x(t)\| \\
&\leq M \|\psi(t_0)x_0\|^\gamma [e_{-\lambda}(t, t_0)]^d, \quad t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0},
\end{aligned} \tag{22}$$

where $M = \max\{1, (M_1/c_1)^d\}$, $\gamma = q/p$, and $d = 1/p$. Hence the trivial solution of system (2) is ψ -exponentially stable. This completes the proof. \square

Remark 9. We obtain the following results from Theorem 8.

- (i) If we set $\psi(t) = 1$ for each $t \in \mathbb{T}$ in Theorem 8, then the zero solution of system (2) is exponentially stable.
- (ii) If the conditions of Theorem 8 hold and $p = q$, then the zero solution of system (2) is globally ψ -exponentially stable.

Also, we can obtain the following result as a discrete version of Theorem 8 for $\mathbb{T} = \mathbb{Z}$.

Corollary 10. Assume that there exists a function $V : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ and constants $p, q, c, c_1, c_2 > 0$ and $\alpha > 1, c < \lambda < 1$ such that the following conditions hold:

- (i) $V(i, x)$ is locally Lipschitzian in the second variable $x \in \mathbb{R}^n$ and $V(i, 0) = 0$ for each $i \in \mathbb{Z}$;
- (ii) $c_1 \|\psi(i)x\|^p \leq V(i, x) \leq c_2 \|\psi(i)x\|^q$ for $(i, x) \in \mathbb{Z}_{i_0} \times \mathbb{R}^n$;
- (iii) $V(i+1, x) \leq (1+c)V(i, x)$ for all $i \in (i_{k-1}, i_k] \cap \mathbb{Z}_{i_0}$;
- (iv) $V(i_k^+, x(i_k) + I_k(x(i_k))) \leq d_k V(i_k, x(i_k))$, where each d_k is a positive constant;
- (v) $0 < i_k - i_{k-1} < \alpha$ and $d_k < (1-\lambda)^{(i_{k+1}-i_k)} e^{-\alpha\lambda}$ for each $k \in \mathbb{N}$.

Then the zero solution of system (2) is ψ -exponentially stable.

We can obtain the following result which can be proved as in the similar manner of Theorem 8.

Corollary 11. Assume that all conditions of Theorem 8 are satisfied with the condition (ii) replaced by (ii)':

- (ii)' $V^\Delta(t, x) \leq 0$ for all $t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0}, k \in \mathbb{N}$.

Then the zero solution of system (2) is also ψ -exponentially stable.

Remark 12. If we set $\psi(t) = 1$ in the condition (i) of Corollary 11, then the zero solution of system (2) is also exponentially stable.

Next, we obtain the following result that the stability properties of dynamic systems can be preserved under certain impulsive perturbations. It is adapted from Theorem 1 in [13].

Theorem 13. Assume that there exist a function $V \in \nu_0$ and constants $p, q, c, c_1, c_2 > 0$ and $\alpha > 0, \lambda > c$ with $-\lambda \in \mathcal{R}^+$ such that the following conditions hold:

- (i) $c_1 \|\psi(t)x\|^p \leq V(t, x) \leq c_2 \|\psi(t)x\|^q$ for $(t, x) \in \mathbb{T}_{t_0} \times \mathbb{R}^n$;
- (ii) $V^\Delta(t, x) \leq -cV(t, x)$ for all $t \in (t_{k-1}, t_k] \cap \mathbb{T}_{t_0}, k \in \mathbb{N}$;
- (iii) $V(t_k^+, x(t_k) + I_k(x(t_k))) \leq (1+d_k)V(t_k, x(t_k))$, where each d_k ($k \in \mathbb{N}$) is a positive constant and $\sum_{k=1}^{\infty} d_k < \infty$.

Then the zero solution of system (2) is ψ -exponentially stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2) with $x(t_0) = x_0$, and $V_*(t) = V(t, x(t))$.

