Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 609525, 26 pages
http://dx.doi.org/10.1155/2014/609525

Research Article

Function Spaces with Bounded L^p Means and Their Continuous Functionals

Massimo A. Picardello[1,2]

[1] *Department of Mathematics, University of Rome "Tor Vergata", Via Ricerca Scientifica, 00133 Rome, Italy*
[2] *Department of Mathematics, University of Maryland, College Park, MD 20742, USA*

Correspondence should be addressed to Massimo A. Picardello; picard@mat.uniroma2.it

Received 7 October 2013; Accepted 14 November 2013; Published 19 February 2014

Academic Editor: Vakhtang M. Kokilashvili

This paper studies typical Banach and complete seminormed spaces of locally summable functions and their continuous functionals. Such spaces were introduced long ago as a natural environment to study almost periodic functions (Besicovitch, 1932; Bohr and Fölner, 1944) and are defined by boundedness of suitable L^p means. The supremum of such means defines a norm (or a seminorm, in the case of the full Marcinkiewicz space) that makes the respective spaces complete. Part of this paper is a review of the topological vector space structure, inclusion relations, and convolution operators. Then we expand and improve the deep theory due to Lau of representation of continuous functional and extreme points of the unit balls, adapt these results to Stepanoff spaces, and present interesting examples of discontinuous functionals that depend only on asymptotic values.

1. Introduction

Families of Banach spaces of locally L^p functions whose L^p means satisfy various boundedness conditions on finite intervals were introduced in [1–3] and references therein as a natural environment to extend the notion of almost periodic functions originally introduced in [4–6].

All the spaces of bounded p-means contain L^p, but usually they consist of functions that are not small at infinity and have norms defined by the asymptotic behaviour of their integral means. Therefore a relevant part of the information carried by these functions is at infinity, where they may become large. Which consequences does this fact yield for convolution operator, and for continuous functionals on these spaces? Should we expect the same behavior that is typical of L^p spaces? This paper presents old and (some) new results and proofs for this question: it is aimed to show that bounded p-mean spaces behave as L^p spaces on the issue of completeness but (some or all of them) are completely different for what concerns isometric properties of translations, convolution operators, separability, representation theorems for continuous linear functionals, and extreme points of the unit balls.

For this goal, we focus our attention onto three significant families of locally L^p spaces on $\mathbb{R}$: Marcinkiewicz spaces $\mathscr{M}^p$, consisting of functions whose values on finite intervals are irrelevant (they can be changed without changing the norm, which is indeed only a seminorm), Stepanoff spaces $\mathscr{S}^p$, whose norm depends only on the maximum L^p content of functions on all finite intervals, and finite p-mean spaces, where the p-means with respect to intervals $[-T, T]$ are bounded with respect to T, for T sufficiently large (say $T \geqslant 1$).

We give an expanded and revised presentation of some known results, mostly taken from the fundamental paper [7], and prove some new ones. The results on Marcinkiewicz spaces and bounded p-mean spaces are taken from [7]; the duality results for Stepanoff spaces in Section 6 are new and so are the examples of discontinuous asymptotic functionals on $\mathscr{S}^p$ and M^p. The analysis of the integral representation of continuous functionals on M^p (Theorem 29) follows again [7] very closely but provides many more details, gives slightly more general statements, and improves several steps. Also the description of extreme points of the unit ball of M^p (Section 7.1) is taken from [7]; the results on extreme points for the unit ball of $\widetilde{\mathscr{M}}^p$ (Section 7.2) are a greatly expanded and somewhat improved revision of the approach of [7], leading to a full characterization. In Section 7.3 we extend this approach to the extreme points of the unit ball of $\mathcal{S}^p$.

2. Spaces with Bounded L^p Means and Convolutions

2.1. Marcinkiewicz Spaces. The Besicovitch-Marcinkiewicz spaces $\mathcal{M}^p(\mathbb{R})$ (that we briefly call Marcinkiewicz spaces) have been introduced by Besicovitch (see [8]), and their completeness was proved in [9] (we present this result in Section 3; a later but independent proof was given in [10]).

Here is their definition. Let $\mathcal{M}^p(\mathbb{R})$ be the space of functions $f \in L^p$ on all compact sets in $\mathbb{R}$, $1 \leqslant p < \infty$, such that

$$\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds < +\infty. \tag{1}$$

We equip $\mathcal{M}^p(\mathbb{R})$, $1 \leqslant p < \infty$, with the following seminorm:

$$\|f\|_{\mathcal{M}^p}^p = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds. \tag{2}$$

The quotient of $\mathcal{M}^p$ with respect to the null space $\mathcal{I}^p$ of the semi-norm is therefore a Banach space, which we denote by $\widetilde{\mathcal{M}}^p$.

Marcinkiewicz spaces have been studied or used in [3, 7, 9–14]. In particular, it has been observed in [11] that all regular bounded Borel measures give rise to bounded convolution operators on $\mathcal{M}^p(\mathbb{R})$, with norm bounded by the norm of the measure, just as it happens for L^p spaces. This follows by the fact that translation operators on $\mathcal{M}^p$ have norm 1 (they are isometries!), and

$$\|\mu * f\| \leqslant \int_{-\infty}^{\infty} \|\lambda_t f\| \, d|\mu|(t). \tag{3}$$

On the other hand, this is not true on other spaces defined by boundedness of p-means, as we will now see.

2.2. Convolution on Marcinkiewicz Spaces. It is obvious that $\mathcal{M}^p$ functions can be convolved with L^∞ functions with compact support, and the convolution integral converges. At first glance, it might look obvious that L^1 functions, and more generally finite Borel measures, are bounded convolution operators on $\mathcal{M}^p$, by the following argument. Let B be a normed or semi-normed space of functions on $\mathbb{R}$, where translations are isometries and, for every $f, h \in B$, the map $x \mapsto f(x)\lambda_x h$ is measurable. Then, for every finite Borel measure μ on $\mathbb{R}$, one has

$$\begin{aligned} \|\mu * f\|_B &= \left\| \int_{-\infty}^{\infty} \lambda_x f(t)\, d\mu(t) \right\|_B \\ &\leqslant \int_{-\infty}^{\infty} \|\lambda_x f\|_B \, d|\mu|(t) \leqslant \|\mu\| \, \|f\|_B. \end{aligned} \tag{4}$$

Unfortunately, here one needs that the vector valued integral $\int_{-\infty}^{\infty} \lambda_x f(t)\, d\mu(t)$ be defined, and this condition is not obvious in our case, since the integrand $x \mapsto \lambda_x f$ is not continuous (see, e.g., [14, 15]). The lack of continuity is usually expressed by saying that $\mathcal{M}^p$ is a semi-homogeneous but not homogeneous Banach space.

However, let us show that the vector valued integral in (4) exists and therefore that convolution with finite Borel measures makes sense on $\mathcal{M}^p$. The proof is taken from [11, Chapter I, Section 4].

Let M be the normed linear space of all finite Borel measures with the norm given by the total variation of μ; that is,

$$\|\mu\|_M = \int_{-\infty}^{+\infty} d|\mu|. \tag{5}$$

Let $f \in \mathcal{M}^p$ and μ be a Radon measure. If I is a finite interval in $\mathbb{R}$, set

$$g_I(t) = \int_I f(t-s)\, d\mu(s). \tag{6}$$

By Hölder's inequality

$$|g_I(t)| = \left(\int_I |f(t-s)|^p d|\mu|(s)\right)^{1/p} \left(\int_I d|\mu|\right)^{1/q}. \tag{7}$$

By Fubini's theorem, the function g_I is in L^p on compact sets. Since

$$\frac{1}{2T}\int_{-T}^{T} |f(t-s)|^p dt \tag{8}$$

is uniformly bounded on I for $T \geqslant T_0 > 0$, Lebesgue's dominated convergence theorem yields

$$\limsup_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \int_I |f(t-s)|^p d|\mu|(s) \leqslant \|f\|_{\mathcal{M}^p}^p \int_I d|\mu|. \tag{9}$$

Hence

$$\|g_I\|_{\mathcal{M}^p} \leqslant \|f\|_{\mathcal{M}^p} \int_I d|\mu|. \tag{10}$$

Let us consider $g_{[0,u]}$ as a function of the variable u with values in $\mathcal{M}^p$, that is, the function

$$\int_0^u f(t-s)\, d\mu(s). \tag{11}$$

Since the measure μ is finite, it follows from inequality (10) that this function verifies the Cauchy condition with respect to the Marcinkiewicz semi-norm as u tends to $\pm\infty$.

Therefore there exists a unique function g of $\mathcal{M}^p$ such that

$$g(t) = \lim_{u,w\to+\infty} \int_{-u}^{w} f(t-s)\, d\mu(s), \tag{12}$$

where the limit is taken in the $\mathcal{M}^p$-norm.

The function g in (12) is the convolution of f and μ, and we write

$$g(t) := (f * \mu)(t) = \int_{-\infty}^{+\infty} f(t-s)\, d\mu(s). \tag{13}$$

Inequality (10) gives a bound for the norm of g:

$$\|g\|_{\mathscr{M}^p} \leqslant \|f\|_{\mathscr{M}^p}\|\mu\|_M. \tag{14}$$

The convolution defined in this way is a linear continuous operator from $\mathscr{M}^p$ to $\mathscr{M}^p$, with the norm bounded by $\|\mu\|_M$.

Since $L^1(\mathbb{R})$ embeds isometrically into the space of finite Borel measures on $\mathbb{R}$, we can convolve every $\rho \in L^1$ with functions in $\mathscr{M}^p$:

$$(f * \rho)(t) = \int_{-\infty}^{+\infty} f(t-s)\,\rho(s)\,ds. \tag{15}$$

In particular,

$$\|f * \rho\|_{\mathscr{M}^p} \leqslant \|f\|_{\mathscr{M}^p}\|\rho\|_1. \tag{16}$$

2.3. Spaces with Upper Bounded p-Means. A related family of spaces are the bounded p-mean spaces M^p introduced again in [3] and studied in [7, 16]. Their norm is defined as

$$\|f\|_{M^p}^p = \sup_{T\geqslant 1} \frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds. \tag{17}$$

Obviously, for every $K > 0$ this norm is equivalent to the norm of the dilated space M_K^p defined by

$$\|f\|_{M_K^p}^p = \sup_{T\geqslant K} \frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds. \tag{18}$$

Since $\|f\|_{\mathscr{M}^p} \leqslant \|f\|_{M^p}$, the space M^p embeds continuously in $\mathscr{M}^p$, but the embedding is not an isometric isomorphism, as the next lemma will show.

But first let us clarify the reason for which the length of the interval $[-T,T]$ in the definition of the bounded p-mean space is bounded away from 0.

Remark 1. The space of all functions f on $\mathbb{R}$ such that $f \in L^p$ on compact sets in $\mathbb{R}$ and

$$|||f||| := \sup_{T>0}\left(\frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds\right)^{1/p} < \infty \tag{19}$$

is just $L^\infty(\mathbb{R})$ and the supremum above is $\|f\|_\infty$.

Indeed, it is obvious that if $f \in L^\infty$, then $|||f||| \leqslant \|f\|_\infty$. For the converse inequality, let us show that if $|||f||| < \infty$, then f is bounded on a subset whose complement has measure zero, for instance, the set of Lebesgue points of f (as f is integrable on compacts). Let us choose a representative of f in its Lebesgue class. If x is a Lebesgue point, then the norm

$$\|f\|_{L^p[-T,T]} = \left(\frac{1}{2T}\int_{-T}^{T} |f(s)|^p ds\right)^{1/p} \tag{20}$$

verifies $\|f(x+s)-f(x)\|_{L^p[-T,T]} < \varepsilon$ for T small enough. By the triangle inequality for this norm,

$$\begin{aligned}\|f\|_{L^p[-T,T]} &\geqslant \left|\left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^p ds\right)^{1/p} - \left(\frac{1}{2T}\int_{-T}^{T} |f(x+s)-f(x)|^p ds\right)^{1/p}\right| \\ &> |f(x)| - \left(\frac{1}{2T}\int_{-T}^{T} |f(x+s)-f(x)|^p ds\right)^{1/p}\end{aligned} \tag{21}$$

if T is small enough. Therefore $|||f||| > |f(x)|$ for almost every Lebesgue point x: since almost every point is a Lebesgue point, $|||f||| > \|f\|_\infty$.

As for the spaces introduced before, also M^p is complete. However, translations are not isometries here (see also [16, inequality (30)]).

Lemma 2. *The translation operator λ_x is bounded on M^p, with norm $w(x) \equiv |||\lambda_x||| \geqslant 1 + |x|/K$.*

Proof. Without loss of generality, let $x > 0$. Choose $0 < \eta < x$ and let χ be the characteristic function of the interval $[x-\eta, x+\eta]$. Suppose that $K \leqslant x+\eta$. Then, for $T \geqslant K$, the p-mean $(1/2T)\int_{-T}^{T} |\chi(s)|^p ds$ is largest for $T = x+\eta$, and its maximum value $\|\chi\|_{M_K^p}$ is $(\eta/(x+\eta))^{1/p}$. On the other hand, the translate $\lambda_x\chi$ is a characteristic function centered at 0, and its norm in M_K^p is 1 if $\eta \geqslant K$ and $(\eta/K)^{1/p}$ otherwise. Therefore

$$\||\lambda_x\chi\||_{M_K^p} = \frac{x+\eta}{\max\{\eta, K\}}\||\chi\||_{M_K^p}. \tag{22}$$

The constant in this inequality is maximum for $\eta = K$ and its largest value is $1 + x/K$. □

2.4. Stepanoff Spaces. Stepanoff functions, introduced in [1], are those measurable functions whose L^p-norm on intervals of length, say, 1, is bounded. The supremum of these norms, $\sup_{x\in\mathbb{R}}\int_x^{x+1}|f|^p$, defines a norm for the Stepanoff space $\mathcal{S}^p$, $1 \leqslant p < \infty$. The Stepanoff space $\mathcal{S}^\infty$ coincides with L^∞ and is not considered in this paper. It is immediate to prove (see Corollary 9 below) that the $\mathscr{M}^p$-norm is bounded by the $\mathcal{S}^p$-norm, and therefore the Stepanoff space embeds into the Marcinkiewicz space.

More generally, for every $L > 0$, one could introduce the following L-Stepanoff norm:

$$\|f\|_{\mathcal{S}^p}^{(L)} = \sup_{x\in\mathbb{R}} \frac{1}{L}\int_x^{x+L} |f|^p. \tag{23}$$

However, we have the following.

Proposition 3. *The norms $\|f\|_{\mathcal{S}^p}^{(L)}$ are equivalent for all $L > 0$. Moreover, if $L_1 < L_2$, then*

$$\left(\frac{L_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)}\|f\|_{\mathcal{S}^p}^{(L_2)} < \left(1+\frac{L_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)}. \tag{24}$$

Proof. For every $L_1 < L_2$,

$$\begin{aligned}\left(\|f\|_{\mathcal{S}^p}^{(L_1)}\right)^p &= \sup_{x\in\mathbb{R}}\frac{1}{L_1}\int_x^{x+L_1}|f|^p(t)\,dt\\ &\leqslant \sup_x\frac{1}{L_1}\int_x^{x+L_2}|f|^p = \frac{L_2}{L_1}\left(\|f\|_{\mathcal{S}^p}^{(L_2)}\right)^p.\end{aligned} \quad (25)$$

Now let $n \in \mathbb{Z}$ be such that

$$(n-1)L_1 \leqslant L_2 < nL_1. \quad (26)$$

Since the Stepanoff norms are clearly invariant under translations, we can limit attention to positive x and n. Now,

$$\frac{1}{nL_1}\int_x^{x+nL_1}|f|^p = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{L_1}\int_{x+(j-1)L_1}^{x+jL_1}|f|^p. \quad (27)$$

That is, the mean over $[x, x+nL_1]$ is the average of the means over $[x+(j-1)L_1, x+jL_1]$. Hence

$$\|f\|_{\mathcal{S}^p}^{(nL_1)} \leqslant \|f\|_{\mathcal{S}^p}^{(L_1)}. \quad (28)$$

Then, by the two previous inequalities,

$$\|f\|_{\mathcal{S}^p}^{(L_2)} < \left(\frac{nL_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(nL_1)} \leqslant \left(\frac{nL_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)}. \quad (29)$$

Now, by (26), $nL_1 < L_1 + L_2$. Therefore (28) yields

$$\|f\|_{\mathcal{S}^p}^{(L_2)} \leqslant \left(\frac{nL_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)} < \left(1+\frac{L_1}{L_2}\right)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)}. \quad (30)$$

On the other hand, let $a_j = (1/L_1)\int_{x+(j-1)L_1}^{x+jL_1}|f|^p$. Then the following inequality is similar to (27):

$$\begin{aligned}\frac{1}{L_2}\int_x^{x+L_2}|f|^p &\geqslant \frac{1}{L_2}\int_x^{x+(n-1)L_1}|f|^p\\ &= \frac{L_1}{L_2}\sum_{j=1}^{n-1}a_j \geqslant \frac{L_1}{L_2}\max_{1\leqslant j\leqslant n-1}a_j.\end{aligned} \quad (31)$$

Hence $\|f\|_{\mathcal{S}^p}^{(L_1)} \geqslant (L_1/L_2)^{1/p}\|f\|_{\mathcal{S}^p}^{(L_1)}$. □

Corollary 4. *If $f \in \mathcal{S}^p$, then its Weyl norm*

$$\|f\|_{W^p} := \lim_{L\to\infty}\sup_{x\in\mathbb{R}}\left(\frac{1}{L}\int_x^{x+L}|f|^p(t)\,dt\right)^{1/p} \quad (32)$$

is finite.

Proof. It follows from Proposition 3 that

$$\limsup_{L\to\infty}\|f\|_{\mathcal{S}^p}^{(L)} \leqslant \|f\|_{\mathcal{S}^p}^{(L_1)} \quad (33)$$

for every L_1. Therefore

$$\limsup_{L\to\infty}\|f\|_{\mathcal{S}^p}^{(L)} \leqslant \liminf_{L\to\infty}\|f\|_{\mathcal{S}^p}^{(L)}. \quad (34)$$

Therefore the limit exists; it is infinite if and only if $\|f\|_{\mathcal{S}^p}^{(L)}$ is infinite for some, hence for all, L. □

The Weyl norm defines a normed space called the Weyl space W^p. This was one of the first bounded mean spaces introduced in order to extend the definition of almost periodic functions [2]. However, it was proved in [10] that the Weyl space is not complete; therefore it will not be considered in this paper.

3. Completeness

It is easily seen that the spaces M^p and $\mathcal{S}^p$ are complete. Instead, it is considerably more difficult to show that $\mathcal{M}^p$, hence $\widetilde{\mathcal{M}}^p$, are complete. The proof given here essentially reproduces the ingenious argument given by Marcinkiewicz in [9], except for correcting minor computational mistakes.

Theorem 5. *The Marcinkiewicz spaces $\mathcal{M}^p$ are complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in the Marcinkiewicz semi-norm. Choose a subsequence $\{n_i\}$ such that

$$\left\|f_m - f_{n_i}\right\|_{\mathcal{M}^p} \leqslant 2^{-i} \quad \text{for every } m > n_i. \quad (35)$$

In particular,

$$\left\|f_{n_{i+1}} - f_{n_i}\right\|_{\mathcal{M}^p} \leqslant 2^{-i} \quad \text{for every } i. \quad (36)$$

For the sake of simplicity, for every $f \in \mathcal{M}^p$ and $\lambda \in \mathbb{R}$, we rewrite the norm in M_λ^p, defined in (18), as follows:

$$\delta_\lambda(f) := \|f\|_{M_\lambda^p} = \sup_{T\geqslant\lambda}\left(\frac{1}{2T}\int_{-T}^{T}|f|^p\right)^{1/p}. \quad (37)$$

Then, for every $f \in \mathcal{M}^p$, $\|f\|_{\mathcal{M}^p} = \inf_{\lambda>0}\delta_\lambda(f)$; therefore $\delta_\lambda(f_n)$ is a Cauchy sequence.

