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Research Article

Multiplicity of Positive Solutions for a Singular Second-Order Three-Point Boundary Value Problem with a Parameter

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This paper is concerned with the following second-order three-point boundary value problem $u''(t)+\beta^2 u(t)+\lambda q(t) f(t,u(t)) = 0$, $t \in (0,1)$, $u(0) = 0$, $u(1) = \delta u(\eta)$, where $\beta \in (0,\pi/2)$, $\delta > 0$, $\eta \in (0,1)$, and λ is a positive parameter. First, Green's function for the associated linear boundary value problem is constructed, and then some useful properties of Green's function are obtained. Finally, existence, multiplicity, and nonexistence results for positive solutions are derived in terms of different values of λ by means of the fixed point index theory.

1. Introduction

For given positive numbers $\eta \in (0,1)$ and $\beta \in (0,\pi/2)$, the existence, multiplicity, and nonexistence of positive solutions for the following boundary value problem (BVP for short)

$$\begin{gathered} u''(t)+\beta^2 u(t)+\lambda q(t) f(t,u(t)) = 0, \quad t \in (0,1) \\ u(0) = 0, \qquad u(1) = \delta u(\eta) \end{gathered} \tag{1}$$

are considered, where λ is a positive parameter, $f \in C([0,1]\times[0+\infty),[0+\infty))$, and $q:(0,1) \to [0,+\infty)$ may be singular at $t = 0$ and 1.

A function $u(t) \in C^2(0,1)$ is said to be a solution of BVP (1) if u satisfies BVP (1). Moreover, if $u(t) > 0$ for any $t \in (0,1)$, then u is said to be a positive solution of BVP (1).

Due to a wide range of applications in physics and engineering, second-order boundary value problems have been extensively investigated by numerous researchers in recent years. The study of multipoint boundary value problems was initiated by Il'in and Moiseev [1]. Gupta studied three-point boundary value problems for nonlinear ordinary differential equations in [2]. Since then, nonlinear three-point boundary value problems have been studied by many authors using the fixed point index theorem, Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder, coincidence degree theory, and fixed point theorem in cones. For details, the readers are referred to [3–7] and the references therein.

In [8], positive solutions for the following three-point boundary value problem at resonance

$$\begin{gathered} x''(t) = f(t,x(t)), \quad t \in (0,1), \\ x'(0) = 0, \qquad x(\eta) = x(1) \end{gathered} \tag{2}$$

were studied. Han's approach is to rewrite the original BVP as an equivalent one so that the Krasnosel'skii-Guo fixed point theorem can be applied and then the existence and multiplicity of positive solutions are investigated.

Then in [9], Han considered the following three-point boundary value problem:

$$\begin{gathered} x''(t)+\beta^2 x(t) = h(t) f(t,x(t)), \quad t \in (0,1), \\ x'(0) = 0, \qquad x(\eta) = x(1), \end{gathered} \tag{3}$$

under some conditions concerning the first eigenvalue of the relevant linear operator, where $\eta \in (0,1)$ is a constant and $h(t)$ is allowed to be singular at $t = 0$ and $t = 1$. The existence of positive solutions is studied by means of fixed point index theory.

Motivated by the above work, here we study the second-order three-point BVP (1). Under certain suitable conditions, the results of existence, multiplicity, and nonexistence of positive solutions for BVP (1) were established via the fixed point index theory.

We make the following assumptions:

(H_1) $0 < \beta < (\pi/2)$, $\sin\beta - \delta\sin\beta\eta > 0$, and $\delta\cos\beta\eta - \cos\beta \geq 0$;

(H_2) $\sin\beta(1-\eta) - \delta\sin\beta\eta > 0$ and $\sin\beta\alpha - \delta\sin\beta\eta > 0$, where $0 < \alpha < (1/2)$;

(H_3) $q(t) \geq 0, q(t) \not\equiv 0$ for $t \in (0,1)$ and $\int_0^1 q(s)ds < \infty$;

(H_4) $f(t,x)$ is nondecreasing in x and $f(t,x) > 0$ for any $(t,x) \in [0,1] \times (0,+\infty)$.

The main results of the present paper are summarized as follows.