It follows from condition (ii) that

$$\begin{aligned}
&[V_*(t) e_c(t, t_k)]^\Delta \\
&= V_*^\Delta(t) e_c(\sigma(t), t_k) + V_*(t) e_c^\Delta(t, t_k) \\
&\leq V_*^\Delta(t) (1 + c\mu(t)) e_c(t, t_k) + cV_*(t) e_c(t, t_k) \\
&\leq -c\mu(t) V_*(t) e_c(t, t_k) \\
&\leq 0, \quad t \in (t_k, t_{k+1}] \cap \mathbb{T}_{t_0}, k \in \mathbb{N}.
\end{aligned} \tag{23}$$

By integrating both sides of (23) from t_k^+ to t and condition (iii), we obtain

$$V_*(t) \leq (1+d_k) V_*(t_k) e_{-c}(t, t_k), \quad t \in (t_k, t_{k+1}] \cap \mathbb{T}_{t_0}. \tag{24}$$

From condition (ii), we have

$$V_*(t) \leq V_*(t_0) e_{-c}(t, t_0), \quad t \in [t_0, t_1] \cap \mathbb{T}_{t_0}. \tag{25}$$

In view of conditions (iii) and (25), we have

$$\begin{aligned}
V_*(t) &\leq (1+d_1) V_*(t_1) e_{-c}(t, t_1), \\
&\leq (1+d_1) V_*(t_0) e_{-c}(t_1, t_0) e_{-c}(t, t_1) \\
&\leq (1+d_1) V_*(t_0) e_{-c}(t, t_0), \quad t \in (t_1, t_2] \cap \mathbb{T}_{t_0}.
\end{aligned} \tag{26}$$

It follows from mathematical induction that

$$\begin{aligned}
&V_*(t) \\
&\leq \prod_{i=1}^k (1+b_i) V_*(t_0) e_{-c}(t, t_0), \quad t \in (t_k, t_{k+1}] \cap \mathbb{T}_{t_0}, k \in \mathbb{N}.
\end{aligned} \tag{27}$$

Thus we obtain

$$\begin{aligned} V_*(t) &\leq \prod_{i=1}^{\infty} (1 + b_i) V_*(t_0) e_{-c}(t, t_0) \\ &\leq \exp\left(\sum_{i=1}^{\infty} b_i\right) c_2 \|\psi(t_0)x_0\|^q e_{-c}(t, t_0) \\ &\leq M_1 c_2 \|\psi(t_0)x_0\|^q e_{-c}(t, t_0), \quad t \in \mathbb{T}_{t_0}, \end{aligned} \tag{28}$$

where $M_1 = \exp(\sum_{i=1}^{\infty} b_i)$.

In view of conditions (i) and (28), we have

$$\|\psi(t)x(t)\| \leq M \|\psi(t_0)x_0\|^\gamma e_{-c}[(t, t_0)]^d, \quad t \in \mathbb{T}_{t_0}, \tag{29}$$

where $M = (c_2 M_1 / c_1)^{1/p}$, $\gamma = q/p$, and $d = 1/p$. The proof is complete. \square

Remark 14. We obtain the following results in [13] from Theorem 13.

- (i) If we set $\psi(t) = 1$ for each $t \in \mathbb{T}$ in Theorem 13, then the zero solution of system (2) is exponentially stable.
- (ii) If we set $\psi(t) = 1$ and $p = q$ in condition (i) of Theorem 13, then the zero solution of system (2) is exponentially stable.

4. Examples

In this section we give two examples which illustrate our results from the previous section. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

Example 15 (see [24, Example 2]). We consider the impulsive dynamic equation on time scales

$$\begin{aligned} x^\Delta(t) &= 3x, \quad t \neq t_k, t \in \mathbb{T}_{t_0}; \\ x(t_k^+) &= x(t_k) + (e^{-8} - 1)x(t_k), \quad t = t_k; \\ x(t_0) &= x_0, \end{aligned} \tag{30}$$

where $t_k = k \in \mathbb{T}$ for each $k \in \mathbb{N}$ and $x_0 \in \mathbb{R}$.

Let $\psi(t) = 1$ and $V(t, x) = x^2$, then it follows that

$$\begin{aligned} V^\Delta(t, x(t)) &= x^\Delta(t)x(\sigma(t)) + x^\Delta(t)x(t) \\ &= x^\Delta(t)(2x + \mu(t)x^\Delta(t)) \\ &= x^2(t)(6 + 9\mu(t)), \quad t \in \mathbb{T}, \end{aligned}$$

$$\begin{aligned} V(t_k^+, x(t_k) + I_k(x(t_k))) &= [x(t_k) + (e^{-8} - 1)x(t_k)]^2 \\ &= e^{-16}x(t_k)^2 \leq e^{-16}V(t_k, x(t_k)). \end{aligned} \tag{31}$$

We consider two cases: $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \{t = i/10 : i \in \mathbb{Z}_+\}$.