Let us choose a sequence λ_i such that

$$\begin{gathered}\lambda_{i+1} > 2\lambda_i,\\ \delta_{\lambda_i}\left(f_{n_{i+1}} - f_{n_i}\right) \leqslant 2^{-i}.\end{gathered} \quad (38)$$

We claim that the sequence f_n converges to the following function f:

$$f(x) = \begin{cases} f_{n_i}(x), & \text{if } \lambda_i \leqslant |x| < \lambda_{i+1},\ i = 1,2,\ldots,\\ 0, & \text{if } |x| < \lambda_1.\end{cases} \quad (39)$$

Indeed, for $m \geqslant 1$ define

$$D_m = \{\lambda_m \leqslant |x| < \lambda_{m+1}\}, \quad (40)$$

and observe that

$$\left(\frac{1}{2\lambda_{m+1}}\int_{D_m}\left|f_{n_{j+1}} - f_{n_j}\right|^p\right)^{1/p} \leqslant \left\|f_{n_{j+1}} - f_{n_j}\right\|_{M^p} \leqslant 2^{-j}, \quad (41)$$

and so

$$\int_{D_m}\left|f_{n_{j+1}} - f_{n_j}\right|^p \leqslant 2^{1-j}\lambda_{m+1}. \quad (42)$$

Now let $\lambda_k \leqslant T < \lambda_{k+1}$. Then

$$\begin{aligned}\int_{-T}^{T} \left|f(x) - f_{n_i}(x)\right|^p dx = {} & \int_{|x|\leqslant\lambda_1} \left|f_{n_i}(x)\right|^p dx \\ & + \sum_{m=1}^{i} \int_{D_m} \left|f(x) - f_{n_i}(x)\right|^p dx \\ & + \sum_{m=i+1}^{k-1} \int_{D_m} \left|f(x) - f_{n_i}(x)\right|^p dx \\ & + \int_{\lambda_k\leqslant|x|\leqslant T} \left|f(x) - f_{n_i}(x)\right|^p dx \\ = {} & I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{43}$$

Let us estimate the four integrals on the right-hand side. Remember that $\delta_{\lambda_1}(f_n)$ is a Cauchy sequence, hence uniformly bounded with respect to n. Therefore there exists a constant $C > 0$ such that, for every i,

$$I_1 = \int_{|x|\leqslant\lambda_1} \left|f_{n_i}(x)\right|^p dx < C. \tag{44}$$

On the other hand, $f = f_{n_m}$ on D_m. So, for $1 \leqslant m < i$, from inequalities (36), (42) and the fact that the measure of D_m is less than $2\lambda_{m+1}$ it follows that

$$\begin{aligned}\left(\int_{D_m} \left|f - f_{n_i}\right|^p dx\right)^{1/p} &= \left(\int_{D_m} \left|f_{n_m} - f_{n_i}\right|^p dx\right)^{1/p} \\ &\leqslant \sum_{j=m}^{i-1} \left(\int_{D_m} \left|f_{n_{j+1}} - f_{n_j}\right|^p dx\right)^{1/p} \\ &< \left(2\lambda_{m+1}\right)^{1/p} \sum_{j=m}^{i-1} 2^{-j/p} \\ &< \frac{2^{-m/p}}{1 - 2^{-1/p}} \lambda_{m+1}^{1/p} < A_p \lambda_{m+1}^{1/p}\end{aligned} \tag{45}$$

for some constant A_p depending only on p. Therefore

$$I_2 < A_p^p \sum_{m=1}^{i} \lambda_{m+1} < A_p^p \lambda_{i+1} \sum_{j=0}^{i-1} 2^{-j} < 2A_p^p \lambda_{i+1} \tag{46}$$

by the first inequality in (38). The same argument for $m > i$ yields

$$\begin{aligned}\left(\int_{D_m} \left|f - f_{n_i}\right|^p dx\right)^{1/p} &\leqslant \sum_{j=i}^{m-1} \left(\int_{D_m} \left|f_{n_{j+1}} - f_{n_j}\right|^p dx\right)^{1/p} \\ &< \left(2\lambda_{m+1}\right)^{1/p} \sum_{j=i}^{m-1} 2^{-j} < 2^{-i+1+1/p} \lambda_{m+1}^{1/p}.\end{aligned} \tag{47}$$

Hence, again by the first inequality (38),

$$I_3 < \sum_{m=i+1}^{k-1} 2^{(1-i)p+1} \lambda_{m+1} < 2^{(1-i)p+1} \lambda_k < 2^{(1-i)p+1} T. \tag{48}$$

Finally, by the same argument,

$$\begin{aligned}&\left(\int_{\lambda_k\leqslant|x|\leqslant T} \left|f_{n_k} - f_{n_i}\right|^p dx\right)^{1/p} \\ &\quad \leqslant (2T)^{1/p} \sum_{m=i+1}^{k-1} \left(\int_{\lambda_k\leqslant x\leqslant T} \left|f_{n_{j+1}} - f_{n_j}\right|^p dx\right)^{1/p} \\ &\quad < 2^{-i+1+1/p} \lambda_k^{1/p},\end{aligned} \tag{49}$$

and so

$$I_4 < 2^{(1-i)p+1} T. \tag{50}$$

Now, by (44), (46), (48), and (50); one has

$$\left(\frac{1}{2T}\int_{-T}^{T} \left|f - f_{n_i}\right|^p\right)^{1/p} < \left(\frac{C}{2T} + A_p^p \frac{\lambda_{i+1}}{T} + 2^{1+(1-i)p}\right)^{1/p}. \tag{51}$$

Therefore, the lim sup with respect to T satisfies the inequality

$$\left\|f_{n_i} - f\right\|_{\mathscr{M}^p} \leqslant 2^{1+1/p} 2^{-i}. \tag{52}$$

Now, finally, if $n_i \leqslant j < n_{i+1}$, this and inequality (35) yield

$$\begin{aligned}\left\|f_j - f\right\|_{\mathscr{M}^p} &\leqslant \left\|f_j - f_{n_i}\right\|_{\mathscr{M}^p} + \left\|f_{n_i} - f\right\|_{\mathscr{M}^p} \\ &\leqslant \left(1 + 2^{1+1/p}\right) 2^{-i} = C' 2^{-i},\end{aligned} \tag{53}$$

and the claim is proved, hence the theorem. □

4. Inclusions and Banach Space Structure

For the goal of understanding duality, it is appropriate to discuss first the inclusions between all these spaces, and their structure.

4.1. Inclusions. First of all, it is obvious from the inclusions between L^p spaces over compact sets that $M^q \subset M^p$ if $1 \leqslant p \leqslant q \leqslant \infty$, and the inclusions are continuous. The same is true for the spaces M^p and $\mathscr{S}^p$, and for the Banach quotient $\widetilde{\mathscr{M}}^p$. All these families of spaces coincide with $L^\infty(\mathbb{R})$ when $p = \infty$, and obviously L^∞ embeds continuously in all of them, but in this paper we do not consider the special case $p = \infty$.

Let us come to more interesting inclusions. It is easy to see that $\mathscr{S}^p$ embeds continuously in M^p and $\mathscr{M}^p$, as follows.

Definition 6. For every locally $L^p(\mathbb{R})$ function f, denote by

$$A_p(T, f) = \left(\frac{1}{2T}\int_{-T}^{T} \left|f(t)\right|^p dt\right)^{1/p} \tag{54}$$

its L^p-averages and by

$$A(T,f)=\frac{1}{2T}\int_{-T}^{T}f(t)\,dt \tag{55}$$

the usual average.

Lemma 7. $\|f\|_{M^p}=\sup_{n\in\mathbb{N},n>0}A_p(n,f)$.

Proof. If $n<x<n+1$, it is clear that

$$\left(\frac{n}{x}\right)^{1/p}A_p(n,f)\leqslant A_p(x,f)\leqslant\left(\frac{n+1}{x}\right)^{1/p}A_p(n,f). \tag{56}$$

Since $x>1$ this yields $\sup_{n>0}A_p(n,f)\leqslant\|f\|_{M^p}\leqslant 2^{1/p}\sup_{n>0}A_p(n,f)$. □

Lemma 8. $\|f\|_{\mathcal{S}^p}\leqslant 2^{1/p}\sup_n\|f\|_{L^p[n,n+1]}$.

Proof. In computing the p power of the norm, that is, $\sup_{x\in\mathbb{R}}\int_x^{x+1}|f|^p$, just split the domain of integration $[x,x+1]$ into the intervals with integer endpoints that overlap it. □

Corollary 9. $\|f\|_{\mathscr{M}^p}\leqslant\|f\|_{M^p}\leqslant\|f\|_{\mathcal{S}^p}$.

Proof. The first inequality is obvious. For the second, by splitting the interval $[-n,n]$ into $2n$ subintervals of length 1, we see that $2nA_p(f,n)^p\leqslant 2n\|f\|^p_{\mathcal{S}^p}$; therefore, by Lemma 7, $\|f\|_{M^p}\leqslant\|f\|_{\mathcal{S}^p}$. □

Corollary 10. *$\mathcal{I}^p$ is closed in M^p.*

Proof. Recall that $\mathcal{I}^p$ is the null space of the semi-norm; hence it is obviously closed in $\mathscr{M}^p$. If $f_n\in\mathcal{I}^p$ converges to f in the norm of $\mathscr{M}^p$, then, by the first inequality of Lemma 8, $f_n\to f$ also in the seminorm of $\mathscr{M}^p$; hence $f\in\mathcal{I}^p$ since $\mathcal{I}^p$ is closed in this seminorm. □

Remark 11. The embedding of $\mathcal{S}^p$ in M^p is proper; it is not an isomorphism, or, equivalently, the norm of $\mathcal{S}^p$ cannot be bounded by a multiple of the M^p-norm. Indeed, we have already observed that translations are isometries on $\mathcal{S}^p$. Instead, $\|\lambda_t(f)\|_{M^p}$ tends to 0 as $t\to\pm\infty$ for every f with compact support.

As a consequence of Corollary 9, one has a continuous embedding $M^p\subset\mathscr{M}^p$. Therefore the embedding is projected onto the Banach quotient $\widetilde{\mathscr{M}}^p$; one has $M^p/\mathcal{I}^p\subset\widetilde{\mathscr{M}}^p$. It turns out that these two quotients coincide. This has been proved in [17, Proposition 2.2(ii)]; here we give a slightly different proof.

Proposition 12. *$\widetilde{\mathscr{M}}^p=\mathscr{M}^p/\mathcal{I}^p$ is isometrically isomorphic to $M^p/\mathcal{I}^p$.*

Proof. We have already observed that the latter quotient is embedded in the former, and, for every $f\in M^p$, one has $\|f\|_{\widetilde{\mathscr{M}}^p}\leqslant\|f\|_{M^p/\mathcal{I}^p}$. Let $\overline{f}$ denote the coset of $f \bmod \mathcal{I}^p$. We only need to show that the coset $\overline{f}$ of every $f\in\mathscr{M}^p$ contains a function in M^p and the norms are equal.

Let $f\in\mathscr{M}^p$ and $a>1$, and denote by f_a the function that coincides with f outside the interval $[-a,a]$ and is zero inside: $f_a=f-f\chi_{[-a,a]}$. Observe that $f\chi_{[-a,a]}\in\mathcal{I}^p$ since it is compactly supported. Moreover, $\|f_a\|_{\mathscr{M}^p}=\|f\|_{\mathscr{M}^p}$ for every a, because the semi-norm of a function in $\mathscr{M}^p$ does not change by adding another function with semi-norm zero. Now, if $a>1$,

$$\|f_a\|_{M^p}=\sup_{T\geqslant1}A_p(T,f_a)=\sup_{T\geqslant a}A_p(T,f_a)\leqslant\sup_{T\geqslant a}A_p(T,f). \tag{57}$$

Therefore

$$\begin{aligned}\|f\|_{\mathscr{M}^p}&=\|f_a\|_{\mathscr{M}^p}=\limsup_{T\to\infty}A_p(T,f)\\&=\inf_{a>1}\sup_{T\geqslant a}A_p(T,f)=\inf_{a>1}\|f_a\|_{M^p}.\end{aligned} \tag{58}$$

As $f_a=f-f\chi_{[-a,a]}$ belongs to the coset of f modulo $\mathcal{I}^p$ and $|f_a|\leqslant|f|$, this shows that, for a large enough, this coset contains functions with finite M^p-norm, and

$$\begin{aligned}\|\overline{f}\|_{M^p/\mathcal{I}^p}&=\inf\{\|f-g\|_{M^p}:g\in\mathcal{I}^p\}\\&\leqslant\inf_{a>1}\|f_a\|_{M^p}=\|f\|_{\mathscr{M}^p}.\end{aligned} \tag{59}$$

□

4.2. Tensor Products

Definition 13. From every sequence B_n of Banach spaces of functions on $\mathbb{R}$, one obtains a product Banach space $\otimes_{\ell^p}B_n$ by taking the completion of all finite linear combinations of functions $f_n\in B_n$ in the norm

$$\|f\|=\left\|\sum_n c_nf_n\right\|=\inf\left(\sum_n|c_n|^p\|f_n\|^p_{B_n}\right)^{1/p}, \tag{60}$$

where the infimum is taken over all possible decompositions of f of this type. Similarly, for every sequence $a_n\in\mathbb{C}$, one introduces a weighted product $\otimes_{\ell^p(\{a_n\})}B_n$ where the norm is defined by

$$\|f\|=\left\|\sum_n c_nf_n\right\|=\inf\left(\sum_n|a_n|^p|c_n|^p\|f_n\|^p_{B_n}\right)^{1/p}. \tag{61}$$

It is clear how to extend these definitions to the case $p=\infty$, or to products over c_0 instead of ℓ^p.

When we consider spaces of functions over disjoint intervals, for instance if $B_n=L^r(I_n)$, where $I_n=[n,n+1]$ or $[2^n,2^{n+1}]$, then the above representations are unique (up to normalization), and we can choose f_n to be the truncation $f\chi_{I_n}$ and $c_n\equiv1$.

Remark 14. It follows from Lemma 8 and Definition 13 that

$$\mathcal{S}^p=\underset{\ell^\infty}{\otimes}L^p[n,n+1]. \tag{62}$$

4.3. The Predual of M^p and Tensor Products. We now consider another Banach space of functions with appropriate L^p averages, the space E^q, introduced in [16], that is defined as follows. Let $1/p + 1/q = 1$, and $\psi_0 = \chi_{[-2,2]}$ and $\psi_k = \chi_{[-2^{k+1},-2^k]} + \chi_{[2^k,2^{k+1}]}$. Then

$$E^q \equiv \mathscr{E}^q(\mathbb{R}) = \left\{ f : \|f\|_{E^q} := \sum_{k=0}^{\infty} 2^{k/p} \|f\psi_k\|_q < \infty \right\}. \quad (63)$$

We also recall that the null space $\mathscr{J}^p$ of the Marcinkiewicz semi-norm, endowed with the norm of M^p, is a Banach space.

Now the following result, proved in [16, Theorems 2 and 3], is easy.

Proposition 15. *(i) One has $E^q = \otimes_{\ell^1(\{2^{n/p}\})} L^q(J_n)$ with $J_0 = [-2,2]$ and $J_n = \{x : 2^n \leqslant |x| \leqslant 2^{n+1}\}$.*

(ii) Also $M^p = \otimes_{\ell^\infty(\{2^{-n/p}\})} L^q(J_n)$.

(iii) If p, q are conjugate indices and $1 < p \leqslant \infty$, then M^p is the dual space of E^q (it will follow from Theorem 60(iii) that M_1 is not a dual space).

(iv) Moreover $\mathscr{J}^p = \otimes_{c_0(\{2^{-n/p}\})} L^q(J_n)$.

(v) If $1 \leqslant p < \infty$, then E^q is the dual space of $\mathscr{J}^p$.

Proof. We give a sketch of the proof. Part (i) follows directly from the definition of E^q. Let $d_k = \|f\psi_k\|_p$ (for $k > 1$ this is just $2^{k/p}$). Then part (ii) is equivalent to the statement that the M^p norm is equivalent to $|||f||| := \sup_{k\geqslant 1} 2^{-k/p} d_k$; we outline the argument of [16, Theorem 2]. Let $A_k := A_p(2^k, |f|^p) = A(2^k, |f|^p)^{1/p}$ and observe that $2^{-k/p} d_k \leqslant A_k \leqslant \|f\|_{\mathscr{M}^p}$. Therefore $|||f||| \leqslant \|f\|_{\mathscr{M}^p}$. For the opposite inequality, let $T \geqslant 1$ and choose k such that $T \in J_{k-1}$. Let $g \equiv f$ on $[-T, T]$ and zero elsewhere. Then

$$\begin{aligned} A_p(T,f) &= A_p(T,g) = A\left(T,|f|^p\right)^{1/p} \\ &\leqslant \left(\frac{2^k}{T}\right)^{1/p} A\left(2^k,|f|^p\right)^{1/p} \\ &\leqslant 2^{1/p} A_k^{1/p} = 2^{(1-k)/p} \left(\sum_{m=1}^{k} d_m^p\right)^{1/p} \\ &\leqslant 2^{(1-k)/p} \left(\sum_{m=1}^{k} 2^k\right)^{1/p} |||f||| < C\,|||f|||. \end{aligned} \quad (64)$$

This proves part (ii). The rest of the proof follows easily from this. □

Remark 16. For $p = 1$, the duality property of Proposition 15(iii) does not hold; M^1 is strictly smaller than the dual of E^∞, because the restrictions of the spaces M^p and E^q to functions supported, say, in a dyadic interval J_n are, respectively, $M^p = L^p(J_n)$ and $E^q = L^q(J_n)$. Similarly, for $p = \infty$ the duality of part (v) does not hold, because the restriction of $\mathscr{J}^\infty$ to the dyadic interval J_n is $L^\infty(J_n)$.

Remark 17. It is easy to see, as in [15, Theorem 3.1.C], that compactly supported functions and Schwartz functions are dense in E^q. It has been observed in [16, Theorem 2.E] that the same is true for the null space $\mathscr{J}^p$ of the semi-norm. Therefore E^q and $\mathscr{J}^p$ are separable, and translation is strongly continuous on them. Instead, M^p, $\widetilde{\mathscr{M}}^p$, and M^p are not separable, as we show in the next theorem.

4.4. Separability. It follows from the tensor product structure of $\mathscr{S}^p$ (Remark 14) that $\mathscr{S}^p$ contains a closed subspace isometric to ℓ^∞ and therefore is not separable. For the same reason, M^p is not separable, by part (ii) of Proposition 15. Following [7, Proposition 2.5], we now show that $\mathscr{M}^p$ and $\widetilde{\mathscr{M}}^p$ are not separable.

Theorem 18. *For all $p \geqslant 1$, the Marcinkiewicz spaces $\mathscr{M}^p$ and $\widetilde{\mathscr{M}}^p$ contain a closed subspace isomorphic to ℓ^∞; hence they are not separable.*

Proof. We start by building a sequence of intervals with larger and larger distance and length. Start with $a_0 = 0$ and $b_0 = 1$ and let $a_n = 2^n b_{n-1}$ and $b_n = 2^n a_n$. Then $b_{n-1}/a_n = 2^{-n}$, so

$$\begin{aligned} &\frac{1}{2a_n}\int_{-a_n}^{a_n}\left(\chi_{[a_{n-1},b_{n-1}]} + \chi_{[-b_{n-1},-a_{n-1}]}\right) \\ &\quad = \frac{1}{a_n}\int_0^{a_n}\chi_{[a_{n-1},b_{n-1}]} < \frac{1}{2^n} \end{aligned} \quad (65)$$

but $a_n/b_n = 2^{-n}$, so

$$\frac{1}{2b_n}\int_{-b_n}^{b_n}\left(\chi_{[a_n,\,b_n]} + \chi_{[-b_n,\,-a_n]}\right) = \frac{1}{b_n}\int_0^{b_n}\chi_{[a_n,b_n]} = 1 - \frac{1}{2^n}. \quad (66)$$

In particular, $\lim_n A(b_n, \chi_{[a_n,b_n]} + \chi_{[-b_n,-a_n]}) = 1$. Hence, if we denote by $\{\mathscr{J}_n, n \in \mathbb{N}\}$ a partition of $\mathbb{N}$ into a sequence of infinite subsets and let $f_n = \sum_{k\in\mathscr{J}_n}(\chi_{[a_k,b_k]} + \chi_{[-b_k,-a_k]})$, then $\limsup_{T\to\infty} A(T,|f|^p) = 1$, because there are infinitely many intervals $[a_k, b_k]$ with $k \in \mathscr{J}_n$, all disjoint, and $A(T,|f|^p) \leqslant A(T,1) = 1$. Then, for every sequence c_n with $\sup_n |c_n| = 1$, the function $f = \sum_{n=1}^{\infty} c_n f_n$ satisfies

$$|f| \leqslant \sum_{j=0}^{\infty}\left(\chi_{[a_j,b_j]} + \chi_{[-b_j,-a_j]}\right), \quad (67)$$

$$\|f\|_{\mathscr{M}^p} \geqslant 1 \quad (68)$$

(the last statement holds because, for every $\varepsilon > 0$, there is n_ε such that $|c_{n_\varepsilon}| > 1 - \varepsilon$, and $\|f\|_{\mathscr{M}^p} \geqslant \|f_{n_\varepsilon}\|_{\mathscr{M}^p}$ because all the f_n are nonnegative).