Theorem 1. *Let* (H_1)–(H_4) *be fulfilled and suppose that*

$$\begin{aligned} f_0 &:= \lim_{x\to 0^+} \min_{t\in(0,1)} \frac{f(t,x)}{x} = \infty, \\ f_\infty &:= \lim_{x\to +\infty} \min_{t\in(0,1)} \frac{f(t,x)}{x} = \infty. \end{aligned} \tag{4}$$

Then, there exists $\lambda^* > 0$ such that BVP (1) *has at least two positive solutions for $\lambda \in (0,\lambda^*)$, at least one positive solution for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$.*

The remainder of this paper is arranged as follows. Green's function of BVP (1) and its properties are given in Section 2, and some preliminaries are also presented. The proof of Theorem 1 is given in Section 3.

2. Preliminaries

In this section we collect some preliminary results that will be used in subsequent sections.

Consider the linear boundary value problem

$$\begin{gathered} -u''(t) - \beta^2 u(t) = h(t), \quad t \in (0,1), \\ u(0) = 0, \qquad u(1) = \delta u(\eta). \end{gathered} \tag{5}$$

Lemma 2. *Assume that* (H_1) *holds. Then, for each $h \in C[0,1]$, BVP* (5) *has a unique solution*

$$u(t) = \int_0^1 G(t,s)\,h(s)\,ds, \tag{6}$$

where

$$G(t,s) = \frac{1}{\beta(\sin\beta - \delta\sin\beta\eta)} \times \begin{cases} [\sin\beta(1-s) + \delta\sin\beta(s-\eta)]\sin\beta t, & 0 \le t \le s \le \eta, \\ \sin\beta s\sin\beta(1-t) + \delta\sin\beta s\sin\beta(t-\eta), & s \le t,\ s \le \eta, \\ \sin\beta(1-s)\sin\beta t, & t \le s,\ \eta \le s, \\ \sin\beta s\sin\beta(1-t) + \delta\sin\beta\eta\sin\beta(t-s), & \eta \le s \le t \le 1. \end{cases} \tag{7}$$

Proof. Suppose that

$$G(t,s) = -\begin{cases} a_1\cos\beta t + a_2\sin\beta t, & 0 \le t \le s \le \eta, \\ a_3\cos\beta t + a_4\sin\beta t, & s \le t,\ s \le \eta, \\ a_5\cos\beta t + a_6\sin\beta t, & t \le s,\ \eta \le s, \\ a_7\cos\beta t + a_8\sin\beta t, & \eta \le s \le t \le 1. \end{cases} \tag{8}$$

According to the definition and properties of Green's function, for any $s \in [0,\eta]$, we have

$$\begin{gathered} a_1\cos\beta s + a_2\sin\beta s = a_3\cos\beta s + a_4\sin\beta s, \\ (-\beta a_1\sin\beta s + a_2\beta\cos\beta s) - (-a_3\beta\sin\beta s + a_4\beta\cos\beta s) \\ = -1, \end{gathered} \tag{9}$$

and thus

$$\begin{aligned} a_1 - a_3 &= \frac{1}{\beta}\sin\beta s, \\ a_2 - a_4 &= -\frac{1}{\beta}\cos\beta s. \end{aligned} \tag{10}$$

Then by using the boundary conditions, we have

$$\begin{gathered} a_1 = 0, \\ a_3\cos\beta + a_4\sin\beta = \delta(a_3\cos\beta\eta + a_4\sin\beta\eta). \end{gathered} \tag{11}$$

Therefore

$$\begin{gathered} a_1 = 0, \\ a_2 = a_4 - \frac{1}{\beta}\cos\beta s = -\frac{\sin\beta(s-1) + \delta\sin\beta(\eta - s)}{\beta(\delta\sin\beta\eta - \sin\beta)}, \\ a_3 = -\frac{1}{\beta}\sin\beta s, \\ a_4 = -\frac{\sin\beta s(\cos\beta - \delta\cos\beta\eta)}{\beta(\delta\sin\beta\eta - \sin\beta)}. \end{gathered} \tag{12}$$

For any $s \in [\eta, 1]$, we have

$$\begin{gathered} a_5 \cos \beta s + a_6 \sin \beta s = a_7 \cos \beta s + a_8 \sin \beta s, \\ (-\beta a_5 \sin \beta s + a_6 \beta \cos \beta s) - (-a_7 \beta \sin \beta s + a_8 \beta \cos \beta s) \\ = -1, \end{gathered} \tag{13}$$

and hence

$$\begin{aligned} a_5 - a_7 &= \frac{1}{\beta} \sin \beta s, \\ a_6 - a_8 &= -\frac{1}{\beta} \cos \beta s. \end{aligned} \tag{14}$$