Case 1 ($\mathbb{T} = \mathbb{R}$). Letting $d_k = e^{-16}$, $c = 6$, $\lambda = 7$, $\alpha = 1.1$, $p = q = 1$, $c_1 = 1/2$, $c_2 = 1$, we note that all conditions of Theorem 8 are satisfied. Hence the zero solution of system (30) is ψ -exponentially stable by Theorem 8.

Case 2 ($\mathbb{T} = \{t = i/10 : i \in \mathbb{Z}_+\}$ with $\mu(i/10) = 1/10$). Then system (30) rewrites

$$\begin{aligned} x^\Delta\left(\frac{i}{10}\right) &= 3x\left(\frac{i}{10}\right), \quad t \neq t_k, t = \frac{i}{10} \in \mathbb{T}; \\ x(k^+) &= x(k) + (e^{-8} - 1)x(k), \quad t = t_k; \\ x(0) &= x_0, \end{aligned} \tag{32}$$

where $t_k = k \in \mathbb{N}$. Then we have

$$\begin{aligned} V^\Delta(t, x) &= V^\Delta\left(\frac{i}{10}, x\right) \\ &= x^2\left(6 + 9\mu\left(\frac{i}{10}\right)\right) \\ &\leq 7V(t, x), \quad t = \frac{i}{10} \in \mathbb{T}. \end{aligned} \tag{33}$$

Letting $d_k = e^{-16}$, $c = 7$, $\lambda = 7.1$, $\alpha = 1.1$, $p = q = 1$, $c_1 = 1/2$, $c_2 = 1$, it follows that all conditions of Theorem 8 are satisfied. Hence the zero solution of system (32) is also ψ -exponentially stable by Theorem 8.

Remark 16. It follows from Example 15 that the zero solution of system (30) without impulses is unstable; however, after impulsive effect, the zero solution becomes ψ -exponentially stable. This implies that impulses may be used to exponentially stabilize dynamic equations on time scales.

We give the following example to illustrate Theorem 13.

Example 17 (see [13, Example]). Let $t_0 \in \mathbb{T}$ and $x(t_0) = (c, d) \in \mathbb{R}^2$. We consider the impulsive dynamic system on time scales

$$\begin{aligned} x_1^\Delta(t) &= \frac{x_2(t)}{1 + x_1^2(t)} - 2x_1(t), \quad t \neq t_k, t \in \mathbb{T}_{t_0}; \\ x_2^\Delta(t) &= \frac{x_1(t)}{1 + x_2^2(t)} - 2x_2(t), \quad t \neq t_k, t \in \mathbb{T}_{t_0}; \\ x_1(t_k^+) &= \sqrt{1 + \frac{1}{k^2}}x_1(t_k), \quad t = t_k; \\ x_2(t_k^+) &= \sqrt{1 + \frac{1}{k^2}}x_2(t_k), \quad t = t_k; \\ x(t_0) &= (x_1(t_0), x_2(t_0)) = (c, d), \end{aligned} \tag{34}$$

where $t_k = k \in \mathbb{T}$ for each $k \in \mathbb{N}$. In case $\mathbb{T} = \{t = i/2 : i \in \mathbb{Z}_+\}$ with $\mu(i/2) = i/2$, then the system (34) rewrites

$$\begin{aligned}x_1^\Delta\left(\frac{i}{2}\right) &= \frac{x_2(i/2)}{1+x_1^2(i/2)} - 2x_1\left(\frac{i}{2}\right), \quad t \neq t_k, \quad t = \frac{i}{2} \in \mathbb{T}; \\x_2^\Delta\left(\frac{i}{2}\right) &= \frac{x_1(i/2)}{1+x_2^2(i/2)} - 2x_2\left(\frac{i}{2}\right), \quad t \neq t_k, \quad t = \frac{i}{2} \in \mathbb{T}; \\x_1(k^+) &= \sqrt{1 + \frac{1}{k^2}} x_1(k), \quad t = t_k; \\x_2(k^+) &= \sqrt{1 + \frac{1}{k^2}} x_2(k), \quad t = t_k; \\x(0) &= (x_1(0), x_2(0)) = (c, d),\end{aligned}\tag{35}$$

where $t_k = k \in \mathbb{N}$. Letting $V(t, x(t)) = x_1^2(t) + x_2^2(t)$ and $\|x(t)\| = x_1^2(t) + x_2^2(t)$ and employing similar manner in [13, Example], it follows that the zero solution of (34) is exponentially stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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