Let now k be the integer such that $a_k \leqslant T < a_{k+1}$. Then, by (67), the fact that $\sum_{j=k+1}^{\infty} \chi_{[a_j,b_j]}$ has support in (T,∞), the disjointness of the intervals, and (68), one has

$$\begin{aligned} A\left(T,|f|^p\right) &\leqslant A\left(T,\left|\sum_{j=0}^{k}\left(\chi_{[a_j,b_j]}+\chi_{[-b_j,-a_j]}\right)\right|^p\right) \\ &\leqslant \frac{1}{T}\int_0^T \chi_{[a_k,b_k]} + \frac{1}{a_k}\int_0^{a_k}\sum_{j=0}^{k-1}\chi_{[a_j,b_j]} \\ &\leqslant 1+\sum_{j=0}^{k-1}\frac{1}{2^{j+2}} \leqslant 2. \end{aligned} \tag{69}$$

So $1 \leqslant \| \sum_{n=1}^{\infty} c_n f_n \|_{\mathscr{M}^p} \leqslant 2^{1/p}$. Therefore the closed subspace of $\mathscr{M}^p$ generated by the sequence $\{f_n\}$ is isomorphic to ℓ^{∞}, and the same argument also works for $\widetilde{\mathscr{M}}^p$. □

5. The Dual Spaces of M^p and $\widetilde{\mathscr{M}}^p$

The Riesz representation theorem shows that all continuous linear functionals on L^p spaces can be represented as (integrals versus) functions in L^q, and so they depend mostly on the values of the L^p functions on compact sets. Our aim here is to show that, on spaces of locally summable functions, that can be large at infinity, some continuous functionals depend on asymptotic values and cannot be represented by functions in the usual integral sense (we shall see that most of them can be represented by integrals of means). Continuous linear functionals on Marcinkiewicz spaces have been studied in [7] on bounded p-mean spaces, in [16]. We present these results here and construct interesting examples of functionals that are not represented by functions; in the next Section, we extend these results to Stepanoff spaces.

5.1. Functionals on Seminormed Spaces. Let us consider the dual space of the Marcinkiewicz space $\mathscr{M}^p$. This is a complete semi-normed space but not a Banach space. It is clear that its continuous linear functionals are precisely those that factor through the null space $\mathscr{I}^p$ of the semi-norm, that is, the dual of the Banach quotient $\widetilde{\mathscr{M}}^p = \mathscr{M}^p/\mathscr{I}^p$.

Indeed, all continuous linear functionals on a semi-normed complete space W vanish on the null space I of the semi-norm, because if $F \in W'$ does not vanish on I, then $F(w) \neq 0$ for some non-zero $w \in I$, but $\|w\|_W = 0$ because I is the null space of the semi-norm; hence there is no constant C such that $|F(w)| < C\|w\|_W$. The converse is obvious.

Since every compactly supported L^p function is in $\mathscr{I}^p$, the dual of $\mathscr{M}^p$ does not contain non-zero functionals that can be represented as L^q functions; that is, it consists of linear functionals that depend only on the asymptotic behaviour of Marcinkiewicz functions. Here are the two most natural ones, defined and continuous on a closed subspace of $\mathscr{M}^p$ and thereby extended to continuous functionals on the whole of $\mathscr{M}^p$ by the Hahn-Banach theorem:

$$L^{\pm}(f) = \lim_{T\to\pm\infty}\frac{1}{T}\int_0^T f(t)\,dt. \tag{70}$$

There are interesting instances of $\mathscr{M}^p$-discontinuous functionals defined on appropriate subspaces of $\mathscr{M}^p$. For instance, the functionals

$$L^{\pm}(f) = \lim_{x\to\pm\infty} f(x), \tag{71}$$

defined on the subspaces $J_p^{\pm}$ of functions vanishing at infinity, are discontinuous. The lack of continuity is equivalent to the fact that the subspaces $J_p^{\pm}$ are not closed in $\mathscr{M}^p$; the proof of this elementary fact will be given in Corollary 50.

Here are some other interesting $\mathscr{M}^p$-discontinuous functionals. For $0 \leqslant \alpha \leqslant 1$, let

$$W_\alpha = \left\{ f \in \mathscr{M}^p : \int_{-T}^{T} |f(t)|^p dt = O\left(T^{\alpha}\right)\right\} \tag{72}$$

as $T \to +\infty$. It is clear that $W_1 = \mathscr{M}^p$ and, for $\alpha < 1$, W_α is contained in the closed subspace of $\mathscr{M}^p$ of the functions with semi-norm 0; in particular, these subspaces are not dense in $\mathscr{M}^p$. Moreover, $W_\alpha \subsetneq W_\beta$ if $\alpha < \beta$.

Since W_α is a subspace of the null space of the semi-norm, the only linear functional that is continuous in the semi-norm of $\mathscr{M}^p$ is the zero functional. We now exhibit some interesting nontrivial (hence discontinuous) linear functionals on W_α. For simplicity, we first describe them in the case $p = 1$:

$$L_\alpha^{\pm}(f) = \lim_{T\to\pm\infty}\frac{1}{T^{\alpha}}\int_0^T f(t)\,dt. \tag{73}$$

Clearly, $L_\beta^{\pm}|_{W_\alpha} = 0$ if $\alpha < \beta$.

Hölder's inequality shows that, for $p > 1$, the correct way to define $U_\alpha^{\pm}$ and $L_\alpha^{\pm}$ is by replacing T^{α} at the denominator with $T^{1-(1-\alpha)/p}$.

In the next sections we expand these ideas to achieve a more complete representation, developed in [7], where the above Hahn-Banach extensions are reinterpreted as Dirac measures on the points at infinity of a suitable Stone-Čech compactification.

5.2. Uniformly Convex Normed Spaces

Definition 19 (see [18]). A normed (or semi-normed) space is uniformly convex if, for every $\varepsilon > 0$ and all vectors f, g in the unit ball such that $\|f-g\| \geqslant \varepsilon$, there exists $\delta(\varepsilon) > 0$ such that $(1/2)\|f+g\| \leqslant 1-\delta(\varepsilon)$. The function $\delta(\varepsilon)$ is called the modulus of convexity; its geometrical meaning is the infimum of the distance from the midpoint of f and g to the unit sphere (the boundary of the ball). Observe that δ is a nondecreasing function of ε.

The following results are stated without proof in [7].

Lemma 20. *(i) Let V be a uniformly convex space and ℓ_f be a continuous linear functional on V of norm 1 that attains its norm at a vector f with $\|f\| = 1$, in the sense that $\ell_f(f) = 1$. Then, for every g in the unit ball B of V with $\|f-g\| \geqslant \varepsilon$, one has $|\ell_f(g)| \leqslant 1-2\delta(\varepsilon)$.*

(ii) The same statement holds if $\|g\| \leqslant \|f\| < 1$, $\|f-g\| \geqslant \varepsilon$ and ℓ_f attains its norm at $f/\|f\|$.

(iii) If, more generally, f and g are any vectors in the unit ball B such that $\|f-g\| \geqslant \varepsilon$ and $\|f\| \geqslant 1-\varepsilon/2$ and ℓ_f attains its norm at $f/\|f\|$, then $\ell_f(g) \leqslant 1-2\delta(\varepsilon/2)$.

Proof. We can restrict attention to the bidimensional subspace of B generated by f and g. The proof is illustrated in Figure 1. For simplicity, we have drawn the figure under the implicit assumption that the restriction of the B-norm to this bidimensional space is the Euclidean norm, and indeed the spotted line that represents the hyperplane $\{h : \ell_f(h) = 1\}$ is drawn as perpendicular to the radius of the unit ball, but the only property that we are using is that all of the ball is on one side of this line, that is, we only use the fact that the ball is convex: that is, the triangular inequality.

Part (i) follows by considering the segment B in Figure 1, drawn from g to the hyperplane $\{\ell_f = 1\}$ and orthogonal to this hyperplane, whose length is $\ell_f(f) - \ell_f(g) = 1 - \ell_f(g)$. This segment is twice as long as its parallel segment A drawn from the mid-point $(f+g)/2$, and in turn A is longer than the distance between the mid-point and the unit sphere, hence longer than $\delta(\varepsilon)$.

For part (ii), it is enough to observe that, whenever $\|g\| \leqslant \|f\| < 1$, the distance from $f/\|f\|$ and g is larger than $\|f-g\|$ and to apply part (i).

To prove (iii) consider the triangle whose vertices are $f/\|f\|$, f, and g. We may as well consider the worst possible case where the segment from f to g has maximal length $\|f-g\| = \varepsilon$. The segment from f to $f/\|f\|$ has length larger than or equal to $\varepsilon/2$. Hence the third side, from $f/\|f\|$ to g, has length $\tilde{\varepsilon}$ not less than $\varepsilon - \varepsilon/2 = \varepsilon/2$. Now part (i) and the monotonicity of δ yield $\ell_f(g) \leqslant 1 - 2\delta(\tilde{\varepsilon}) \leqslant 1 - 2\delta(\varepsilon/2)$. □

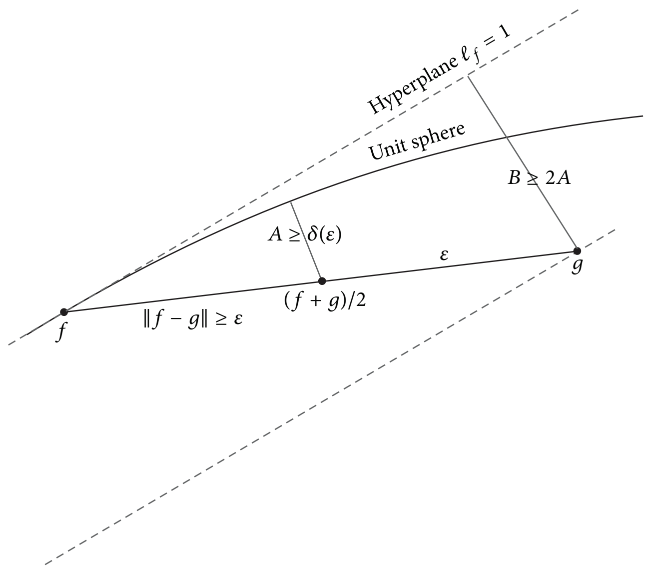

Figure 1: Estimate for $\ell_f(g)$ if $\|f-g\| \geqslant \varepsilon$ and f has norm 1.

Proposition 21 (see [18, 19]). *For $1 < p < \infty$, $L^p[0,1]$ is uniformly convex. Its modulus of convexity δ_p satisfies*

$$\delta_p(\varepsilon) = 1 - \sqrt[p]{1 - \left(\frac{\varepsilon}{2}\right)^p} \tag{74}$$

if $p \geqslant 2$, and

$$\delta_p(\varepsilon) = \delta_q(\varepsilon) \tag{75}$$

if $1 < p < 2$ and q is the conjugate index.

5.3. The Dual of M^p: Integral Representation of Norm-Attaining Continuous Functionals. Now we describe the dual of the spaces of bounded p-means, studied in [7, 16]; in particular, we describe an integral representation, obtained in [7], for those continuous linear functions that attain their norm. All the forthcoming results on integral representation are taken from [7]; our proofs are more detailed and expanded than those in the original paper.

We start with some easy comments on functionals that attain their norm.

Lemma 22. *(i) Let V be a Banach space and V' its dual space. Then every element of V, regarded as a functional on V', attains its norm on some element $a \in V'$. In particular, all continuous functionals on reflexive spaces attain their norms.*

(ii) Every real finitely additive finite Borel measure on a Borel space X, regarded as a continuous functional on $L^\infty(X)$, attains its norm. The norm is attained on a function that has modulus 1 on the support of the measure. The same is true for complex-valued finitely additive measures on $\mathbb{R}$ provided that they are absolutely continuous with respect to Lebesgue measure.

(iii) If X is not compact, finitely additive measures are continuous functionals on $C \cap L^\infty(X)$ (by restriction from functionals on $L^\infty(X)$: so, not all continuous functionals on this space are given by countably additive measures. A real finitely additive measure μ attains its norm on $C \cap L^\infty(X)$ if and only if it is positive.

(iv) Not every (countably additive) finite (real or complex) Borel measure on $\mathbb{R}$, regarded as a continuous functional on $C(\mathbb{R})$, attains its norm, but it attains its norm if it is positive (up to multiplication by a constant).

Proof. We first observe that, for every $b \in V$, there is $a \in V'$ such that $\langle a, b\rangle = \|b\|$. This is trivially true in the one-dimensional subspace $\widetilde{V}$ generated by b, for some linear functional $\tilde{a}$ on $\widetilde{V}$; the requested element $a \in V'$ is the norm-preserving Hahn-Banach extension of $\tilde{a}$ to a functional on the whole of V.

Then, for every $b \in V$, $\langle b, a\rangle = \langle a, b\rangle = \|b\|$; therefore b, as a functional on V', attains its norm. This proves part (i).

The real finitely additive finite Borel measures μ on a Borel space X are the continuous dual of $L^\infty(X)$. Let χ_+ and χ_- be the characteristic functions of the supports of the positive and negative parts of μ, respectively. Then $\|\mu\| = |\mu|(X) = \mu(\chi_+ - \chi_-)$. This proves the first half of part (ii), and it also proves parts (iii) and (iv); a real countably additive measure attains its norm if and only if it is positive (because it attains its norm only on the function $\chi_+ - \chi_-$, which is discontinuous unless one of its two terms vanishes), or, slightly more generally, if it is a constant multiple of a positive measure.

Let us show that an absolutely continuous measure on $L^\infty(\mathbb{R})$ attains its norm as a functional on L^∞. Indeed, if $\mu(E) = \int_{\mathbb{R}} \chi_E h\, dx$ for every measurable set E and for some $h \in L^1$, then

$$\begin{aligned} \|\mu\| &= |\mu|(\mathbb{R}) = \int_{\mathbb{R}} |h(x)|\, dx \\ &= \int_{\mathbb{R}} e^{-i\,\mathrm{phase}\, h(x)}\, h(x)\, dx = \mu\left(e^{-i\,\mathrm{phase}\, h}\right). \end{aligned} \tag{76}$$

In general, though, discrete non-positive measures on $\mathbb{R}$ do not attain their norm; for instance, let $\{q_n\}$ be an enumeration of the rationals and let $\mu = \sum_n (-1)^n 2^{-n}\delta_{q_n}$; since $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$, there is no *continuous* function f such that $f(q_n) = (-1)^n$, and so μ cannot attain its norm as a functional on $C(\mathbb{R})$.

To finish the proof, let us provide examples of continuous functionals on $C \cap L^\infty(\mathbb{R})$ that are not represented by countably additive measures. Consider the closed subspace of $C \cap L^\infty$ of functions that have a finite limit for, say, $x \to +\infty$. The limit $\lim_{x\to+\infty} f(x)$ is continuous on this subspace, and, by Hahn-Banach theorem, extends to a continuous functional on all of $C \cap L^\infty$ that vanishes on all compactly supported functions. Represented as a measure μ, this functional vanishes on all bounded sets but $\mu(\mathbb{R}) = 1$; therefore μ is finitely but not countably additive. □

Definition 23. For every locally L^p function g on $\mathbb{R}$, let $\natural_p$ be the operator on L^p defined by

$$\natural_p g := g^\natural(x) = \frac{|g(x)|^p}{g(x)} = |g(x)|^{p-1} e^{-i\,\mathrm{phase}\,(g(x))} \tag{77}$$

if $g(x) \neq 0$, and $g^\natural = 0$ otherwise (here as usual, for $z \in \mathbb{C}$, $z = \rho e^{i\theta}$, we write $\theta = \mathrm{phase}\,(z)$).

Lemma 24. *If $p, q \in (1,\infty)$ are conjugate indices, and $g \in L^p$ on an interval (or a measurable subset of $\mathbb{R}$). Then $g^\natural := \natural_p g \in L^q$ and $\|g^\natural\|_q^q = \|g\|_p^p$. Moreover, the operator $\natural_p$ is the inverse of $\natural_q$.*

Proof. If $g \in L^p$, one has $g^\natural \in L^q$, because $1/p + 1/q = 1$ implies that $(p-1)q = p$, and so $\|g^\natural\|_q = \|g\|_p^{p/q} = \|g\|_p^{p-1}$.

Finally, if $g \in L^p$, then

$$\natural_q \natural_p g(x) = \natural_q\left(\frac{|g|^p}{g}\right) = \frac{|g|^{(p-1)q}}{|g|^p/g} = \frac{g|g|^p}{|g|^p} = g, \tag{78}$$

again because $(p-1)q = p$. □

Proposition 25. *Let $p, q \in (1,\infty)$ be conjugate indices, let μ be a σ-additive finite Borel measure on $[1,\infty]$ and $\phi \in M^q(\mathbb{R})$. If A is the average operator introduced in Definition 6 and ℓ is the functional defined on M^p by*

$$\ell(h) = \int_1^\infty A(T, h\phi)\, d\mu(T), \tag{79}$$

then ℓ is continuous on M^p, and

$$\begin{aligned} \|\ell\| &\leqslant \int_1^\infty A\left(T, |\phi|^q\right)^{1/q} d|\mu|(T) \leqslant \|\phi\|_{M^p}\|\mu\|, \\ \int_1^\infty & A\left(T, |\phi|^q\right) d\mu(T) \leqslant \|\ell\|\, \|\phi\|_{M^q}^{q-1}. \end{aligned} \tag{80}$$

Proof. The first inequality follows from Hölder's inequality, because

$$\begin{aligned} |\ell(h)| &\leqslant \int_1^\infty A\left(T, |\phi|^q\right)^{1/q} A\left(T, |h|^p\right)^{1/p} d\mu(T) \\ &\leqslant \int_1^\infty A\left(T, |\phi|^q\right)^{1/q} d\mu(T)\, \|h\|_{M^p}. \end{aligned} \tag{81}$$

If $\phi \in M^q$, by Lemma 24 one has $\phi^\natural \in M^p$, $\phi\phi^\natural = |\phi|^q$, and $\|\phi^\natural\|_{M^p} = \|\phi\|_{M^q}^{q/p} = \|\phi\|_{M^q}^{q-1}$. Therefore

$$\int_1^\infty A\left(T, |\phi|^q\right) d\mu(T) = \ell\left(\phi^\natural\right) \leqslant \|\ell\| \left\|\phi^\natural_{M^p}\right\| \leqslant \|\ell\|\, \|\phi\|_{M^q}^{q-1}. \tag{82}$$

This is the second inequality of the statement. □

Definition 26. Denote by $U(M^p)$ the subset of M^p of all functions of norm 1, by K_p the Cartesian product $[1,\infty] \times U(M^p)$, and by $\beta(K_p)$ its Stone-Čech compactification.

Lemma 27. *For $1 < p < \infty$ and $f \in M^p$ define the weighted mean operator $f^\dagger$ on K_p as*

$$f^\dagger(T, \phi) = A(T, f\phi). \tag{83}$$

Then $f^\dagger$ is an isometric isomorphism from M^p to $C \cap L^\infty(K_p)$ and therefore also from M^p to $C(\beta(K_p))$. In particular, the $\mathbb{R}$-subspace of M^p that consists of real valued functions is isometrically isomorphic to the spaces of real valued functions in $C \cap L^\infty(K_p)$ and $C(\beta(K_p))$.

Proof. Observe again that, since $1/p + 1/q = 1$, one has $pq = p + q$ hence $(p-1)q = p$; that is, $p - 1 = p/q$. Therefore Lemma 24 shows that if $\phi_0 = f^\natural/\|f\|_{M^p}^{p-1}$ (notation as in Definition 23), then

$$\|\phi_0\|_{M^q} = 1. \tag{84}$$

This yields an inequality between the two norms:

$$\left\|f^\dagger\right\|_\infty \geqslant \sup_{T\geqslant 1} A(T, f\phi_0) = \frac{\sup_{T\geqslant 1} A\left(T, |f|^p\right)}{\|f\|_{M^p}^{p-1}} = \|f\|_{M^p}. \tag{85}$$

Let us prove the converse inequality. Observe that, again by Hölder's inequality (86) for every $(T, \phi) \in K_p$ one has

$$\begin{aligned} \left|f^\dagger(T,\phi)\right| &= |A(T, f\phi)| \leqslant A\left(T, |f|^p\right)^{1/p} A\left(T, |\phi|^q\right)^{1/q} \\ &= A_p(T, f)\, A_q(T, \phi). \end{aligned} \tag{86}$$

This and (84) imply that

$$\sup_T \left| f^\dagger (T, \phi) \right| \leqslant \|\phi\|_{M^q} \|f\|_{M^p} = \|f\|_{M^p}. \quad (87)$$

Hence

$$\left\| f^\dagger \right\|_\infty \leqslant \|f\|_{M^p}. \quad (88)$$

This proves the converse inequality. The last statement follows from the fact that every continuous and bounded function on K_p has a unique continuous extension on the Stone-Čech compactification, and the extension is an isometry (see [20, Chapter 6]). □

Corollary 28 (the dual of M^p). *The space of continuous functionals on M^p, $1 < p < \infty$, is isometrically isomorphic to the space $M(\beta(K_p))$ of countably additive regular finite Borel measure on $\beta(K_p)$ and to the space $m(K_p)$ of finitely additive regular finite Borel measures on K_p, in the sense that every functional $\ell \in (M^p)'$ can be uniquely written as*

$$\ell(f) = \int_{\beta(K_p)} f^\dagger (T, \phi) \, d\tilde{\nu} = \int_{K_p} f^\dagger (T, \phi) \, d\nu, \quad (89)$$

with $\tilde{\nu} \in M(\beta(K_p))$ and $\nu \in m(K_p)$ such that $\|\ell\| = \|\tilde{\nu}\| = \|\nu\|$, and conversely.

The extreme points in the unit ball of $(M^p)'$ are the Dirac measures on $\beta(K_p)$, or the extreme points in the unit ball of $m(K_p)$; these correspond to the functionals $f \rightarrow f^\dagger(T, \phi)$ where ϕ is an extreme point in the unit ball of M^q and $T \in [1, \infty)$, plus the purely finitely additive measures in the sense of the forthcoming Definition 31. (Necessary and sufficient conditions for extremality in the unit ball of M^q, for $1 < q < \infty$, will be given later in Theorem 60.)