By using the boundary conditions, we have

$$\begin{gathered} a_5 = 0, \\ a_7 \cos \beta + a_8 \sin \beta = \delta \left(a_5 \cos \beta \eta + a_6 \sin \beta \eta \right). \end{gathered} \tag{15}$$

Then

$$\begin{aligned} a_5 &= 0, \\ a_6 &= \frac{\sin \beta (1-s)}{\beta (\delta \sin \beta \eta - \sin \beta)}, \\ a_7 &= -\frac{1}{\beta} \sin \beta s, \\ a_8 &= \frac{\delta \sin \beta \eta \cos \beta s - \sin \beta s \cos \beta}{\beta (\delta \sin \beta \eta - \sin \beta)}. \end{aligned} \tag{16}$$

Consequently, we can get Green's function $G(t, s)$, and the lemma is proved. □

Lemma 3. *There exist a continuous function $g : [0, 1] \to [0, \infty)$ and a constant $\gamma \in (0, 1]$ such that*

(i) *if (H_1) holds, $0 \le G(t, s) \le g(s)$, $t, s \in [0, 1]$;*

(ii) *if (H_1) and (H_2) hold, $G(t, s) \ge \gamma g(s)$, $(t, s) \in [\alpha, 1 - \alpha] \times [0, 1]$.*

Proof. Firstly, it is obvious that $G(t, s) \ge 0$ for any $(t, s) \in [0, 1] \times [0, 1]$.

Next, we will give the continuous function $g(s)$ and the constant γ.

Let

$$\phi(s) = s(1-s), \qquad H(t, s) = \mu \phi(s) - G(t, s). \tag{17}$$

In the first step, we try finding the upper bounds.

We only need to show that there exists $\mu = \mu^* > 0$ such that

$$H(t,s)_{s \ge t} \ge 0, \qquad H(t,s)_{s \le t} \ge 0, \quad (t, s) \in [0, 1] \times [0, 1]. \tag{18}$$

Case 1. $s \in [0, \eta]$.

If $s = 0$, then $G(t, s) = 0$ and $\phi(s) = 0$; the conclusion is true.

If $s \in (0, \eta]$, then

$$\begin{aligned} H(t,s)_{s \ge t} &= \mu s (1-s) - \frac{[\sin \beta (1-s) + \delta \sin \beta (s - \eta)] \sin \beta t}{\beta (\sin \beta - \delta \sin \beta \eta)} \\ &\ge \mu t (1-s) - \frac{\sin \beta (1-s) \sin \beta t}{\beta (\sin \beta - \delta \sin \beta \eta)} \\ &\ge \left[\mu - \frac{\sin \beta (1-s) \sin \beta t}{(1-s)(\sin \beta - \delta \sin \beta \eta) \beta t} \right] t (1-s) \\ &\ge \left[\mu - \frac{\sin \beta (1-\eta)}{(1-\eta)(\sin \beta - \delta \sin \beta \eta)} \right] t (1-s), \end{aligned} \tag{19}$$

so, for

$$\mu \ge \mu_1 := \frac{\sin \beta (1-\eta)}{(1-\eta)(\sin \beta - \delta \sin \beta \eta)}, \tag{20}$$

we have $H(t,s)_{s \ge t} \ge 0$. Consider

$$\begin{aligned} H(t,s)_{s \le t} &= \mu s (1-s) - \frac{[\sin \beta (1-t) + \delta \sin \beta (t - \eta)] \sin \beta s}{\beta (\sin \beta - \delta \sin \beta \eta)} \\ &\ge \mu s (1-s) - \frac{[\sin \beta (1-s) + \delta \sin \beta (1-s)] \sin \beta s}{\beta (\sin \beta - \delta \sin \beta \eta)} \\ &\ge \mu s (1-s) - \frac{(1+\delta) \sin \beta s \sin \beta (1-s)}{\beta (\sin \beta - \delta \sin \beta \eta)} \\ &\ge \left[\mu - \frac{(1+\delta) \sin \beta s \sin \beta (1-s)}{\beta s (\sin \beta - \delta \sin \beta \eta)(1-s)} \right] s (1-s) \\ &\ge \left[\mu - \frac{(1+\delta) \sin \beta (1-\eta)}{(\sin \beta - \delta \sin \beta \eta)(1-\eta)} \right] s (1-s), \end{aligned} \tag{21}$$

so, for

$$\mu \ge \mu_2 := \frac{(1+\delta) \sin \beta (1-\eta)}{(\sin \beta - \delta \sin \beta \eta)(1-\eta)}, \tag{22}$$

we have $H(t,s)_{s \le t} \ge 0$.