Proof. Recall that, for every Borel space X, the dual space of $L^\infty(X)$ is $m(X)$. By restriction, the dual space of $C \cap L^\infty(K_p)$ is again $m(K_p)$. More precisely, as $C \cap L^\infty$ is a norm-closed subspace of L^∞, its dual space is the quotient of $m(K_p) = (L^\infty(K_p))'$ obtained by identifying two finitely additive measures that give rise to the same functional when restricted to $C \cap L^\infty(K_p)$; apart from this equivalence, the dual of $C \cap L^\infty(K_p)$ is isometrically isomorphic to $m(K_p)$. Then the isometric isomorphism between $C \cap L^\infty(K_p)$ and $C(\beta(K_p))$ induces an isometry between the respective dual spaces $M(\beta(K_p))$ and $m(K_p)$.

The characterization of extreme points, whose details are left to the reader, follows from this. □

On the basis of the isomorphism between $M(\beta(K_p))$ and $m(K_p)$, from now on, with abuse of notation, we shall write the measure in $M(\beta(K_p))$ corresponding to $\nu \in m(K_p)$ again as ν. The representing measure can be described more precisely for functionals attaining their norm, as follows.

Theorem 29 (integral representation of functionals on M^p attaining their norm). *Let p, q be conjugate indices, with $1 < p, q < \infty$, and ℓ a continuous functional on M^p that attains its norm. Then, for some $\phi \in U(M^q)$ (notation as in Definition 26) and for some finite finitely additive positive measure μ on $[1, \infty]$, one has*

$$\ell(f) = \int_1^\infty f^\dagger (T, \phi) \, d\mu(T) \quad (90)$$

for every $f \in M^p$. Moreover, $\mu \geqslant 0$, $\|\mu\| = \|\ell\|$, $A(T, |\phi|^p) = 1$ on the support of μ and ℓ attains its norm on $\phi^\natural$.

Conversely, let ℓ be a functional as in (90), with respect to a finitely additive measure μ. Then this integral representation of ℓ is unique (except for the identification mentioned in the proof of Corollary 28), and ℓ is continuous on M^p. Moreover, ℓ attains its norm if and only if the measure μ is positive, and $A(T, |\phi|^p) = 1$ on the support of μ.

Proof. Without loss of generality, assume $\|\ell\| = 1$. By Lemma 27 the dual of M^p is isometric to the space of countably additive measures on $\beta(K_p)$; therefore, for some $\nu \in M(\beta(K_p))$ with $\|\nu\| = 1$ and for all $f \in M^p$, one has

$$\ell(f) = \int_{\beta(K_p)} f^\dagger d\nu. \quad (91)$$

Let $g \in M^p$ be a function on which ℓ attains its norm: $\ell(g) = 1$. Since $1 = \|\ell\| = \sup\{|\ell(\nu)| : \|\nu\|_{M^p} = 1\}$, one has $\|g\|_{M^p} = 1$, and by Lemma 24

$$\left\| g^\natural \right\|_{M^q} = \|g\|_{M^p} = 1. \quad (92)$$

Denote by F the subset of $\beta(K_p)$ where $|g^\dagger|$ attains its maximum value (i.e., 1, by Proposition 25). Consider the family Φ of all nets (i.e., ultrafilters) (T_α, ϕ_α) in K_p that converge to points of F.

As g and ϕ have norm 1, it follows by (86) that, for every $(T, \phi) \in K_p$,

$$\begin{aligned} \left| g^\dagger (T, \phi) \right| &= \left| A(T, g\phi) \right| \leqslant \left| A\left(T, |g|^p\right) \right|^{1/p} \left| A\left(T, |\phi|^q\right) \right|^{1/q} \\ &\leqslant \|g\|_{M^p} \|\phi\|_{M^q} = 1. \end{aligned} \quad (93)$$

Therefore, if $\{(T_\alpha, \phi_\alpha)\} \in \Phi$, then the interval $[-T_\alpha, T_\alpha]$ must verify the condition

$$\lim_\alpha \left| A\left(T_\alpha, |g|^p\right) \right|^{1/p} = 1 = \|g\|_{M^p}. \quad (94)$$

Since the measure ν in (91) has mass 1, by (93) ν must be supported in F. We make the following *Claim 1*: for every $f \in M^p, z \in F$ and for every net $\{(T_\alpha, \phi_\alpha)\}$ in Φ that converges to z, one has $f^\dagger(z) = 1 = \lim_\alpha f^\dagger(T_\alpha, g^\natural)$ (notation as in Definition 23), and so

$$\left\| f^\dagger \right\|_\infty \leqslant \sup_{T \geqslant 1} \left| f^\dagger \left(T, g^\natural\right) \right|. \quad (95)$$

Indeed, remember that $gg^\natural = |g|^p$ by Definition 23, and choose any point in F and let (T_α, ϕ_α) be a net in Φ that converges to it. Then $A(T_\alpha, gg^\natural) = (1/2T_\alpha)\int_{-T_\alpha}^{T_\alpha}|g|^p \to 1 = \|g\|_{M^p}^p$ by (94). This proves Claim 1.

In the remainder of this proof, we keep notation more compact by writing $L^q_\alpha := L^q([-T_\alpha, T_\alpha]\ , 1/(2T_\alpha))$. Denote by $\widetilde{\ell}_\alpha$ the continuous functional on L^q_α given by

$$\widetilde{\ell}_\alpha(f) = A(T_\alpha, gf) = \frac{1}{2T_\alpha}\int_{-T_\alpha}^{T_\alpha} gf\,dx. \tag{96}$$

Then, by (94),

$$\left\|\widetilde{\ell}_\alpha\right\| = \|g\|_{L^p_\alpha} = \left(\frac{1}{2T_\alpha}\int_{-T_\alpha}^{T_\alpha}|g|^p dx\right)^{1/p} \longrightarrow 1 = \|g\|_{M^p}. \tag{97}$$

We make the following *Claim 2*: the functional $\widetilde{\ell}_\alpha$ attains its norm at $g^\natural/\|g^\natural\|_{L^q_\alpha}$.

Indeed, $\|g^\natural\|_{L^q_\alpha} = \|g\|_{L^p_\alpha}^{p/q}$ by Lemma 24, and $pq - p = q$ because p and q are conjugate indices. Therefore

$$\begin{aligned}\widetilde{\ell}_\alpha\left(\frac{g^\natural}{\|g^\natural\|_{L^q_\alpha}}\right) &= A\left(T_\alpha, \frac{gg^\natural}{\|g^\natural\|_{L^q_\alpha}}\right) = \frac{1}{2T_\alpha}\int_{-T_\alpha}^{T_\alpha}\frac{|g|^p}{\|g\|_{L^p_\alpha}^{p/q}}dx \\ &= \|g\|_{L^p_\alpha}^{p-p/q} = \|g\|_{L^p_\alpha} = \left\|\widetilde{\ell}_\alpha\right\|.\end{aligned} \tag{98}$$

This proves Claim 2.

Now observe that

$$\begin{aligned}\lim_\alpha\left|A(T_\alpha, g\phi_\alpha)\right| &= 1 = \lim_\alpha A\left(T_\alpha, gg^\natural\right) \\ &= \lim_\alpha A\left(T_\alpha, |g|^p\right) = \lim_\alpha \|g\|_{L^p_\alpha}^p.\end{aligned} \tag{99}$$

Indeed, the last identity for $\|g\|_{L^p_\alpha}^p$ has been proved in (94). On the other hand, $|A(T_\alpha, g\phi_\alpha)| = g^\dagger(T_\alpha, \phi_\alpha) \to 1$ by continuity of $g^\dagger$. It follows from these two identities that

$$\lim_\alpha\left(\left|A(T_\alpha, g\phi_\alpha)\right| - A\left(T_\alpha, |g|^p\right)\right) = 0. \tag{100}$$

Next, we prove the following *Claim 3*: $\limsup_\alpha A(T_\alpha, \phi_\alpha - g^\natural) = 0$.

Indeed, suppose that $\limsup_\alpha A(T_\alpha, \phi_\alpha - g^\natural) > \varepsilon$ for some $\varepsilon > 0$. Then, for infinitely many values of α, one has $\|\phi_\alpha - g^\natural\|_{L^q_\alpha} > \varepsilon$. Then it follows by part (iii) of Lemma 20 that

$$\left|\widetilde{\ell}_\alpha(\phi_\alpha)\right| \leqslant \left\|\widetilde{\ell}_\alpha\right\|\left(1 - 2\delta\left(\frac{\varepsilon}{2}\right)\right) = \|g\|_{L^p_\alpha}\left(1 - 2\delta\left(\frac{\varepsilon}{2}\right)\right). \tag{101}$$

On the other hand, $\widetilde{\ell}_\alpha(g^\natural) = A(T_\alpha, |g|^p) \geqslant \|\widetilde{\ell}_\alpha\|\|g^\natural\|_{L^q_\alpha} = \|g\|_{L^p_\alpha}\|g^\natural\|_{L^q_\alpha} = \|g\|_{L^p_\alpha}^{1+p/q}$, by Lemma 24. In particular, $\widetilde{\ell}_\alpha(g^\natural) > 0$. Therefore,

$$\begin{aligned}&\left|\widetilde{\ell}_\alpha\left(g^\natural\right) - \widetilde{\ell}_\alpha(\phi_\alpha)\right|\widetilde{\ell}_\alpha\left(g^\natural\right) - \left|\widetilde{\ell}_\alpha(\phi_\alpha)\right| \\ &\quad\geqslant \|g\|_{L^p_\alpha}^{1+p/q} - \|g\|_{L^p_\alpha}\left(1 - 2\delta\left(\frac{\varepsilon}{2}\right)\right).\end{aligned} \tag{102}$$

Since $\|g\|_{L^p_\alpha} \to 1$ by (94), the left-hand side is bounded below by $\delta(\varepsilon)$ for infinitely many α's. This contradicts (100), thereby proving Claim 3.

By applying again Hölder's inequality (86) to Claim 3, we finally obtain

$$\lim_\alpha A(T_\alpha, f\phi_\alpha) = \lim_\alpha A\left(T_\alpha, fg^\natural\right) \tag{103}$$

for every $f \in M^p$. Hence, for every $z \in F$ and every net converging to z,

$$f^\dagger(z) = \lim_\alpha A\left(T_\alpha, fg^\natural\right). \tag{104}$$

Now, by (90) and (95) and the fact that ν has support in F,

$$|\ell(f)| \leqslant \left|\int_{\beta(K_p)}\left|\widetilde{f}\right|d\nu\right| = \left|\int_F\left|\widetilde{f}\right|d\nu\right| \leqslant \sup_{T\geqslant 1}\left|f^\dagger\left(T, g^\natural\right)\right|. \tag{105}$$

The functional $\tau(f^\dagger) = \sup_{T\geqslant 1}|f^\dagger(T, g^\natural)|$ is a nonnegative homogeneous subadditive functional on $C(\beta(K_p))$. The previous inequality and Hahn-Banach theorem yield a norm-preserving extension of ℓ to a continuous functional on $C(\beta(K_p))$, which is a countably additive measure $\widetilde{\nu}$ on $\beta(K_p)$, such that $\|\widetilde{\nu}\| = 1$ and $|\langle\widetilde{\nu}, f\rangle| \leqslant \tau(f)$ for every $f \in C(\beta(K_p))$. The previous inequality implies that $\widetilde{\nu}$ has support in $\beta([1,\infty] \times g^\natural)$. Since $\beta([1,\infty] \times g^\natural)$ is isomorphic to $\beta([1,\infty])$, one has $\beta([1,\infty] \times g^\natural) = \beta([1,\infty]) \times g^\natural$ (in particular, $\phi = g^\natural$, and ℓ attains its maximum on $g = \phi^\natural$, by the last statement in Lemma 24).

Define a *finitely* additive measure on $[1,\infty]$ by restriction: for every Borel set E in $[1,\infty]$ let $\mu(E) = \widetilde{\nu}(E \times g^\natural)$. Then $\|\mu\| = 1$ and (91) becomes

$$\ell(f) = \int_1^\infty f^\dagger\left(T, g^\natural\right)d\mu(T). \tag{106}$$

As $\|g^\natural\|_{M^p} = 1$, the integrand has modulus less than or equal to 1 for every f in the unit ball of M^p. On the other hand, $1 = \ell(g) = \int_1^\infty A(T, |g|^p)\,d\mu(T)$. As $\|\mu\| = 1$, the measure μ must be positive. Moreover, $A(T, |g|^p) = 1$ on the support of μ.

Conversely, let us write $\ell(f) = \int_{K_p} f^\dagger d\nu = \langle\nu, f^\dagger\rangle$, where ν is the finitely additive Borel measure on K_p given by $\nu = \mu \times \delta_\phi$. This integral representation is uniquely associated to an integral representation over $\beta(K_p)$, of the form $\ell(f) = \int_{\beta(K_p)} f^\dagger d\widetilde{\nu}$. The measure $\widetilde{\nu}$ is unique because the map $f \to f^\dagger$ is onto $C(\beta(K_p))$; hence also ν is unique on $C \cap L^\infty(K_p)$, so, there is only one such integral representation for ℓ. By the isometric isomorphism of Corollary 28, ℓ attains its norm on M^p if and only if the functional given by $\nu = \mu \times \delta_\phi$ on $C \cap L^\infty([1,\infty] \times \phi)$ attains its norm. This means that ℓ attains its norm if and only if μ attains its norm on $C \cap L^\infty[1,\infty]$. By Lemma 22, a sufficient condition for this is $\mu \geqslant 0$. If we restrict attention to the space M^p over the reals (consisting of real valued functions), then this condition in Lemma 22 is also necessary. But the necessity holds in general also for complex spaces, as we have shown in the first half of the proof. That proof also shows that $A(T, |\phi|^p) = 1$ on the support of μ and $\ell(\phi^\natural) = 1$. □

Remark 30. The previous proof makes use of the Stone-Čech compactification of the cartesian product $[1,\infty) \times U(M^p)$. It is important to remember that, in general, the Stone-Čech compactification of a cartesian product does not coincide with the product of the compactification (for an example, see [20, Chapter 6, problem 6N2]), and so the points of the compactification have not be written as pairs, as is done, for the aim of brevity, in the original reference [7, Lemma 4.3].

We now apply the Yosida-Hewitt decomposition theorem for finitely additive measures [21]. We need the following definition.

Definition 31. A Borel measure μ on a topological space X is purely finitely additive (p.f.a.) if whenever ν is a nonnegative countably additive Borel measure on X bounded by μ, in the sense that $0 \leqslant \nu \leqslant |\mu|$, then $\nu = 0$.

The Yosida-Hewitt decomposition theorem [21, Theorem 1.24] states that, for every finitely additive Borel measure μ, there exists a unique pair of Borel measure μ_1, μ_2 with μ_1 countably additive and μ_2 purely finitely additive, such that $\mu = \mu_1 + \mu_2$. If μ is nonnegative, then so are μ_1 and μ_2. As a consequence one has the following.

Lemma 32. *If μ is a purely finitely additive positive measure vanishing on sets of Lebesgue measure zero, then $\mu(f) = 0$ for every measurable function f that vanishes at infinity.*

Proof. Since the finitely additive measures vanishing on sets of Lebesgue measure zero are the Banach dual of L^∞, then the restriction of such a measure μ to the space C_0 of continuous functions vanishing at infinity yields a continuous functional on C_0, hence a countably additive measure. This restriction is dominated by μ, and so it must be zero if μ is purely finitely additive. □

Corollary 33. *For $1 < p < \infty$, let ℓ be a norm attaining functional on $\mathcal{J}^p$. Then, for some $\phi \in U(M^q)$ and for some positive countably additive measure μ on $[1,\infty]$ with $\|\mu\| = \|\ell\|$, one has*

$$\ell(f) = \int_1^\infty f^\dagger(T,\phi)\, d\mu(T) \qquad (107)$$

for every $f \in \mathcal{J}^p$.

Proof. Let $\widetilde{\ell}$ be a norm-preserving Hahn-Banach extension of ℓ to M^p. By Theorem 29, we know that

$$\widetilde{\ell}(f) = \int_1^\infty f^\dagger(T,\phi)\, d\mu(T) \qquad (108)$$

for some finitely additive measure μ and $\phi \in U(M^q)$ defined as in Proposition 25. Now the previous Lemma states that the purely finitely additive measure μ_2 in the Yosida-Hewitt decomposition $\mu = \mu_1 + \mu_2$ satisfies the identity

$$\int_1^\infty f^\dagger(T,\phi)\, d\mu_2(T) = 0 \qquad (109)$$

for every $f \in \mathcal{J}^p$, because $\lim_{T\to\infty} f^\dagger(T,\phi) = 0$ by definition of null space.

Let G_ϕ be a function on which ℓ attains its norm. Then, by (92), $\ell(G_\phi) = \|G_\phi\| = 1$. Therefore $\int_1^\infty f^\dagger(T,\phi)\, d\mu(T) = \ell(G_\phi) = 1$. Moreover, by (93), the integrand is bounded by 1. Since $\|\mu_1\| \leqslant 1$, the measure μ_1 must be positive and of norm 1. □

We now extend this result by proving that, on $\mathcal{J}^p$, the condition that the functional attains its norm is not needed.

Theorem 34. *For $1 < p < \infty$, all continuous linear functionals on $\mathcal{J}^p$ (attaining their norm or not) can be represented as in* (79).

Proof. By Proposition 15(iii), $M^p = (E^q)'$ for $1 < p < \infty$, and part (v) of the same Proposition states that $E^q = (\mathcal{J}^p)'$. Then, by Lemma 22(i) (with $V = \mathcal{J}^p$ and $V' = M^p = (E^q)'$) every element of $E^q = (\mathcal{J}^p)'$ attains its norm as a functional on $(E^q)' = M^p$. Now the statement follows from Corollary 33. □

Our next goal is to provide a similar integral representation for the dual of the Marcinkiewicz Banach quotient $\widetilde{\mathcal{M}}^p = \mathcal{M}^p/\mathcal{J}^p$, $1 < p < \infty$. For this we need to remind and extend some previous results.

Proposition 35. *Let $1 < p < \infty$.*

(i) M^p is the bidual $(\mathcal{J}^p)''$.

(ii) Regard the predual $(\mathcal{J}^p)'$ of M^p as a subspace of $(M^p)'$; then

$$(M^p)' = (\mathcal{J}^p)' \oplus \mathcal{J}^{p\perp}. \qquad (110)$$

(iii) For every $\ell_1 \in (\mathcal{J}^p)'$ and $\ell_2 \in \mathcal{J}^{p\perp}$, $\|\ell_1 + \ell_2\| = \|\ell_1\| + \|\ell_2\|$.

(iv) For every coset $\overline{f} \in M^p/\mathcal{J}^p$ there exists $f \in M^p$ such that $\|f\|_{M^p} = \|\overline{f}\|_{M^p/\mathcal{J}^p} = \min\{\|f+g\|_{M^p} : g \in \mathcal{J}^p\}$.

Proof. Part (i) follows obviously from (iii) and (v) of Proposition 15, and part (ii) is a direct consequence. Without loss of generality, we prove part (iii) in the case where $\|\ell_1\| = \|\ell_2\| = 1$. For every $\varepsilon > 0$ there exist $f_1 \in \mathcal{J}^p$, $f_2 \in M^p$ with

$$\|f_i\| = 1, \qquad \ell_i(f_i) > 1 - \varepsilon\,. \qquad (111)$$

The fact that $\|f_i\|_{M^p} = 1$ amounts to say that

$$\sup_{T\geqslant 1} A\left(T, |f_i|^p\right) = 1. \qquad (112)$$

On the other hand, since $\ell_1 \in (\mathcal{J}^p)'$, the representation (79) holds.

Since $f_1 \in \mathcal{J}^p$, we can approximate it (and therefore replace it) by a compactly supported function, which, by abuse of notation, we denote again by f_1. More precisely, we choose f_1 supported in $[-a, a]$ where $a > 0$ is so large that

(I) if μ denotes the representing finitely additive measure in (79), then $\mu(a, \infty) < \varepsilon$;

(II) $A(T, |f_1|^p) < \varepsilon^p$ for every $T > a$ (this is possible because $f_1 \in \mathcal{J}^p$).

Let C be the complement $\mathbb{R} \setminus [-a, a]$. It follows from (II) that

$$\|f_2\chi_C\| = \sup_{T\geqslant 1} A\left(T, |f_2|^p \chi_C\right)^{1/p} = \sup_{T\geqslant a} A\left(T, |f_2|^p \chi_C\right)^{1/p} \geqslant \varepsilon. \tag{113}$$

Since f_1 has norm 1, the function $f = f_1 + f_2\chi_C$ satisfies the inequality

$$\|f\| \geqslant 1 + \varepsilon. \tag{114}$$

On the other hand, by (79) and Hölder's inequality, as in (93),

$$\begin{aligned} |\ell_1(f_2\chi_C)| &\leqslant \int_1^\infty |A(T, f_2\chi_C\phi)|\, d\mu(T) \\ &= \int_a^\infty |A(T, f_2\phi)|\, d\mu(T) \\ &\leqslant \int_a^\infty A\left(T, |\phi|^q\right)^{1/q} d\mu(T) \cdot \|f_2\|_{M^p}. \end{aligned} \tag{115}$$

But $A(T, |\phi|^q)^{1/q} \geqslant 1$ by (84), and $\|f_2\| = 1$. Hence, by (I), $|\ell_1(f_2\chi_C)| \geqslant \varepsilon$. Now observe that $\ell_2(f_1) = 0$ as $f_1 \in \mathcal{J}^p$ and $\ell_2 \in \mathcal{J}^{p\perp}$. Moreover, $\ell_2(f_2\chi_C) = \ell_2(f_2) - \ell_2(f_2\chi_{[-a,a]}) = \ell_2(f_2)$ because the compactly supported function $f_2\chi_{[-a,a]}$ belongs to $\mathcal{J}^p$ and so it is in the kernel of $\ell_2 \in \mathcal{J}^{p\perp}$. By all this and (111) we have

$$\begin{aligned} (\ell_1 + \ell_2)(f) &\geqslant \ell_1(f_1) - \varepsilon + \ell_2(f_2\chi_C) \\ &= \ell_1(f_1) + \ell_2(f_2) - \varepsilon \geqslant 2 - 3\varepsilon. \end{aligned} \tag{116}$$

It now follows from (114) and (116) that

$$\|\ell_1 + \ell_2\| \geqslant \inf_{\varepsilon > 0} \frac{2 - 3\varepsilon}{1 + \varepsilon} = 2. \tag{117}$$

This proves part (iii). Part (iv) is a bit more technical. Indeed, in the terminology of [22], the statement of part (ii) says that $\mathcal{J}^p$ is an M-ideal in M^p, and then [22, Corollary 5.6] shows that for every $f \in M^p$ there is some $g \in \mathcal{J}^p$ such that

$$\|f - g\|_{M^p} = \inf\{\|f - h\|_{M^p} : h \in \mathcal{J}^p\} = \|f + \mathcal{J}^p\|_{M^p/\mathcal{J}^p}. \tag{118}$$

This proves that any coset F in the quotient $M^p/\mathcal{J}^p$ has a representative f such that $\|F\|_{M^p/\mathcal{J}^p} = \|f\|_{M^p}$, hence (iv). □

5.4. The Dual and Predual of $\mathcal{M}^p$: Integral Representation of Norm-Attaining Continuous Functionals. Computing the predual of $\widetilde{\mathcal{M}}^p$ is now an easy job; by Proposition 15 and Proposition 12 it is clear that the predual of $\widetilde{\mathcal{M}}^p = \mathcal{M}^p/\mathcal{J}^p = M^p/\mathcal{J}^p$ is exactly the annihilator of $\mathcal{J}^p$ in E^p (the predual of M^p, considered here as a subspace of the dual of M^p).

It is slightly more difficult to extend to integral representation theorem for norm-attaining functionals to the Banach quotient $\widetilde{\mathcal{M}}^p$. We begin by remarking the following immediate consequence of Propositions 12 and 35(ii).

Corollary 36. *Let $1 < p < \infty$, and denote as before by $\mathcal{J}^{p\perp}$ the annihilator of $\mathcal{J}^p$ in $(M^p)'$. Then the following hold.*

(i) The dual of $\widetilde{\mathcal{M}}^p$ is isometrically isomorphic to $\mathcal{J}^{p\perp}$.

(ii) The norm of a functional $\ell \in \widetilde{\mathcal{M}}^p \approx M^p/\mathcal{J}^p$ is the same as the norm of the lifting of ℓ to a functional on M^p. In particular, ℓ attains its norm on $\widetilde{\mathcal{M}}^p$ if and only if its lifting attains its norm on M^p.

Let us now begin to assemble the ingredients of our integral representation theorem.

Definition 37. A finitely additive measure μ on a measure space X is supported at infinity if $\mu(E) = 0$ for every bounded measurable set $E \subset X$.

Remark 38. It follows from Lemma 32 that a purely finitely additive measure is supported at infinity.

Corollary 39. *If $\overline{f} \in \widetilde{\mathcal{M}}^p$ and $\overline{\phi} \in \widetilde{\mathcal{M}}^q$, then the limit of $A(T, f\phi)$ as $T \to \infty$ does not depend on the representatives of f and ϕ in the respective $\mathcal{J}^p$ cosets.*

Proof. This is an easy consequence of Hölder's inequality (93). □

Now we can state the representation theorem for the dual of $\widetilde{\mathcal{M}}^p$. Its proof here is considerably simpler than in the original reference [7, Theorem 5.2].

Theorem 40. *Let $1 < p < \infty$ and denote by q its conjugate index. Let ℓ be a norm-attaining functional on $\widetilde{\mathcal{M}}^p$. Then there exist $\phi \in \widetilde{\mathcal{M}}^q$ and a positive finitely additive measure μ on $[1, \infty)$ supported at infinity such that, for all $f \in \widetilde{\mathcal{M}}^p$,*

$$\ell(f) = \int_1^\infty A(T, f\phi)\, d\mu(T) \tag{119}$$

(observe that, although the integrand involves functions instead of $\mathcal{J}^p$ cosets, the statement is well posed because ℓ vanishes on $\mathcal{J}^p$ by Corollary 36 and for large T the integrand does not depend on the choice of coset representatives by Corollary 39).

Proof. By the previous Corollary, ℓ is identified with a continuous functional on M^p vanishing on $\mathcal{J}^p$ and attaining its norm. Then Theorem 29 yields the following integral representation:

$$\ell(f) = \int_1^\infty A(T, f\phi)\, d\mu(T) \tag{120}$$

for all $f \in M^p$ (with this realization, all functions belong to M^p or M^q and we do not need to pay attention to equivalence classes mod $\mathcal{I}^p$, in accordance with the remark at the end of the statement).

We only need to prove that the representing measure μ is supported at infinity. Recall from the proof of Theorem 29 that, in this integral representation, the function $\phi \equiv g^\natural \in M^q$ is built as in Definition 23 in terms of the function $g \in M^p$ where ℓ attains its norm and has the property that $g\phi = |g|^p \geqslant 0$; of course g is not the zero function except in the trivial case $\ell = 0$.

Suppose that, for some $a > 0$, the interval $[-a, a]$ has positive μ-measure, and consider the truncation $g_a = g\chi_{[-a,a]}$. Take a large enough so that g_a is not identically zero (this is possible since g is not identically zero; we are disregarding the trivial case $\ell = 0$). Then $g_a \in \mathcal{I}^p$ and $A(T, g_a\phi) = A(T, |g_a|^p)$ is a positive function on $-a \leqslant T \leqslant a$. Therefore $\ell(g_a) = \int_1^\infty A(T, g_a\phi)\, d\mu(T) > 0$. This contradicts the assumption that ℓ vanishes on $\mathcal{I}^p$. □

These theorems on the integral representation of almost every continuous functionals shed light upon the examples of functionals on M^p and $\mathcal{M}^p$ supported at infinity, which we built in Section 5.1. Those examples are all functionals of the type $\ell(f) = \int_1^\infty A(T, f\phi)\, d\mu(T)$ where the representing measure is supported at infinity. In other words, here μ is a finitely additive positive finite measure given by the restriction of a countably additive positive finite Borel measure supported in $\beta(K_p) \setminus K_p$, where K_p is the product space introduced in Definition 26 and $\beta(K_p)$ is its Stone-Čech compactification. As observed at the beginning of Section 5.1, all continuous functionals on $\mathcal{M}^p$ vanish on the null space of the seminorm, hence they must depend only on asymptotic values, and so, if they have an integral representation of the type $\ell(f) = \int_1^\infty A(T, f\phi)\, d\mu(T)$, the measure μ must be supported at infinity. Instead, functionals on M^p can be represented by measures that have a part at finite (necessarily countably additive, by the Yosida-Hewitt representation theorem and Remark 38). In the next section we extend this analysis to Stepanoff spaces.

5.5. Correlation Functionals. By the representation theorems proved in this section we know that $\widetilde{\mathcal{M}}^q$ is not the dual space of $\widetilde{\mathcal{M}}^p$. However, the following construction of functionals, called *correlation functionals* in [11], shows that at least it is possible to embed $\widetilde{\mathcal{M}}^q$ as a quotient space of the dual of $\widetilde{\mathcal{M}}^p$.

Let V be a separable linear subspace of $\mathcal{M}^p$ (one such subspace is $\mathcal{I}^p$, by Remark 17). Let $g \in \mathcal{M}^q$ and consider a sequence $\{f_n\}$ dense in V. By Hölder's inequality in $L^p([-T, T], dx/2T)$, one has

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, g(t)\, dt \leqslant \|f_1\|_{\mathcal{M}^p} \|g\|_{\mathcal{M}^q}. \tag{121}$$

Therefore there is an increasing unbounded sequence $\{T_{1,k}\}$ such that the limit

$$\langle g, f_1\rangle_{\{T_{1,k}\}} := \lim_{k\to\infty} \frac{1}{2T_{1,k}} \int_{-T_{1,k}}^{T_{1,k}} f_1(t)\, g(t)\, dt \tag{122}$$

exists, and $|\langle g, f_1\rangle_{\{T_{1,k}\}}| \leqslant \|f_1\|_{\mathcal{M}^p} \|g\|_{\mathcal{M}^q}$.

Let us now extract from $\{T_{1,k}\}$ a subsequence $\{T_{2,k}\}$ such that the limit

$$\langle g, f_2\rangle_{\{T_{2,k}\}} := \lim_{k\to\infty} \frac{1}{2T_{2,k}} \int_{-T_{2,k}}^{T_{2,k}} f_2(t)\, g(t)\, dt \tag{123}$$

exists. Here again, one has $|\langle g, f_2\rangle_{\{T_{2,k}\}}| \leqslant \|f_2\|_{\mathcal{M}^p} \|g\|_{\mathcal{M}^q}$.

We iterate this process to build a family of nested subsequences $\{T_{j,k}\}$ that define limits $\langle g, f_j\rangle_{\{T_{j,k}\}}$ satisfying the Hölder inequality above for the Marcinkiewicz seminorms. Then the diagonal sequence $\{T_k := T_{k,k}\}$ gives rise to a limit

$$\left\langle g, f_j\right\rangle := \lim_{k\to\infty} \frac{1}{2T_k} \int_{-T_k}^{T_k} f_j(t)\, g(t)\, dt \tag{124}$$

that exists for every j and satisfies $|\langle g, f_j\rangle| \leqslant \|f_j\|_{\mathcal{M}^p} \|g\|_{\mathcal{M}^q}$. Therefore $\langle g, \cdot\rangle$ is a continuous functional on the subspace of $\mathcal{M}^p$ generated by the sequence $\{f_n\}$, hence on V by density. Clearly this functional vanishes on the null space $\mathcal{I}^p$ of the semi-norm; hence it defines a continuous functional on $V/\mathcal{I}^p \subset \widetilde{\mathcal{M}}^p$. By Hahn-Banach extension, we produce in this way a continuous functional of $\widetilde{\mathcal{M}}^p$ that depends only on limits of means, even when these limits are not defined directly by integration.

5.6. Summary and Open Problems. We have proved representation theorems for continuous functionals on M^p and $\mathcal{M}^p$ $(1 < p < \infty)$ as integrals with respect to measures ν over $\beta(K_p)$. For the norm-attaining functionals, the representing measure splits as the product of a finitely additive positive measure μ on $[1, \infty)$ times a delta measure δ_ϕ on the unit ball $B(M^q)$, and so the representation becomes more specific: a μ-average over $T \in [1, \infty)$ of ϕ-weighted means over intervals of length $2T$.

In particular, the convex hull of these functionals contains those whose measure ν, regarded as a finitely additive measure on K_p, splits as a product.

Is there a characterization of those functionals arising from measures μ that are not countably additive and are supported at infinity, as for instance the Banach limits?

We have seen that the same problem is not interesting in the case $p = \infty$, since the representation of continuous functionals on $\mathcal{M}^\infty = L^\infty$ as finitely additive measures is already known. But can we prove interesting representation theorems for continuous functionals on $\mathcal{M}^1$?

6. Duality for Stepanoff Spaces

6.1. The Predual of $\mathcal{S}^p$. Before considering $(\mathcal{S}^p)'$, we need to examine the Banach space structure of $\mathcal{S}^p$.

Observe that $L^p(\mathbb{R})$ embeds continuously in $\mathcal{S}^p$. Indeed, if $f_n = f\chi_{[n,n+1]}$,

$$\|f\|_{\mathcal{S}^p} = \sup_n \|f_n\|_p \leqslant \left(\sum_n \|f_n\|_p^p\right)^{1/p} = \|f\|_{L^p(\mathbb{R})}. \tag{125}$$

Similarly, consider the product $\mathcal{T}^q \equiv \otimes_{\ell^1} L^q[n, n+1]$, that is, the space of all $g = \sum_n c_n g_n$, where $\{c_n\} \in \ell^1$ and $g_n \in L^q[n, n+1]$ with $1/p + 1/q = 1$ and $\|g_n\|_q = 1$. $\mathcal{T}^q$ is a Banach space with respect to the norm given by the infimum of $\sum_n |c_n|$ over all such representations. By the same argument, $\mathcal{T}^q$ embeds continuously in $L^q(\mathbb{R})$. Observe that both inequalities, hence both embeddings, are proper except for the cases $\mathcal{S}_\infty = L^\infty(\mathbb{R})$ and $\mathcal{T}_1 = L^1(\mathbb{R})$.

It is clear that the dual Banach space of $\mathcal{S}^p$ contains $\mathcal{T}^q$. Indeed, the functions $g = \sum_n c_n g_n$, with $\{c_n\} \in \ell^1$ and $g_n \in L^q[n, n+1]$, define continuous functionals on $\mathcal{S}^p$ with norm $\|g\|_{(\mathcal{S}^p)'} = \sum_n |c_n|$, since, for every $f \in \mathcal{S}^p$,

$$\left|\int_{-\infty}^{\infty} g(t) f(t)\, dt\right| \leqslant \sum_n |c_n| \left\|f\chi_{[n,n+1]}\right\|_p \leqslant \sum_n |c_n| \|f\|_{\mathcal{S}^p} \tag{126}$$

by Lemma 8. More generally, every $g \in L^q(\mathbb{R})$ with compact support defines a continuous functional on $\mathcal{S}^p$ by the rule $F_g(f) = \int_{-\infty}^{\infty} g(t) f(t)\, dt$. Indeed, if $\operatorname{supp} f \subset [-m, m]$, then

$$\begin{aligned} \left|F_g(f)\right| &\leqslant \sum_{n=-m}^{m-1} \int_n^{n+1} \left|g(t) f(t)\right| dt \\ &\leqslant 2m\|g\|_q \max_{-m\leqslant n\leqslant m-1} \|f\|_{L^p[n,n+1]} \\ &\leqslant 2m\|g\|_q \|f\|_{\mathcal{S}^p}. \end{aligned} \tag{127}$$

It is clear that the restriction to $[n, n+1]$ of every function g such that the functional F_g is defined on $\mathcal{S}^p$ must belong to $L^q[n, n+1]$. In general, however, L^q functions whose support is not compact do not yield continuous functionals on $\mathcal{S}^p$, unless they belong to $\mathcal{T}^q$ (we have already observed that $\mathcal{T}^q$ is properly contained in $L^q(\mathbb{R})$, except for $q = 1$). For instance, choose a non-negative real sequence $\{a_n\} \in \ell^q \setminus \ell^1$ and let $g = \sum_{n=-\infty}^{\infty} a_n \chi_{[n,n+1]}$. It is clear that $\|g\|_q = \|\{a_n\}\|_{\ell^q} < \infty$ but $\sum_{n=-\infty}^{\infty} \|g\|_{L^q[n,n+1]} = \|\{a_n\}\|_{\ell^1} = \infty$. Moreover, for every n choose $f_n \in L^p[n, n+1]$ such that $\int_n^{n+1} g(t) f_n(t)\, dt = \|g\|_{L^q[n,n+1]}$. For simplicity, let us consider first the case $q = 1$. Then $|f_n(x)| = 1$ almost everywhere, and the function f that coincides with f_n on every interval $[n, n+1)$ has norm 1 in $\mathcal{S}^p$, but $F_g(f) = \sum_{n=-\infty}^{\infty} \|g\|_{L^1[n,n+1]} = \infty$. In general, if $q \geqslant 1$, then $|f_n(x)| = |g|^{q-1}$ almost everywhere, $\|f_n\|_{L^q[n,n+1]} = |a_n|^{q-1}$ and the function f built as above by glueing together the consecutive f_n's has finite $\mathcal{S}^p$-norm given by $\max_n |a_n|^{q-1}$, but, as before, $F_g(f) = \sum_{n=-\infty}^{\infty} \|g\|_{L^q[n,n+1]} = \infty$.

We include these remarks in the next statement.

Theorem 41. *$\mathcal{S}^p$ is the dual space of the Banach space*

$$\mathcal{T}^q \equiv \bigotimes_{\ell^1} L^q[n, n+1]. \tag{128}$$

A continuous functional F on $\mathcal{S}^p$ is represented by a function g in the sense that

$$F(f) \equiv F_g(f) = \int_{-\infty}^{\infty} g(t) f(t)\, dt \tag{129}$$

if and only if $g \in \mathcal{T}^q$. In this case, the following quasi-isometry holds up to a factor 2: $\|F_g\|_{\mathcal{S}^{p\prime}} \approx \sum_{n=-\infty}^{\infty} \|g\|_{L^q[n,n+1]} = \|g\|_{\mathcal{T}^q}$. In other words, the subspace of the dual of $\mathcal{S}^p$ of functionals that can be represented by a function is the bidual of $\mathcal{T}^q$.

Proof. Let F be a continuous functional on $\mathcal{T}^q$. On all functions $f \in \mathcal{T}^q$ with support in $[n, n+1]$, F is represented by a function h_n in L^p with support in $[n, n+1]$: $F(f) = \int h_n f$. Therefore, for all functions $f \in \mathcal{T}^q$ with support in $[-m, m]$, $F(f) = \int g_m f$ where $g_m = \sum_{n=-m}^{m-1} h_n$. Now observe that, by the way the norm in $\mathcal{T}^q$ is defined, compactly supported L^q functions are dense in $\mathcal{T}^q$. Therefore every continuous functional on $\mathcal{T}^q$ is represented by a locally L^p function g. Again by the way the norm is defined, it is clear that the norm of this functional F_g is given by $\|F_g\| = \sup_n \|g_n\|_q$; in other words, by Lemma 8, the norm of F_g is quasi-isometric with the norm of g in $\mathcal{S}^p$. Thus the dual of $\mathcal{T}^q$ is $\mathcal{S}^p$.

On the other hand, we have already observed that if a functional on $\mathcal{S}^p$ is represented by a function, then this function must belong to $\mathcal{T}^q$ and the correspondence is quasi-isometric. □

Since $\mathcal{S}^p$ embeds continuously in M^p, we know by Proposition 15 that the predual E^q of M^p embeds continuously in $\mathcal{T}^q$. The following is an independent simple proof of the embedding of E^q into $\mathcal{T}^q$.

Lemma 42. *One has $\|f\|_{\mathcal{T}^q} \leqslant \|f\|_{E^q}$. Conversely, there is no $C > 0$ such that $\|f\|_{E^q} \leqslant C\|f\|_{\mathcal{T}^q}$.*

Proof. Let $f \in E^q$ and, as before, $f_n = f\chi_{[n,n+1]}$. Let ψ_k be the characteristic functions of the dyadic intervals introduced in the definition of E^q. Hölder's inequality for ℓ^1 yields

$$\sum_{n=2^k}^{2^{k+1}-1} \|f_n\|_q \leqslant 2^{k/p} \left(\sum_{n=2^k}^{2^{k+1}-1} \|f_n\|_q^q\right)^{1/q} = 2^{k/p} \|f\psi_k\|_q. \tag{130}$$

Therefore

$$\begin{aligned} \|f\|_{E^q} &= \sum_{k=0}^{\infty} 2^{k/p} \|f\psi_k\|_q \geqslant \sum_{k=0}^{\infty} \sum_{|n|=2^k}^{2^{k+1}-1} \|f_n\|_q \\ &= \sum_{n=-\infty}^{\infty} \|f_n\|_q \geqslant \|f\|_{\mathcal{T}^q}. \end{aligned} \tag{131}$$

In these inequalities we have made use of Hölder's inequality, which is not an equality if $f \in \mathcal{T}^q$ because then the sequence $\{\|f_n\|_q\}$ cannot be constant. This yields the fact that the converse inequality does not hold. □

6.2. $\mathcal{S}^p$ as a Bidual. In analogy with the null space $\mathcal{F}^p$ of the Marcinkiewicz semi-norm, we now introduce a similar subspace in $\mathcal{S}^p$.

Definition 43. We shall write

$$\mathcal{J}^p \equiv \left\{ f \in \mathcal{S}^p : \lim_n \|f\|_{L^p[n,n+1]} = 0 \right\}. \quad (132)$$

Remark 44. Clearly, $\mathcal{J}^p \approx \otimes_{c_0} L^p[n,n+1]$, and $\mathcal{J}^p$ is a closed subspace of $\mathcal{S}^p$.