Case 2. $s \in [\eta, 1]$.

If $s = 1$, then $G(t, s) = 0$ and $\phi(s) = 0$; the conclusion is true.

If $s \in [\eta, 1)$, then

$$\begin{aligned} H(t,s)_{s\geq t} &= \mu s(1-s) - \frac{\sin\beta(1-s)\sin\beta t}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\geq \mu t(1-s) - \frac{\sin\beta(1-s)\sin\beta t}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\geq \left[\mu - \frac{\sin\beta(1-s)\sin\beta t}{\beta t(1-s)(\sin\beta - \delta\sin\beta\eta)}\right] t(1-s) \\ &\geq \left[\mu - \frac{\sin\beta t}{t(\sin\beta - \delta\sin\beta\eta)}\right] t(1-s) \\ &\geq \left[\mu - \frac{\beta}{\sin\beta - \delta\sin\beta\eta}\right] t(1-s), \end{aligned} \tag{23}$$

so, for

$$\mu \geq \mu_3 := \frac{\beta}{\sin\beta - \delta\sin\beta\eta}, \tag{24}$$

we have $H(t,s)_{s\geq t} \geq 0$. Consider

$$\begin{aligned} &H(t,s)_{s\leq t} \\ &= \mu s(1-s) - \frac{\sin\beta s\sin\beta(1-t) + \delta\sin\beta\eta\sin\beta(t-s)}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\geq \mu s(1-s) - \frac{\sin\beta s\sin\beta(1-s) + \delta\sin\beta\eta\sin\beta(1-s)}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\geq \left[\mu - \frac{(\sin\beta s + \delta\sin\beta\eta)\sin\beta(1-s)}{(\sin\beta - \delta\sin\beta\eta)\beta s(1-s)}\right] s(1-s) \\ &\geq \left[\mu - \frac{(1+\delta)\sin\beta s}{(\sin\beta - \delta\sin\beta\eta)s}\right] s(1-s) \\ &\geq \left[\mu - \frac{(1+\delta)\sin\beta\eta}{(\sin\beta - \delta\sin\beta\eta)\eta}\right] s(1-s), \end{aligned} \tag{25}$$

so, for

$$\mu \geq \mu_4 := \frac{(1+\delta)\sin\beta\eta}{(\sin\beta - \delta\sin\beta\eta)\eta}, \tag{26}$$

we have $H(t,s)_{s\leq t} \geq 0$.

Thus, we take $\mu = \mu^* \geq \max\{\mu_1, \mu_2, \mu_3, \mu_4\}$ and then $H(t,s) \geq 0$ for $(t,s) \in [0,1]\times[0,1]$, and accordingly $\mu^*\phi(s) \geq G(t,s)$.

Set $g(s) := \mu^*\phi(s)$ and then $G(t,s) \leq g(s)$, $t, s \in [0,1]$.

In the next step, we try finding the lower bounds.

We only need to show that there exists $\mu = \mu_* > 0$ such that

$$H(t,s)_{s\geq t} \leq 0, \qquad H(t,s)_{s\leq t} \leq 0, \qquad (t,s) \in [\alpha, 1-\alpha]\times[0,1]. \tag{27}$$

Case 1. $s \in [0, \eta]$.

If $s = 0$, then $G(t,s) = 0$ and $\phi(s) = 0$; the conclusion is true.

If $s \in (0, \eta]$, then

$$\begin{aligned} H(t,s)_{s\geq t} &= \mu s(1-s) - \frac{[\sin\beta(1-s) + \delta\sin\beta(s-\eta)]\sin\beta t}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\leq \frac{\mu}{4} - \frac{\sin\beta(1-\eta) - \delta\sin\beta\eta}{\beta(\sin\beta - \delta\sin\beta\eta)}\sin\beta\alpha, \end{aligned} \tag{28}$$

so, for

$$\mu \leq \mu_5 := 4\sin\beta\alpha\frac{\sin\beta(1-\eta) - \delta\sin\beta\eta}{\beta(\sin\beta - \delta\sin\beta\eta)}, \tag{29}$$