The next lemma is proved by the same argument of Theorem 41. As a consequence, $\mathcal{S}^p$ is the second dual of $\mathcal{J}^p$.

Lemma 45. *$\mathcal{T}^p \approx \otimes_{l^1} L^p[n,n+1]$ is (isometrically isomorphic to) the dual of $\mathcal{J}^p \approx \otimes_{c_0} L^p[n,n+1]$.*

Lemma 46. *$\mathcal{J}^p \subset \mathcal{F}^p$, and the inclusion is proper.*

Proof. The inclusion means that $\lim_N (1/2N)\int_{-N}^{N}|f| = 0$ provided that $\lim_n \int_n^{n+1}|f|^p = 0$. For the sake of simplicity, we first show that this is true for $p = 1$. Indeed, $(1/2N)\int_{-N}^{N}|f|$ is the average of $a_n \equiv \|f\|_{L^1[n,n+1]}$ as n ranges from $-N$ to $N-1$; then, as the sequence a_n goes to 0, so do its averages. In general, for $1 < p < \infty$, $(1/2N)\int_{-N}^{N}|f|^p = (1/2N)\sum_{n=-N}^{N-1}\|f\|^p_{L^p[n,n+1]}$. If we extract the pth root of both sides this equality becomes an inequality: $(1/2N^{1/p})(\int_{-N}^{N}|f|^p)^{1/p} \leqslant (1/2N^{1/p})\sum_{n=-N}^{N-1}\|f\|_{L^p[n,n+1]}$ (because the ℓ^p norm is dominated by the ℓ^1 norm). Therefore the previous argument still applies.

It is easy to show that the inclusion is proper: a function that belongs to $\mathcal{F}^p$ but not to $\mathcal{J}^p$ is $\sum_{n=0}^{\infty}\chi_{[2^n,2^n+1]}$. □

6.3. The Dual of $\mathcal{S}^p$. We now turn our attention to the dual of $\mathcal{S}^p$. As we did with $\mathcal{M}^p$, we first exhibit some examples of linear functionals that depend only on asymptotic values, and then we prove a representation theorem. For this goal, we introduce some interesting subspaces of $\mathcal{S}^p$, as follows.

Definition 47. (i) $J_p^{\pm}$ is the subspace of $\mathcal{S}^p$ of all functions that have limits at $\pm\infty$;

(ii) $I_p^{\pm}$ is the subspace of $\mathcal{S}^p$ of all functions f such that the sequence $\int_n^{n+1} f(t)\,dt$ has limits at $\pm\infty$;

(iii) $N_p^{\pm}$ is the subspace of $\mathcal{S}^p$ of all functions f such that the sequence $\|f\|_{L^p[n,n+1]}$ has limits at $\pm\infty$.

Remark 48. It is clear that J_p is contained in N_p and N_p is contained in I_p. The other inclusions are false. The function $\sum_{n>0} n^{1/p}\chi_{[n,n+1/n]}$ belongs to N_p but not to J_p. The function $\sum_{n=-\infty}^{\infty}(-1)^n\chi_{[n,n+1]}$ belongs to N_p but not to I_p. The function with values n in $[n,n+1/2]$ and $-n$ in $[n+1/2,n+1]$ belongs to I_p but not N_p. A variant of I_p and of N_p is obtained by integrating with respect to any sequence of finite Borel measure over $[n,n+1]$ instead of Lebesgue measure; the same inclusion properties hold for this variant.

Lemma 49. *$J_p^{\pm}$ is not closed in $\mathcal{S}^p$, and the functionals*

$$J_{\pm}(f) = \lim_{x\to\pm\infty} f(x), \quad (133)$$

defined on $J_p^{\pm}$, are not continuous in the norm of $\mathcal{S}^p$.

Proof. Let $E_n = [n, n+2^{-|n|}]$, χ_n the characteristic function of E_n and $f_j = \sum_{n=-j}^{j}\chi_n$. Each f_j is in $\mathcal{S}^p$ and has compact support, hence it belongs to $J_p^{\pm}$. It is immediately verified that the sequence f_j converges in $\mathcal{S}^p$ to $f = \sum_{n=-\infty}^{\infty}\chi_n$. This f does not have limits at infinity; hence it does not belong to $J_p^{\pm}$. For the same reason, the functional $J_p^{\pm}(f) = \lim_{x\to\pm\infty} f(x)$ is not continuous; $J_p^{\pm}(f_j) = 0$ for every j but $J_p^{\pm}(f) \neq 0$. □

The same argument yields the following.

Corollary 50. *The spaces $J_p^{\pm}$ of functions vanishing at infinity are not closed in $\mathcal{M}^p$, and the functionals*

$$J_{\pm}(f) = \lim_{x\to\pm\infty} f(x) \quad (134)$$

are not continuous in the semi-norm of $\mathcal{M}^p$.

The previous lemma shows that the spaces $J_p^{\pm}$ do not yield natural linear functionals that extend continuously to $\mathcal{S}^p$. On the other hand, the spaces $I_p^{\pm}$ allow to construct continuous functionals on $\mathcal{S}^p$ which do not depend on values over finite intervals, that is, that do not belong to $\otimes_{\ell^1} L^q[n,n+1]$. This can be done as follows. Given a subspace W of a Banach space V and a continuous functional F on W (continuous in the V-norm), denote by $\widetilde{F}$ its (many) Hahn-Banach extensions to V. For instance, the Hahn-Banach extensions to ℓ^∞ of the continuous functional $F(\{x_n\}) = \lim_n x_n$, defined on the subspace of convergent sequences, are usually called *Banach limits*. In the same way we can now define on $\mathcal{S}^p$ some Banach limits induced by the subspace I_p. The following result is now clear.

Corollary 51. *The functionals $F_{\pm}(f) = \lim_{n\to\pm\infty}\int_n^{n+1} f(t)dt$, defined on the subspaces $I_p^{\pm} \subset \mathcal{S}^p$, are continuous in the norm of $\mathcal{S}^p$. Their Hahn-Banach extensions to $\mathcal{S}^p$ are continuous functionals not in $\otimes_{\ell^1} L^q[n,n+1]$. More generally, other functionals with this property are the Hahn-Banach extensions of $F_{\pm}(f) = \lim_{n\to\pm\infty}\int_n^{n+1} f(t)\,d\mu_n(t)$, where μ_n is a finite Borel measure on $[n,n+1]$.*

Proof. The only thing left to prove is the continuity of $F_{\pm}$ in the $\mathcal{S}^p$-norm, which is obvious since

$$\left|\int_n^{n+1} f(t)\,dt\right| \leqslant \|f\|_{L^1[n,n+1]} \leqslant \|f\|_{L^p[n,n+1]} \leqslant \|f\|_{\mathcal{S}^p}. \quad (135)$$

It is obvious that $F_{\pm}(f)$ does not depend on values of f on compact sets; hence it cannot be expressed as an integral of the type $\int_{-\infty}^{\infty} g(t)f(t)\,dt$. □

Remark 52. Since $\mathcal{S}^p$ embeds continuously in M^p and M^p embeds continuously in $\mathcal{M}^p$, the limit functionals on $\mathcal{M}^p$ described in (70) are also continuous functionals on $\mathcal{S}^p$ and M^p.

6.4. A Summary of Duality and Inclusions. Let us summarize the inclusions between these families of spaces and their duals. We have shown that

$$\mathcal{S}^p \hookrightarrow M^p = (E^q)' \hookrightarrow \mathcal{M}^p, \qquad E^q \hookrightarrow \mathcal{T}^q. \tag{136}$$

These embeddings are proper. By the usual embedding of topological vector spaces into their biduals:

$$E^q \hookrightarrow (M^p)' \hookrightarrow (\mathcal{S}^p)'. \tag{137}$$

For the same reason, since $(\mathcal{T}^q)' = \mathcal{S}^p$,

$$E^q \hookrightarrow \mathcal{T}^q \hookrightarrow (\mathcal{S}^p)'. \tag{138}$$

The last embedding yields the part of the dual of $\mathcal{S}^p$ consisting of functionals represented by functions. The embedding $(M^p)' \hookrightarrow (\mathcal{S}^p)'$ encompasses the previous construction of Banach limit functionals depending only on asymptotic values. It is intriguing to exhibit explicit examples of continuous functionals on $\mathcal{S}^p$ that are not continuous on M^p. For instance, an interesting subspace of $(M^p)'$ is the bidual of its predual $\mathcal{I}^p$; not all these functionals are represented by functions (as functionals on M^p), because most functions in $\mathcal{I}^p$ are not small at infinity and do not belong to E^q. For a similar reason, they are not represented by functions when they act on $\mathcal{S}^p$. So here we have other exotic functionals on $\mathcal{S}^p$; we leave to the reader to verify that they are different from the Banach limits considered before.

6.5. Integral Representation of Continuous Functionals on $\mathcal{S}^p$. We now extend to the Stepanoff spaces the integral representation theorem for continuous functionals attaining their norms that we proved in Theorem 29 for M^p and in Theorem 40 for $\widetilde{\mathcal{M}}^p$. The proof is similar; we resume it skipping many details.

Definition 53. For $f \in \mathcal{S}^p(\mathbb{R})$ and $T \in \mathbb{R}$, put

$$I(T, f) = \int_T^{T+1} |f(x)|\, dx. \tag{139}$$

The next statement follows immediately from Lemma 24.

Corollary 54. *If $1 < p, q < \infty$ are conjugate indices and $g \in \mathcal{S}^p(\mathbb{R})$, then for all $T \in \mathbb{R}$ the auxiliary function $g^\natural$ introduced in Definition 23 satisfies $I(T, |g^\natural|^q) = I(T, |g|^p)$; hence $g^\natural \in \mathcal{S}^q$ and $\|g^\natural\|_{\mathcal{S}^q}^q = \|g\|_{\mathcal{S}^p}^p$.*

By making use of Corollary 54 we prove the next result in the same way as Proposition 25.

Proposition 55. *Let p, q be conjugate indices, with $1 < p, q < \infty$. Let μ be a σ-additive finite Borel measure on $[1, \infty]$ and $\phi \in \mathcal{S}^q(\mathbb{R})$. If F is the functional on $\mathcal{S}^p$ given by*

$$F(h) = \int_{-\infty}^{\infty} I(T, h\phi)\, d\mu(T), \tag{140}$$

then

$$\|F\| \leqslant \int_{-\infty}^{\infty} I\left(T, |\phi|^q\right)^{1/q} d\mu(T), \qquad \int_{-\infty}^{\infty} I\left(T, |\phi|^q\right) d\mu(T) \leqslant \|F\|\, \|\phi\|_{M^q}^{q-1}. \tag{141}$$

Definition 56. Denote by $U(\mathcal{S}^p)$ the unit sphere, that is, the subset of $\mathcal{S}^p$ of all functions of norm 1, by C_p the cartesian product $[1, \infty] \times U(\mathcal{S}^p)$, and by $\beta(C_p)$ its Stone-Čech compactification.

Lemma 57. *For $f \in \mathcal{S}^p$, let us define a function $f^\ddagger$ on C_p by*

$$f^\ddagger(T, \phi) = I(T, f\phi). \tag{142}$$

Then $f^\ddagger$ is an isometric isomorphism from $\mathcal{S}^p$ to $C \cap L^\infty(C_p)$ and therefore also from $\mathcal{S}^p$ to $C(\beta(C_p))$.

Proof. By Lemma 24 the function $\phi_0 = f^\natural / \|f\|_{\mathcal{S}^p}^{p-1}$ satisfies

$$\|\phi_0\|_{\mathcal{S}^q} = 1, \tag{143}$$

because $p - 1 = p/q$. Therefore

$$\left\|f^\ddagger\right\|_\infty \geqslant \sup_{T \geqslant 1} I(T, f\phi_0) = \frac{\sup_{T \geqslant 1} I\left(T, |f|^p\right)}{\|f\|_{\mathcal{S}^p}^{p-1}} = \|f\|_{\mathcal{S}^p}. \tag{144}$$

For the opposite inequality, we make use again of Hölder's inequality, this time in the following form: for every $(T, \phi) \in C_p$ one has

$$\left|f^\ddagger(T, \phi)\right| = |I(T, f\phi)| \leqslant I\left(T, |f|^p\right)^{1/p} I\left(T, |\phi|^q\right)^{1/q} = I_p(T, f)\, I_q(T, \phi). \tag{145}$$

This and (143) imply that

$$\sup_T \left|f^\ddagger(T, \phi)\right| \leqslant \|\phi\|_{\mathcal{S}^q} \|f\|_{\mathcal{S}^p} = \|f\|_{\mathcal{S}^p}. \tag{146}$$

Hence

$$\left\|f^\ddagger\right\|_\infty \leqslant \|f\|_{\mathcal{S}^p}. \tag{147}$$

The rest of the proof is as in Lemma 27. □

Theorem 58. *Let p, q be conjugate indices, with $1 < p, q < \infty$, and ℓ a continuous functional on $\mathcal{S}^p$ that attains its norm. Then, for some ϕ in the unit sphere $U(\mathcal{S}^q)$ (notation as in Definition 56) and for some finitely additive measure μ on $[1, \infty]$, one has*

$$\ell(f) = \int_1^{\infty} f^\ddagger(T, \phi)\, d\mu(T) \tag{148}$$

for every $f \in \mathcal{S}^p$.

Proof. We follow the guidelines of the proof of the same result for M^p in Theorem 29. Again, we can assume $\|\ell\| = 1$, and, by Lemma 57, we know that, for some $\nu \in M(\beta(C_p))$ with $\|\nu\| = 1$ and for all $f \in \mathcal{S}^p$,

$$\ell(f) = \int_{\beta(C_p)} f^{\ddagger} d\nu. \tag{149}$$

Let $g \in \mathcal{S}^p$ be a function on which ℓ attains its norm: $\ell(g) = 1$. We have seen in (92) that $\|g^{\natural}\|_{M^p} = 1$.

Denote by W the subset of C_p where $|g^{\dagger}(T,\phi)|$ attains its maximum value 1. Consider the family Φ of all nets (T_α, ϕ_α) in C_p that converge to points of W.

As g and ϕ have norm 1, as in the proof of Theorem 29 it follows by Hölder's inequality (145) that if $\{(T_\alpha, \phi_\alpha)\} \in \Phi$, then the interval $[-T_\alpha, T_\alpha]$ must verify the condition

$$\lim_\alpha \left| I\left(T_\alpha, |g|^p\right) \right|^{1/p} = 1 = \|g\|_{\mathcal{S}^p} \tag{150}$$

and that ν must be supported in W.

The following facts are obtained as in the proof of Theorem 29.

(i) For every $f \in \mathcal{S}^p$, $z \in W$ and for every net $\{(T_\alpha, \phi_\alpha)\}$ in Φ that converges to z, one has $f^{\ddagger}(z) = \lim_\alpha f^{\ddagger}(T_\alpha, g^{\natural})$, and so

$$\left\| f^{\ddagger} \right\|_\infty \leqslant \sup_T \left| f^{\ddagger}\left(T, g^{\natural}\right) \right| \tag{151}$$

(this is now a consequence of the fact that $I(T_\alpha, gg^{\natural}) = \int_{T_\alpha}^{T_\alpha+1} |g|^p$ tends to $\|g\|_{\mathcal{S}^p}^p = 1$ by (150)).

(ii) Denote now by $\tilde{\ell}_\alpha$ the continuous functional on $L^q[T_\alpha, T_\alpha + 1]$ given by

$$\tilde{\ell}_\alpha(f) = I(T_\alpha, gf) = \int_{T_\alpha}^{T_\alpha+1} gf\, dx. \tag{152}$$

Then, by (150),

$$\left\| \tilde{\ell}_\alpha \right\| = \|g\|_{L^p[T_\alpha, T_\alpha+1]} = \left(\int_{T_\alpha}^{T_\alpha+1} |g|^p dx \right)^{1/p} \longrightarrow 1 = \|g\|_{\mathcal{S}^p}. \tag{153}$$

(iii) The functional $\tilde{\ell}_\alpha$ attains its norm at $g^{\natural}/\|g^{\natural}\|_{L^q[T_\alpha, T_\alpha+1]}$.

(iv) Also $\lim_\alpha |I(T_\alpha, g\phi_\alpha)| = 1 = \lim_\alpha |I(T_\alpha, gg^{\natural})| = \lim_\alpha \|g\|_{L^p[T_\alpha, T_\alpha+1]}^p$.

(v) Moreover $\limsup_\alpha I(T_\alpha, \phi_\alpha - g^{\natural}) = 0$. This follows as in the proof of Claim 3 in Theorem 29, by using now the uniform convexity of the spaces $L^q[T_\alpha, T_\alpha + 1]$.

By applying again Hölder's inequality (145) to the identity that we have just proved in point (v) above, we finally obtain $\lim_\alpha I(T_\alpha, f\phi_\alpha) = \lim_\alpha I(T_\alpha, fg^{\natural})$ for every $f \in \mathcal{S}^p$. Hence, for every $p \in W$ and every net converging to p,

$$\tilde{f}(p) = \lim_\alpha I\left(T_\alpha, fg^{\natural}\right). \tag{154}$$

Now, by (90) and (151) and the fact that ν has support in W,

$$|\ell(f)| \leqslant \left| \int_{\beta(C_p)} \left|\tilde{f}\right| d\nu \right| = \left| \int_W \left|\tilde{f}\right| d\nu \right| \leqslant \sup_{T \geqslant 1} \left| \tilde{f}\left(T, g^{\natural}\right) \right|. \tag{155}$$

The rest of the proof is as in Theorem 29. □

7. Extreme Points in the Unit Balls

Compact convex sets K in many Banach spaces (and more generally, in topological vector spaces) have plenty of extreme points. Indeed, the celebrated Krein-Milman theorem states that if the dual space separates points, then K is the closed convex hull of its extreme points. In particular, this is what happens for the unit ball of L^p spaces when $p > 1$ (including the case $p = \infty$, which is compact in the weak* topology by another well-known fact, the Banach-Alaoglu theorem). Therefore the Krein-Milman theorem applies to the unit ball of a normed (or semi-normed) space if the linear functionals that are weak* continuous separate points. On the other hand, the Hahn-Banach theorem shows that the dual of a locally convex space X separates points. So, if X is a normed space that is the dual of another normed space V, then X separates points on V; hence V, regarded as a subspace of X', separates points of X and of course the functionals in this subspace are weak* continuous. Therefore the unit ball of a Banach space X that is the dual of a normed space is the closed convex hull of its extreme points. This property generally fails if X is not a dual space. For instance, if ν is a finite measure on a measure space X which has no atoms, that is, such that every set E with $\nu(E) > 0$ splits as the disjoint union $E = E_1 \cup E_2$ with $0 < \nu(E_i) < \nu(E)$, then the unit ball of $L^1(X, \nu)$ has no extreme points, because every f of L^1-norm 1 is a proper convex combination of its (renormalized) truncations to two disjoint subsets of positive finite measure. Instead, the characteristic function of an atom is clearly an extreme point.

In this section we study the extreme points of the unit balls of the other spaces considered in this paper. We follow again [7] for the spaces M^p and $\widetilde{\mathcal{M}}^p$. Then we handle the easier case of $\mathcal{S}^p$, never considered before.

Remark 59. To simplify the presentation, we shall check extremality in the following form. A vector f in the unit ball B of a normed space V is an extreme point of B if and only if there does not exists $g \neq 0$ in V such that $f \pm g \in B$. Indeed, if such g exists then f is the mid-point of the chord connecting $f + g$ and $f - g$; hence it is not extreme in B. Conversely, if f is not extreme in B then it is an interior point of some chord in B, hence it is the mid-point of some other chord.

As a consequence, the unit ball of a semi-normed but not normed space has no extreme points, since every f of semi-norm less than or equal to 1 is the average of $f \pm g$, and $\|f \pm g\| \leqslant \|f\| + \|g\| = \|f\| \leqslant 1$ whenever $\|g\| = 0$. This makes extremality a trivial issue on $\mathcal{M}^p$.

7.1. Extreme Points in the Unit Ball of M^p. We start with the space M^p, studied in [7].

Theorem 60. *(i) Let $1 < p < \infty$ and let f belong to the unit ball of M^p. Denote by δ the modulus of convexity of L^p, as in Definition 19. If $A_p(T_n, f) := A(T_n, |f|^p)^{1/p} > 1-\delta((c/T_n)^{1/p})$ for some $c > 0$ and some unbounded sequence T_n, then f is an extreme point of the unit ball $S(M^p)$.*

(ii) If $1 < p < \infty$ and f is an extreme point in the unit ball $S(M^p)$, then for every $c > 0$ there is an unbounded sequence T_n such that $A_p(T_n, f) := A(T_n, |f|^p)^{1/p} > 1 - (c/T_n)^{1/p}$.

(iii) The unit ball of M^1 does not contain any extreme points (in particular, M_1 is not a dual space).