we have $H(t,s)_{s\geq t} \leq 0$. Consider

$$\begin{aligned} H(t,s)_{s\leq t} &= \mu s(1-s) - \frac{[\sin\beta(1-t) + \delta\sin\beta(t-\eta)]\sin\beta s}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\leq \left[\mu - \frac{(\sin\beta\alpha - \delta\sin\beta\eta)\sin\beta s}{(\sin\beta - \delta\sin\beta\eta)\beta s}\right] s \\ &\leq \left[\mu - \frac{(\sin\beta\alpha - \delta\sin\beta\eta)\sin\beta\eta}{(\sin\beta - \delta\sin\beta\eta)\beta\eta}\right] s, \end{aligned} \tag{30}$$

so, for

$$\mu \leq \mu_6 := \frac{(\sin\beta\alpha - \delta\sin\beta\eta)\sin\beta\eta}{(\sin\beta - \delta\sin\beta\eta)\beta\eta}, \tag{31}$$

we have $H(t,s)_{s\leq t} \leq 0$.

Case 2. $s \in [\eta, 1]$.

If $s = 1$, then $G(t,s) = 0$ and $\phi(s) = 0$; the conclusion is true.

If $s \in [\eta, 1)$, then

$$\begin{aligned} H(t,s)_{s\geq t} &= \mu s(1-s) - \frac{\sin\beta(1-s)\sin\beta t}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\leq \left[\mu - \frac{\sin\beta(1-s)\sin\beta\alpha}{\beta(1-s)(\sin\beta - \delta\sin\beta\eta)}\right](1-s) \\ &\leq \left[\mu - \frac{\sin\beta(1-\eta)\sin\beta\alpha}{\beta(1-\eta)(\sin\beta - \delta\sin\beta\eta)}\right](1-s), \end{aligned} \tag{32}$$

so, for

$$\mu \leq \mu_7 := \frac{\sin\beta(1-\eta)\sin\beta\alpha}{\beta(1-\eta)(\sin\beta - \delta\sin\beta\eta)}, \tag{33}$$

we have $H(t,s)_{s\geq t} \leq 0$. Consider

$$\begin{aligned} H(t,s)_{s\leq t} &= \mu s(1-s) \\ &\quad - \frac{\sin\beta s \sin\beta(1-t) + \delta \sin\beta\eta \sin\beta(t-s)}{\beta(\sin\beta - \delta\sin\beta\eta)} \\ &\leq \left[\mu - \frac{\sin\beta s\sin\beta\alpha}{\beta s(\sin\beta - \delta\sin\beta\eta)}\right] s \\ &\leq \left[\mu - \frac{\sin\beta\sin\beta\alpha}{\beta(\sin\beta - \delta\sin\beta\eta)}\right] s, \end{aligned} \tag{34}$$

so, for

$$\mu \leq \mu_8 := \frac{\sin\beta\sin\beta\alpha}{\beta(\sin\beta - \delta\sin\beta\eta)}, \tag{35}$$

we have $H(t,s)_{s\leq t} \leq 0$.

Let $\mu = \mu_*$, where $0 < \mu_* \leq \min\{\mu_5, \mu_6, \mu_7, \mu_8\}$, we have $H(t,s) \leq 0$ for $(t,s) \in [\alpha, 1-\alpha] \times [0,1]$. Thus, $\mu_*\phi(s) \leq G(t,s)$, namely

$$\gamma g(s) \leq G(t,s), \quad (t,s) \in [\alpha, 1-\alpha] \times [0,1], \tag{36}$$

where $\gamma := \mu_*/\mu^* \in (0,1]$.

This completes the proof of the lemma. □

Let $E = C[0,1]$ be equipped with norm $\|u\| = \max_{t\in[0,1]}|u(t)|$; then $(E, \|\cdot\|)$ is a real Banach space.

Define the cone P by

$$P = \left\{u \in E : u(t) \geq 0, \min_{t\in[\alpha,1-\alpha]} u(t) \geq \gamma\|u\|\right\}; \tag{37}$$

then P is a nonempty closed subset of E.

For $u, v \in E$, we write $u \leq v$ if $u(t) \leq v(t)$ for any $t \in [0,1]$. For any $r > 0$, let $K_r = \{u \in E : \|u\| < r\}$ and $\partial K_r = \{u \in E : \|u\| = r\}$.