Proof. By Remark 59, to prove (i) it is enough to show that the only g in M^p such that $\|f \pm g\| \leqslant 1$ is the zero function. Indeed, if not, choose T_0 such that $\int_{-T_0}^{T_0} |g|^p = d > 0$. For every $T > 0$, denote by $L(p,T)$ the Banach space $(L^p[-T,T], dx/2T)$; obviously

$$\begin{aligned}\|(f+g)-(f-g)\|_{L(p,T)} &= \left(\frac{1}{2T}\int_{-T}^{T}|2g|^p\right)^{1/p} \\ &\geqslant \left(2^{1-1/p}\right)\left(\frac{d}{T}\right)^{1/p}.\end{aligned} \tag{156}$$

Since $L(p,T)$ is uniformly convex (Proposition 21), it follows from this inequality and Lemma 20(iii) that, for every $T \geqslant T_0$, one has

$$A\left(T,|f|^p\right)^{1/p} \leqslant 1-\delta\left(\left(2^{-1/p}\right)\left(\frac{d}{T}\right)^{1/p}\right). \tag{157}$$

This contradicts the assumption in (i).

To prove (ii), choose and fix $T_0 > 0$. Suppose that for some $c > 0$ there is no unbounded sequence T_n such that $A(T_n, |f|^p)^{1/p} > 1 - (c/T_n)^{1/p}$. This means that, for every $T > T_0$, one has

$$A\left(T,|f|^p\right)^{1/p} + \left(\frac{c}{T}\right)^{1/p} \leqslant 1. \tag{158}$$

Now let $g = (2c)^{1/p}\chi_{[T_0,T_0+1]}$, and observe that

$$\begin{aligned}A\left(T,g^p\right)^{1/p} &\leqslant \left(\frac{1}{2T}\,2c\int_{T_0}^{T}dx\right)^{1/p} \\ &\leqslant \left(\frac{c}{T}\int_{T_0}^{T_0+1}dx\right)^{1/p} = \left(\frac{c}{T}\right)^{1/p}.\end{aligned} \tag{159}$$

On the other hand, by Minkowski's inequality,

$$A\left(T,|f\pm g|^p\right)^{1/p} \leqslant A\left(T,f^p\right)^{1/p} + A\left(T,g^p\right)^{1/p}, \tag{160}$$

and it follows from (159) and (158) that $A(T, |f \pm g|^p)^{1/p} \leqslant 1$. By Remark 59, f is not an extreme point of $S(M^p)$ if we choose $T_0 = 1$.

To prove (iii), let $f \in S(M^1)$; without loss of generality assume $\|f\|_{M^p} = 1$. We want to show that f is not an extreme point. If $\int_{-T_0}^{T_0} |f| = 0$ for every $T_0 > 1$ then $f = 0$, and so it is not an extreme point. Then we can assume that, for some $T_0 > 1$, $\int_{-T_0}^{T_0} |f| = a > 0$. We know that the unit ball of $L^1[-T_0, T_0]$ has no extreme point; so, by Remark 59, there is some $g \in L^1[-T_0, T_0]$ such that $\int_{-T_0}^{T_0} |f \pm g| = a$. Extend this g to the whole of $\mathbb{R}$ by setting it equal to zero outside $[-T_0, T_0]$. Then, for every $T \geqslant T_0$, one has $A(T, |f \pm g|) \leqslant a/2T < 1/2$; in other words, $f \pm g$ belongs to $S(M^1)$, and therefore f is not an extreme point.

Therefore f is not an extreme point, once again by Remark 59. □

Corollary 61. *The unit ball of $\mathcal{J}^p$ has no extreme points.*

Proof. If $f \in \mathcal{J}^p$, then $A(T, f) \to 0$ as $T \to \infty$. Therefore $S(\mathcal{J}^p)$ has no extreme points for $1 < p < \infty$ by part (ii) of Theorem 60. The case $p = 1$ follows directly form part (iii) of the same theorem. □

By part (i) of Theorem 60, the constant function 1 is an extreme point in the unit ball of M^p; more generally, any function such that $A(T, |f|^p) = 1$ for every $T > 0$ is an extreme point. It is easy to characterize such functions.

Proposition 62. *Let $f \in M^p$, $1 \leqslant p < \infty$. Then $A(T, |f|^p) = 1$ for almost every $T > 0$ if and only if $|f(x)|^p + |f(-x)|^p = 1$ for almost every $x \geqslant 0$.*

Proof. If $|f(x)|^p + |f(-x)|^p = 1$ almost everywhere then, clearly, $A(T, |f|^p) = (1/2T)\int_{-T}^{T} |f|^p = 1$. Conversely, the integral of $|f|^p$ is absolutely continuous; by differentiation one has

$$-\frac{1}{2T^2}\int_{-T}^{T}|f|^p + \frac{1}{2T}\left(|f(T)|^p + |f(-T)|^p\right) = 0 \tag{161}$$

for almost every $T \geqslant 0$. This is the same as

$$|f(T)|^p + |f(-T)|^p = \frac{1}{T}\int_{-T}^{T}|f|^p = 2A\left(T,|f|^p\right) \equiv 2 \tag{162}$$

for almost every $T \geqslant 1$. □

7.2. Extreme Points in the Unit Ball of $\widetilde{\mathcal{M}}^p$. Let us now deal with the extreme points in the Marcinkiewicz Banach quotient $\widetilde{\mathcal{M}}^p$. Although all the arguments and ideas are already present in [7, Theorems 3.8 and 3.10], the characterization of extreme points that we prove in what follows was not given in this reference, where only a sufficient condition for extremality is obtained. Actually, in reworking the arguments of [7], we take the opportunity to correct some flaws therein, the first of which is already in the statement. Indeed, Theorem 3.10 in this reference makes use of inequalities involving $A(T, |f|^p)$ for $f \in \widetilde{\mathcal{M}}^p$. However, the elements of $\widetilde{\mathcal{M}}^p$ are not individual functions but classes of equivalence thereof, modulo the null space $\mathcal{J}^p$. In these cosets, the quantity $\lim_{T\to+\infty} A(T, |f|^p)$

(used in [7, Theorem 3.8]) is well defined, but $A(T, |f|^p)$ is not, because it depends on the coset representative. Instead, we need to project it to the quotient. In the next proofs, we shall consider representatives in the equivalence classes modulo $\mathcal{J}^p$ and make use of the following simple remark.

Remark 63. For every $1 \le p < \infty$, the $\mathcal{M}^p$ (semi-)norm of a function. The $\mathcal{M}^p$ (semi-)norm of a function f is equal to the $\widetilde{\mathcal{M}}^p$ norm of its equivalence class modulo $\mathcal{J}^p$. Indeed, for $g \in \mathcal{J}^p$, one has $\|f+g\|_{\mathcal{M}^p} \leqslant \|f\|_{\mathcal{M}^p} + \|g\|_{\mathcal{M}^p} = \|f\|_{\mathcal{M}^p}$, and $\|f+g\|_{\mathcal{M}^p} \geqslant \|f\|_{\mathcal{M}^p} - \|g\|_{\mathcal{M}^p} = \|f\|_{\mathcal{M}^p}$.

We need a preliminary lemma that clarifies some comments in our reference ([7], Remark at page 161).

Lemma 64. *Let $\|f\|_{\mathcal{M}^p} = 1$ and for $0 < \alpha < 1$ write*

$$O_\alpha = \left\{T > 0 : \widetilde{A}\left(T, |f|^p\right) < 1 - \alpha\right\}. \tag{163}$$

There exist two unbounded sequences of positive numbers a_n, b_n, with $b_n < a_{n+1}$ for every n, such that the connected components of O_α are the open intervals (b_n, a_{n+1}). By passing to suitable increasing subsequences, which we still denote by $\{a_n\}$ and $\{b_n\}$, we have $a_n < b_n < a_{n+1}$ for every n, and the disjoint union $\bigcup_{n=1}^{\infty}[a_n, b_n]$ is contained in the complement E_α of O_α:

$$E_\alpha = \left\{T > 0 : A\left(T, |f|^p\right) \geqslant 1 - \alpha\right\}. \tag{164}$$

Proof. Since $T \mapsto A(T, |f|^p)$ is continuous, O_α is open, hence a countable union of open intervals. Because $\|f\|_{\mathcal{M}^p} = 1$, O_α is not all of the real line. As the union of overlapping open intervals is again an open interval, O_α is a disjoint countable union of intervals that we write as $O_\alpha = \bigcup_n (b_n, a_{n+1})$ for suitable $b_n < a_{n+1}$. Then the complement E_α is given by $E_\alpha = \bigcup_n [a_n, b_n]$ (here $a_n < b_n$). Again as $\|f\|_{\mathcal{M}^p} = 1$, E_α is not compact; therefore, by passing to a subsequence, we may assume that $E_\alpha \supset \bigcup_n [a_n, b_n]$ with a_n, b_n monotonically increasing and unbounded. □

We are now ready to characterize the extreme points of the unit ball of $\widetilde{\mathcal{M}}^p$. Part (i) of the next theorem is a slightly more detailed proof of [7, Theorem 3.11]; parts (ii) extends results in [7, Theorems 3.8 and 3.10], where a clever argument is aimed to show that extremality in the unit sphere of $\widetilde{\mathcal{M}}^p$ is critically related to the rate of speed of those subsequences $A(T_n, |f|^p)$ that converge to their maximum limit 1. Our proof of part (ii) is a considerable revision of the argument in [7].

Theorem 65. *(i) The unit ball $B(\widetilde{\mathcal{M}}^1)$ does not contain any extreme point.*

(ii) For $1 < p < \infty$, let f be in the unit ball $B(\widetilde{\mathcal{M}}^p)$. Then f is an extreme point of $S(\widetilde{\mathcal{M}}^p)$ if there exists an unbounded positive sequence $\{T_n\}$ with T_{n+1}/T_n bounded, such that $\lim_{n\to\infty} A(T_n, |f|^p) = 1$.

Proof of Theorem 65. We can as well restrict attention to the unit sphere, that is, to functions f of norm 1. Let us prove part (i). Since $\|f\|_{\widetilde{\mathcal{M}}^1} = 1$, any representative in its equivalence class modulo $\mathcal{J}^1$, that we still call f, satisfies $\|f\|_{\mathcal{M}^1} = 1$ by Remark 63, and the integral $\int_{-T}^{T}|f|$ diverges with T. Hence, for every T, there exists $T' > T$ such that the integral of $|f|$ on $[-T', T'] \setminus [-T, T]$ is equal to 1 (from now on, in this part of the proof, the symbol $\|\cdot\|$ means $\|\cdot\|_{\mathcal{M}^1}$). So we can build an exhausting family of nested intervals B_n as follows. Let $B_1 = [-T_1, T_1]$ with $\int_{-T_1}^{T_1}|f| = 1$, and $B_{n+1} = [-T_{n+1}, T_{n+1}] \setminus [T_n, T_n]$ with $\int_{B_{n+1}}|f| = 1$. The sequence T_n tends to infinity because f is locally summable; indeed, if $\lim T_n = R < \infty$, then $\int_{-R}^{R}|f| = \infty$, a contradiction. Consider the function g equal to $f/2$ on the intervals B_n with n odd, and $-f/2$ on the B_n's with n even; note that $\|g\| = (1/2)\|f\| = 1/2$. For every T, let n be such that $T_n \leqslant T < T_{n+1}$. Then $|f+g| - |f| = -(1/2)|f|$ on B_{2j} and $(1/2)|f|$ on B_{2j+1}, $j \in \mathbb{N}$. Since $\int_{B_j}|f| = 1$, we now have $\sum_{j=1}^{n}\int_{B_j}(|f+g| - |f|) = 1/2$ if n is odd and 0 if n is even. Instead, $\sum_{j=1}^{n}\int_{B_j}(|f-g| - |f|) = -1/2$ if n is odd and 0 if n is even. In both cases,

$$\left|A\left(T, |f \pm g|\right) - A\left(T, |f|\right)\right| \leqslant \frac{1}{2T}\int_{B_{n+1}}|f| = \frac{1}{2T}. \tag{165}$$

Letting $T \to +\infty$ we see that $\|f+g\| \leqslant 1$. By Remark 63, the same is true for the norm in $\widetilde{\mathcal{M}}^1$; thus f is not an extreme point of $\widetilde{\mathcal{M}}^1$. This proves (i).

Let us now turn our attention to part (ii). Again, $\|f\|_{\widetilde{\mathcal{M}}^p} = \lim_{T\to\infty} A(T, |f|^p) = 1$ for every representative f of the equivalence class modulo $\mathcal{J}^p$. Therefore there exist increasing sequences $T_n \to \infty$ such that $\lim_{T_n\to\infty} A(T, |f|^p) = 1$. So, for every $\alpha < 1$, T_n belongs to $E_\alpha = \{T > 0 : A(T, |f|^p) \geqslant 1 - \alpha\}$ for large n. We begin by assuming the existence, for some $\alpha < 1$, of such a sequence $\{T_n\}$ for which the ratio T_{n+1}/T_n is bounded. Again by Remark 59, if f is not an extreme point then $\limsup_{T\to+\infty} A(T, |f \pm g|^p) = 1$ for some $g \in \mathcal{M}^p$, $g \neq 0$. Instead, we shall prove that any such g is the zero element of $\widetilde{\mathcal{M}}^p$; that is, $\limsup_{T\to+\infty} A(T, |g|^p) = 0$. Indeed, we shall prove more, namely,

$$\lim_{T\to+\infty} A\left(T, |g|^p\right) = 0. \tag{166}$$

We first prove that $A(T_n, |g|^p)$ tends to 0. If this were false, then, by passing to a subsequence, we could assume that $A(T_n, |g|^p) > \varepsilon$ for some $\varepsilon > 0$ and all n; we can choose ε as small as we wish. Since the spaces $L(p, T_n) := L^p([-T_n, T_n], dx/2T_n)$ are uniformly convex with the same modulus of convexity (Proposition 21), Definition 19 yields a $\delta = \delta(\varepsilon)$ such that $\|f\|^p_{L(p,T_n)} = A(T_n, |f|^p) < (1-\delta)^p \sim 1 - p\delta$. If we now choose $\alpha = p\delta(\varepsilon/2)$, then, by Proposition 21, α is small if ε is small, and this contradicts the hypothesis. Now let us extend (166) to every other $T > 0$. Let n be such that $T_n < T < T_{n+1}$. Then

$$\begin{aligned} A\left(T, |g|^p\right) &\leqslant \frac{1}{2T}\int_{-T_{n+1}}^{T_{n+1}}|g|^p = \frac{T_{n+1}}{T}\, A\left(T_{n+1}, |g|^p\right) \\ &\leqslant \frac{T_{n+1}}{T_n}\, A\left(T_{n+1}, |g|^p\right). \end{aligned} \tag{167}$$

Since T_{n+1}/T_n is bounded, (166) follows from the same inequality that we have already proved for the T_n's. Thus the condition that T_{n+1}/T_n be bounded is necessary for f to be an extreme point.

Now we prove that this condition is also sufficient. We must show that if f is an extreme point of the unit sphere, there exist arbitrarily small $\alpha > 0$ such that no divergent sequence T_n with T_{n+1}/T_n bounded satisfies $\lim_{n\to\infty} A(T_n, |f|^p) > 1-\alpha$ (so, without loss of generality, from now on we shall restrict attention to $0<\alpha<1/2$).

By Lemma 64, this amounts to show that, for some arbitrarily small α, there are two divergent sequences a_n, b_n such that

$$\left\{T>0 : A\left(T,|f|^p\right) \geqslant 1-\alpha\right\} = \bigcup_{n=1}^{\infty}[a_n, b_n] \tag{168}$$

with $a_n < b_n < a_{n+1}$ for every n and a_{n+1}/b_n unbounded; by passing to a subsequence, we assume that $a_{n+1}/b_n \to \infty$. We must show that under these assumptions f is not an extreme point.

Since $\|f\|_{\mathscr{M}^p} = 1$, for infinitely many n there exists $T_n \in [a_n, b_n]$ such that $A(T, |f|^p) \geqslant 1-\alpha/2$; for the sake of simplicity, by passing again to a subsequence we may assume that this is true for every n. Fix such α for the moment, and let

$$c_n = \max\left\{T \in [a_n, b_n] : A\left(T, |f|^p\right) = 1-\frac{\alpha}{2}\right\}. \tag{169}$$

Then, by semicontinuity and the definition of the sets B_n, one has

$$A\left(c_n, |f|^p\right) = 1-\frac{\alpha}{2}, \tag{170a}$$

$$1-\alpha \leqslant A\left(T,|f|^p\right) < 1-\frac{\alpha}{2} \quad \text{for } c_n < T < b_n, \tag{170b}$$

$$A\left(b_n, |f|^p\right) = 1-\alpha, \tag{170c}$$

$$A\left(T,|f|^p\right) < 1-\alpha \quad \text{for } b_n < T < a_{n+1}. \tag{170d}$$

Let $C_n = [c_n, b_n]$ and $C = \bigcup_{n=1}^{\infty} C_n$. The remainder of the proof is easier if the c_n's satisfy the condition b_n/c_n bounded, but this may not be the case. Then choose and fix d_n such that $c_n \leqslant d_n < b_n$ and b_n/d_n bounded, and write $D_n = [d_n, b_n]$ and $D = \bigcup_{n=1}^{\infty} D_n$. This part of the proof is rather involved; for the sake of clarity, we present its various parts as separate lemmas.

The proof splits into the following two cases:

$$\lim_n A\left(b_n, \left|f\chi_{C_n}\right|^p\right) = 0 \quad \text{or} \quad \limsup_n A\left(b_n, \left|f\chi_{C_n}\right|^p\right) > 0. \tag{171}$$

Case (a). We have $\lim_n A(b_n, |f\chi_{C_n}|^p) = 0$.

By passing to a subsequence, we may assume that $A(b_n, |f\chi_{C_n}|^p)$ tends to zero arbitrarily fast. Since a_n/b_{n-1} is unbounded and $b_n > a_n$, also b_n/b_{n-1} is unbounded. Then, if b_n/c_n is bounded by passing to a further subsequence we may as well assume that

$$\lim_n A\left(b_n, |f\chi_C|^p\right) = 0. \tag{172}$$

Instead, if b_n/c_n is unbounded, since $b_{n-1} < a_n < c_n < b_n$, also b_n/b_{n-1} is unbounded, and again we may assume (172). Then the same obviously holds for $A(b_n, |f\chi_D|^p) \leqslant A(b_n, |f\chi_C|^p)$.

Under assumption (a) we now prove the first preliminary fact.

Lemma 66. *Consider any representative of the coset $f \in \widetilde{\mathscr{M}}^p$, and by abuse of notation denote it again by f. Then $h := f\chi_D$ has norm zero in $\widetilde{\mathscr{M}}^p$; that is, $h \in \mathscr{I}^p$.*

Proof of the Lemma. Since h vanishes in $[b_n, d_{n+1}]$, for every $b_n < T \leqslant d_{n+1}$ one has

$$A\left(T, |h|^p\right) = \frac{1}{2T}\int_{-b_n}^{b_n} |h|^p \leqslant A\left(b_n, |h|^p\right). \tag{173}$$

On the other hand, for $d_{n+1} < T \leqslant b_{n+1}$,

$$\begin{aligned} A\left(T, |h|^p\right) &\leqslant \frac{b_{n+1}}{T}\frac{1}{2b_{n+1}}\int_{-b_{n+1}}^{b_{n+1}} |h|^p = \frac{b_{n+1}}{T} A\left(b_{n+1}, |h|^p\right) \\ &\leqslant \frac{b_{n+1}}{d_{n+1}} A\left(b_{n+1}, |h|^p\right). \end{aligned} \tag{174}$$

Since the sequence b_n/d_n is bounded, the statement now follows from (172). □

Remark 67. In [7, Theorem 3.10], at the beginning of the proof of condition (i), it is stated without proof that $f\chi_C$ is equivalent to 0 mod$\mathscr{I}^p$, that is, that $f\chi_C$ has norm zero. The fact that $f\chi_C$ has norm zero plays a major role in that proof. However, this does not follow without further assumptions from (172) and the condition that a_n/b_{n-1} diverges. Indeed, we now show that in general this is not true without the additional assumption that the sequence c_n/b_n decays faster than $A(b_n, |f\chi_C|^p)$ (see the proof of the previous lemma). A convenient assumption is therefore that b_n/c_n is bounded.

Here is an example where b_n/c_n is unbounded, a_n/b_{n-1} diverges, and $f\chi_C \notin \mathscr{I}^p$. Take $b_n = 2^{2^n}$, $a_n = (b_n + b_{n-1})/2 - 1$, and $c_n = a_n + 1$. Then a_n/b_{n-1} diverges but $c_n = (b_n + b_{n-1})/2$, and the function χ_C is alternatively zero and one on intervals of the same length. Therefore $\|\chi_C\| = 1/2$ instead of 0. This is why we need to change the argument of the proof of [7, Theorem 3.10] and introduce $d_n \geqslant c_n$ such that b_n/d_n be bounded. The proof becomes more difficult and the argument more sophisticated, but still follows the guidelines of the brilliant idea of [7].

As a consequence of Lemma 66, by changing the coset representative f we can for the moment assume that

$$f\chi_{D_n} = 0. \tag{175}$$

That is $h \equiv 0$. This assumption is used for the first inequality of the following lemma. The second inequality is proved for the sake of completeness and it will not be used below.

Lemma 68. *If* (175) *holds,*

(i)

$$\frac{b_n - d_n}{b_n} < \frac{\alpha}{2(1-\alpha)} < \frac{b_n - c_n}{c_n}; \tag{176}$$

(ii)

$$\frac{b_n - c_n}{b_n} \leqslant \frac{\alpha}{2-\alpha}. \tag{177}$$

Proof of the Lemma. Let us prove the first inequality in (i). By (170b),

$$1-\alpha \leqslant A\left(d_n, |f|^p\right) < 1 - \frac{\alpha}{2} \tag{178}$$

and by (170c) and the assumption that f vanishes in D_n,

$$A\left(b_n, |f|^p\right) = \frac{1}{2b_n}\int_{-d_n}^{d_n} |f|^p = 1-\alpha. \tag{179}$$

Therefore, by the second inequality in (178),

$$\left(\frac{1}{2d_n} - \frac{1}{2b_n}\right)\int_{-d_n}^{d_n} |f|^p < \frac{\alpha}{2}. \tag{180}$$

On the other hand,

$$\begin{aligned}\left(\frac{1}{2d_n} - \frac{1}{2b_n}\right)\int_{-d_n}^{d_n} |f|^p &= \frac{b_n - d_n}{b_n}\frac{1}{2d_n}\int_{-d_n}^{d_n} |f|^p \\ &= \frac{b_n - d_n}{b_n} A\left(d_n, |f|^p\right).\end{aligned} \tag{181}$$

So, by (180) and the first inequality in (178),

$$\frac{b_n - d_n}{b_n} < \frac{\alpha}{2A\left(d_n, |f|^p\right)} \leqslant \frac{\alpha}{2(1-\alpha)}. \tag{182}$$

This proves the first inequality, and we now turn our attention to the second. Observe that $(1/2b_n)\int_{-c_n}^{c_n} |f|^p \leqslant \widetilde{A}(b_n, |f|^p) = 1-\alpha$ by (170c). Hence, by (170a),

$$\left(\frac{1}{2c_n} - \frac{1}{2b_n}\right)\int_{-c_n}^{c_n} |f|^p \geqslant \frac{\alpha}{2}. \tag{183}$$

Now

$$\begin{aligned}\left(\frac{1}{2c_n} - \frac{1}{2b_n}\right)\int_{-c_n}^{c_n} |f|^p &= \frac{b_n - c_n}{c_n}\frac{1}{2b_n}\int_{-c_n}^{c_n} |f|^p \\ &\leqslant \frac{b_n - c_n}{c_n}(1-\alpha).\end{aligned} \tag{184}$$

Combining the last two inequalities we obtain

$$\frac{b_n - c_n}{c_n} \geqslant \frac{\alpha}{2(1-\alpha)} \tag{185}$$

that is the second inequality.

Inequality (ii) is proved analogously; by (170a) and (170c),

$$\frac{1}{2c_n}\int_{-c_n}^{c_n} |f|^p dx - \frac{1}{2b_n}\int_{-b_n}^{b_n} |f|^p dx = \frac{\alpha}{2}. \tag{186}$$

If we split $\int_{-b_n}^{b_n} = \int_{-c_n}^{c_n} + \int_{-C_n \cup C_n}$ and discard the last integral, this becomes

$$\left(\frac{1}{2c_n} - \frac{1}{2b_n}\right)\int_{-c_n}^{c_n} |f|^p \leqslant \frac{\alpha}{2}. \tag{187}$$

As before, the left hand side equals

$$\frac{b_n - c_n}{c_n} A\left(c_n, |f|^p\right) = \frac{b_n - c_n}{c_n}\left(1 - \frac{\alpha}{2}\right), \tag{188}$$

and part (ii) follows. □

Corollary 69. *If* (175) *holds, for every sequence* n_k *(or more precisely, for every subsequence of the subsequence introduced before, in the part of the proof between identities* (168) *and* (170a)*) the function* $h := \sum_{k=1}^{\infty} \chi_{C_{n_k}}$ *satisfies* $\|h\|_{\mathscr{M}^p} = \limsup_{T\to\infty} A(T, h^p) > 0$.

Proof of the Corollary. We know that the sequence c_{n_k} of the left ends of the segments C_{n_k} diverges (because $c_{n_k} > a_{n_k}$). Let n be one of the indices n_k. If $T = c_n$, then, by Lemma 68(i),

$$\begin{aligned}A(T, h^p) &= \frac{1}{2c_n}\sum_{n_k \leqslant n}\int_{c_{n_k}}^{b_{n_k}} dx = \frac{1}{2c_n}\sum_{n_k \leqslant n}\left(b_{n_k} - c_{n_k}\right) \\ &> \frac{b_n - C_n}{C_n} > \frac{\alpha}{2(1-\alpha)} > 0.\end{aligned} \tag{189}$$

□

Lemma 70. *Let* $0 < \alpha < 1/2$ *and* $\widetilde{f} \in \widetilde{\mathscr{M}}^p$ *with* $\|\widetilde{f}\|_{\widetilde{\mathscr{M}}^p} = 1$. *Without loss of generality, choose any coset representative* f *of* $\widetilde{f}$ *modulo* $\mathscr{I}^p$ *such that* $f\chi_{D_n} = 0$. *Then the following inequalities hold:*

(i) for $0 < a < (\alpha/(2-\alpha))^{1/p}$ *and for every integer* $n > 0$,

$$A\left(T, \left|f\left(1 \pm a\chi_{C_n}\right)\right|^p\right) < 1 + \frac{b_n}{a_{n+1}}; \tag{190}$$

(ii) if $f\chi_{D_n} \equiv 0$, *then, for* $0 < a^p < 1/(1-\alpha)$,

$$A\left(T, \left|f \pm a\chi_{D_n}\right|^p\right) < 1 + \frac{b_n}{a_{n+1}}. \tag{191}$$

Proof of the Lemma. Remember that $\|f\|_{\mathscr{M}^p} = \|\widetilde{f}\|_{\widetilde{\mathscr{M}}^p} = 1$ by Remark 63. Let $a^p = \alpha/(2-\alpha)$, and observe that $a^p < 1$ since $0 < \alpha < 1/2$. Obviously the statement of part (i) for $T \leqslant c_n$ follows from Remark 63, as $\chi_{D_n} = 0$ in $(-c_n, c_n)$. The same observation applies to part (ii) for $T \leqslant d_n$. For $c_n < T \leqslant b_n$ we know by (170c) and (170d) that $A(T, |f|^p) < 1 - \alpha/2$.

Since $A(T,|f|^p)$ is the norm of $|f|^p$ in $L^1([-T,T],dx/2T)$, the triangular inequality yields

$$\begin{aligned} A\left(T,\left|f\left(1\pm a\chi_{C_n}\right)\right|^p\right) &\leqslant A\left(T,|f|^p\right)+a^p A\left(T,\left|f\chi_{C_n}\right|^p\right) \\ &\leqslant \left(1+a^p\right)A\left(T,|f|^p\right) \\ &\leqslant \left(1-\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2-\alpha}\right)<1. \end{aligned} \tag{192}$$

Instead, for $T\geqslant a_{n+1}$,

$$\begin{aligned} A\left(T,\left|f\left(1\pm a\chi_{C_n}\right)\right|^p\right) &\leqslant 1+a^p\,\frac{b_n}{a_{n+1}}A\left(b_n,|f|^p\right) \\ &\leqslant 1+\frac{\alpha}{2-\alpha}\frac{b_n}{a_{n+1}}. \end{aligned} \tag{193}$$

Since $\alpha/(2-\alpha)<1$ for $0<\alpha<1$, this proves part (i). For part (ii) we use again the condition $f\chi_{D_n}=0$; that is, f and χ_{D_n} have disjoint supports. Then, for $d_n<T\leqslant b_n$, by (170b) one has

$$\begin{aligned} A\left(T,\left|f\pm a\chi_{D_n}\right|^p\right) &= A\left(T,|f|^p\right)+2a^p\,\frac{T-d_n}{2T} \\ &\leqslant 1-\frac{\alpha}{2}+a^p\,\frac{b_n-d_n}{b_n} \\ &\leqslant 1-\frac{\alpha}{2}+a^p\,\frac{\alpha}{2(1-\alpha)}\leqslant 1 \end{aligned} \tag{194}$$

(the first inequality holds because the map $T\mapsto(T-d_n)/T$ is increasing, the middle one by the first inequality of Lemma 68(i) and the last by the fact that $\alpha<1/2$). The same inequality holds if $b_n<T\leqslant a_{n+1}$; the only change is that the fraction $(T-d_n)/2T$ is replaced by $(T-d_n)/2b_n$, but this is smaller than $(b_n-d_n)/2b_n$, so the above chain of inequalities still holds.

Finally, for $T>a_{n+1}$, again by the fact that the supports are disjoint (and, of course, by the triangular inequality of the L^1 norm),

$$\begin{aligned} A\left(T,\left|f\pm a\chi_{D_n}\right|^p\right) &< 1+2a^p\,\frac{b_n-d_n}{2T} \\ &< 1+2a^p\,\frac{b_n}{2a_{n+1}}<1+\frac{b_n}{a_{n+1}}. \end{aligned} \tag{195}$$

□

Proof of Theorem 65 (continued). Build a sequence of integers as follows: $n_1=1$, and, for $k\geqslant 1$, choose n_{k+1} so that, for $T>b_{n_{k+1}}$,

$$A\left(T,\left|\sum_{j=1}^{k}a\chi_{C_{n_j}}\right|^p\right)\leqslant\rho_k, \tag{196}$$

where ρ_k is any positive sequence that tends to zero at infinity, and a is as in the statement of part (ii) of Lemma 70. To be more accurate, the sequence that we have built should be chosen as a subsequence of the subsequence introduced before, in the part of the proof between identities (168) and (170a), and here we use a sloppy but easier notation. Let $g=ah$, where $h=\sum_{j=1}^{\infty}\chi_{C_{n_j}}$ is the function defined in Corollary 69. Choose and fix T and choose k such that $b_{n_k}\leqslant T<b_{n_{k+1}}$. Remember that $A(T,|f\pm g|^p)^{1/p}$ is the norm of $f\pm g$ in $L^p([-T,T],dx/2T)$, and use the triangular inequality and Lemma 70 to obtain

$$\begin{aligned} A\left(T,|f\pm g|^p\right)^{1/p} &\leqslant A\left(T,\left|f\pm a\chi_{C_{n_k}}\right|^p\right)^{1/p} \\ &\quad + A\left(T,\left|\sum_{j=1}^{k-1}a\chi_{C_{n_j}}\right|^p\right)^{1/p} \\ &< \left(1+\frac{b_{n_k}}{a_{n_k+1}}\right)^{1/p}+\rho_k^{1/p}. \end{aligned} \tag{197}$$

Therefore

$$\|f\pm g\|_{\widetilde{\mathcal{M}}^p}=\limsup_{T\to\infty}\widetilde{A}\left(T,|f\pm g|^p\right)^{1/p}=1. \tag{198}$$

Since $g\notin\mathcal{F}^p$ by Corollary 69, then it has non-zero norm in $\widetilde{\mathcal{M}}^p$, and the last identity implies that f is not an extreme point. This completes the proof in Case (a).

Case (b). Consider $\limsup_n A(b_n,|f\chi_{C_n}|^p)>0$.

In this case, the function $u:=\sum_{n=1}^{\infty}f\chi_{C_n}$ satisfies

$$\begin{aligned} \limsup_T A\left(T,|u|^p\right) &\geqslant \limsup_n A\left(b_n,\sum_{j=1}^{n}\left|f\chi_{C_j}\right|^p\right) \\ &\geqslant \limsup_n A\left(b_n,\left|f\chi_{C_n}\right|^p\right)>0. \end{aligned} \tag{199}$$

That is, h has positive norm in $\widetilde{\mathcal{M}}^p$ (now we do not need any longer to assume that $f\chi_{D_n}=0$).

We proceed in analogy with (197). For all $b_{n-1}\leqslant T\leqslant b_n$ we now have, by part (i) of Lemma 70 and (170c),

$$\begin{aligned} &A\left(T,|f\pm g|^p\right)^{1/p} \\ &\leqslant A\left(T,\left|f\pm af\chi_{C_n}\right|^p\right)^{1/p}+A\left(T,\left|\sum_{j=1}^{n-1}f\chi_{C_j}\right|^p\right) \\ &< \left(1+\frac{b_n}{a_{n+1}}\right)^{1/p}+\frac{b_n}{b_{n-1}}A\left(b_n,\left|\sum_{j=1}^{n-1}f\chi_{C_j}\right|^p\right)^{1/p} \\ &< \left(1+\frac{b_n}{a_{n+1}}\right)^{1/p}+\left(\frac{b_n}{a_{n+1}}(1-\alpha)\right)^{1/p}. \end{aligned} \tag{200}$$

Since $b_n/a_{n+1}\to 0$, this implies that $\limsup_T\inf_{g\in\mathcal{F}^p}A(T,|f\pm g|^p)\leqslant 1$, so $f\pm g$ is in the unit ball of $\widetilde{\mathcal{M}}^p$ and f is not an extreme point of this ball. □

7.3. Extreme Points in the Unit Ball of $\mathcal{S}^p$. The problem of extremality is apparently easier for the spaces $\mathcal{S}^p$. Here, however, we have an ambiguity; we have used two equivalent norms in $\mathcal{S}^p$ (see Lemma 8), and precisely

$$\|f\|_{\mathcal{S}^p} = \left(\sup_{x\in\mathbb{R}} \int_x^{x+1} |f|^p\right)^{1/p},$$
$$\|f\|_{\ell^\infty\otimes L^p[n,n+1]} = \left(\sup_{n\in\mathbb{Z}} \int_n^{n+1} |f|^p\right)^{1/p}. \tag{201}$$

Let us denote by $\widetilde{\mathcal{S}^p}$ the Banach space defined by the latter norm. The two spaces are equivalent as Banach spaces, but their unit balls are not the same and their extreme points need not be the same. We characterize the extreme points in the unit ball of $\widetilde{\mathcal{S}^p}$.

Lemma 71. *Let f belong to the unit ball $B(\widetilde{\mathcal{S}^p})$ for some p, $1 \leqslant p < \infty$, and for $n \in \mathbb{Z}$ write $f_n = f|_{[n,n+1]}$. Then f is an extreme point in $B(\widetilde{\mathcal{S}^p})$ if and only if f_n is an extreme point in the unit ball of $L^p[n, n+1]$ for every n.*

Proof. If, for some n, f_n is not an extreme point in the unit ball of $L^p[n, n+1]$, then there is some $g_n \in L^p[n, n+1]$ such that $\|f_n \pm g_n\|_{L^p[n,n+1]} \leqslant 1$. Consider the function $g \in \widetilde{\mathcal{S}^p}$ that coincides with g_n on $[n, n+1]$ and vanishes elsewhere. Then $\|\widetilde{f \pm g}\|_{L^p[n,n+1]} \leqslant 1$ for every n; therefore $\|f \pm g\|_{\widetilde{\mathcal{S}^p}} \leqslant 1$ and f is not an extreme point in $S(\widetilde{\mathcal{S}^p})$.

Conversely, suppose that f_n is an extreme point in the unit ball of $L^p[n, n+1]$ for every n. If f is not an extreme point in $B(\widetilde{\mathcal{S}^p})$, then there is a function $g \neq 0$ in $\widetilde{\mathcal{S}^p}$ such that $f \pm g$ has norm not larger than 1 in $\widetilde{\mathcal{S}^p}$. Then the norm of $f_n \pm g_n$ in $L^p[n, n+1]$ is less than or equal to 1 for every n, but g_m is non-zero for some m. Therefore f_m is not an extreme point in the unit ball of $L^p[m, m+1]$, a contradiction. □

Corollary 72. *(i) For $1 < p < \infty$, a function f in the unit ball of $\widetilde{\mathcal{S}^p}$ is an extreme point if and only if, for every $n \in \mathbb{Z}$, $\|f\|_{L^p[n,n+1]} = 1$.*

(ii) The unit ball of $\widetilde{\mathcal{S}^1}$ has no extreme point.

Proof. Part (i) follows from Lemma 71 since all functions in the unit sphere of L^p are extreme points of its unit ball, as L^p is strictly convex (even more, it is uniformly convex, by Proposition 21). Part (ii) follows similarly, because, as observed at the beginning of this section, the unit ball of L^1 has no extreme points. □

In the case of the norm of $\mathcal{S}^p$, we prove that the above condition is sufficient for extremality.

Proposition 73. *Let f belong to the unit ball $B(\mathcal{S}^p)$ for some p, $1 \leqslant p < \infty$, and for $x \in \mathbb{R}$ write $f_x = f|_{[x,x+1]}$. If f_x is an extreme point in the unit ball $B(L^p[x, x+1])$ for every x, then f is an extreme point in the unit ball $B(\mathcal{S}^p)$.*

Proof. If $f \in B(\mathcal{S}^p)$ is not an extreme point, then $\|f \pm g\|_{\mathcal{S}^p} \leqslant 1$ for some g that is not zero on some set of positive measures and such that $\|g\|_{\mathcal{S}^p} \leqslant 1$. But then, for all x, $\|f \pm g\|_{L^p[x,x+1]} \leqslant 1$ and $\|g\|_{L^p[x,x+1]} \leqslant 1$, and there exists y such that $g \neq 0$ in $[y, y+1]$; therefore f_y is not an extreme point in the unit ball of $L^p[y, y+1]$. □

7.4. Open Problems. Theorem 60 gives necessary conditions and sufficient conditions for extremality in the unit ball of M^p. Can extreme points be characterized by a necessary and sufficient condition?

We have characterized the extreme points in the unit ball of $\widetilde{\mathcal{S}^p}$. Can the extreme points in the unit ball of $\mathcal{S}^p$ be characterized analogously?

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author acknowledges several conversations with Federica Andreano on the contents of this paper.

References

[1] W. Stepanoff, "Über einige Verallgemeinerungen der fast periodischen Funktionen," *Mathematische Annalen*, vol. 95, no. 1, pp. 473–498, 1926.

[2] H. Weyl, "Integralgleichungen und fastperiodische Funktionen," *Mathematische Annalen*, vol. 97, no. 1, pp. 338–356, 1927.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Cambridge University Press, Cambridge, UK, 1932.

[4] H. Bohr, "Zur Theorie der fastperiodischenFunktionen, I," *Acta Mathematica*, vol. 45, pp. 29–127, 1924.

[5] H. Bohr, "Zur Theorie der fastperiodischen Funktionen, II," *Acta Mathematica*, vol. 46, pp. 101–214, 1925.

[6] H. Bohr, "Zur Theorie der fastperiodischen Funktionen, III," *Acta Mathematica*, vol. 47, pp. 237–281, 1926.

[7] K. S. Lau, "On the Banach spaces of functions with bounded upper means," *Pacific Journal of Mathematics*, vol. 91, no. 1, pp. 153–172, 1980.

[8] A. S. Besicovitch, "On generalized almost periodic functions," *Proceedings of the London Mathematical Society*, vol. S2-25, no. 1, pp. 495–512.

[9] J. Marcinkiewicz, "Un remarque sur les éspace de A.S. Besicovitch," *Comptes Rendus de l'Académie des Sciences*, vol. 208, pp. 157–159, 1939.

[10] H. Bohr and E. Fölner, "On some type of functional spaces," *Acta Mathematica*, vol. 76, pp. 31–155, 1944.

[11] J.-P. Bertrandias, "Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p," *Bulletin de la Société Mathématique de France*, vol. 5, pp. 1–106, 1966.

[12] K. S. Lau, "The class of convolution operators on the Marcinkiewicz spaces," *Annales de l'Institut Fourier*, vol. 31, pp. 225–243, 1981.

[13] F. Andreano and R. Grande, "Convolution on spaces of locally summable functions," *Journal of Function Spaces and Applications*, vol. 6, no. 2, pp. 187–203, 2008.

[14] F. Andreano and M. A. Picardello, "Approximate identities on some homogeneous Banach spaces," *Monatshefte für Mathematik*, vol. 158, no. 3, pp. 235–246, 2009.

[15] H. G. Feichtinger, "Banach convolution algebras of functions. II," *Monatshefte für Mathematik*, vol. 87, no. 3, pp. 181–207, 1979.

[16] H. G. Feichtinger, "An elementary approach to Wiener's third Tauberian theorem for the Euclidean n-space," in *Symposia Mathematica*, vol. 29, pp. 267–301, Academic Press, New York, NY, USA, 1987.

[17] K. S. Lau and J. K. Lee, "On generalized harmonic analysis," *Transactions of the American Mathematical Society*, vol. 259, no. 1, pp. 75–97, 1980.

[18] J. A. Clarkson, "Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40, no. 3, pp. 396–414, 1936.

[19] O. Hanner, "On the uniform convexity of L^p and l^p," *Arkiv för Matematik*, vol. 3, pp. 239–244, 1956.

[20] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Graduate Texts in Mathematics, Springer, New York, NY, USA, 1976.

[21] K. Yosida and E. Hewitt, "Finitely additive measures," *Transactions of the American Mathematical Society*, vol. 72, pp. 46–66, 1952.

[22] E. Alfsen and E. Effros, "Structure in real Banach spaces, part I," *Annals of Mathematics*, vol. 96, pp. 98–128, 1972.