Define the operator $T : P \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s) q(s) f(s, u(s))\, ds. \tag{38}$$

Lemma 4. *Assume that (H_1)–(H_4) hold; then the operator $T : P \to P$ is completely continuous.*

Proof. For for all $u \in P$, it follows from the definition of T and Lemma 3 that

$$\begin{aligned} 0 \leq Tu(t) &= \int_0^1 G(t,s) q(s) f(s,u(s))\, ds \\ &\leq \int_0^1 g(s) q(s) f(s,u(s))\, ds, \quad t \in [0,1]. \end{aligned} \tag{39}$$

So,

$$\|Tu\| \leq \int_0^1 g(s) q(s) f(s,u(s))\, ds. \tag{40}$$

In view of Lemma 3 and (40), we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) q(s) f(s,u(s))\, ds \\ &\geq \gamma \int_0^1 g(s) q(s) f(s,u(s))\, ds, \quad t \in [\alpha, 1-\alpha]. \end{aligned} \tag{41}$$

And so

$$\min_{\alpha\leq t\leq 1-\alpha} Tu(t) \geq \gamma\|Tu\|, \tag{42}$$

which shows that $T(P) \subset P$. By the Ascoli-Arzela theorem, it is easy to show that $T : P \to P$ is completely continuous.

In view of Lemmas 2 and 3, it is easy to see that $u \in E$ is a solution of BVP (5) if and only if $u \in E$ is a fixed point of the operator λT.

The proofs of our main results are based on the fixed point index theory. The following three well-known lemmas in [10, 11] are needed in our argument. □

Lemma 5. *Let E be a Banach space and $P \subset E$ a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $T : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If there exists $x_0 \in P \setminus \{\theta\}$ such that $x - Tx \neq \mu x_0$, for all $x \in P \cap \partial\Omega$ and $\mu \geq 0$, then the fixed point index $i(T, P \cap \Omega, P) = 0$.*

Lemma 6. *Let E be a Banach space and $P \subset E$ a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $T : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If $\inf_{x\in P\cap\partial\Omega}\|Tx\| > 0$ and $\mu Tx \neq x$, for $x \in P \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index $i(T, P \cap \Omega, P) = 0$.*

Lemma 7. *Let E be a Banach space and $P \subset E$ a cone in E. Assume that Ω is a bounded open subset of E with $\theta \in \Omega$. Suppose that $T : P \cap \overline{\Omega} \to P$ is a completely continuous operator. If $Tx \neq \mu x$ for all $x \in P \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index $i(T, P \cap \Omega, P) = 1$.*

3. Proofs of the Main Results

For convenience, we firstly introduce the following notations.

$\Phi = \{(\lambda, u) : \lambda > 0$ and $u \in P$ is a positive solution of BVP (1)$\}$;

$\Lambda = \{\lambda > 0 :$ there exists $u \in P$ such that $(\lambda, u) \in \Phi\}$;

$\lambda^* = \sup\Lambda$; $\lambda_* = \inf\Lambda$; $A = \int_\alpha^{1-\alpha} g(s)q(s)ds$.

Lemma 8. *Suppose that (H_1)–(H_3) hold and $f_0 = \infty$. Then $\Phi \neq \emptyset$.*

Proof. Let $R > 0$ be fixed; then we can choose $\lambda_0 > 0$ small enough such that $\lambda_0 \sup_{u\in P\cap\overline{K}_R}\|Tu\| < R$. It is easy to see that

$$\lambda_0 Tu \neq \mu u, \quad \forall u \in P \cap \partial K_R, \ \mu \geq 1. \tag{43}$$

By Lemma 7, it follows that

$$i(\lambda_0 T, P \cap K_R, P) = 1. \tag{44}$$

From $f_0 = \infty$, it follows that there exists $r \in (0, R)$ such that

$$f(t,x) \geq \frac{1}{\lambda_0 \gamma^2 A} x, \quad \forall x \in [0,r], \ t \in [\alpha, 1-\alpha]. \tag{45}$$

We may suppose that $\lambda_0 T$ has no fixed point on $P \cap \partial K_r$. Otherwise, the proof is finished. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in \partial K_1$. We claim that

$$u \neq \lambda_0 Tu + \mu e, \quad \forall u \in P \cap \partial K_r, \ \mu \geq 0. \tag{46}$$

In fact, if not, there exist $u_1 \in P \cap \partial K_r$ and $\mu_1 \geq 0$ such that $u_1 = \lambda_0 T u_1 + \mu_1 e$; then $\mu_1 > 0$. For $u_1 \in P \cap \partial K_r$ and $\mu_1 > 0$, by Lemma 3 and (45), for $t \in [\alpha, 1-\alpha]$, we have

$$\begin{aligned} u_1(t) &= (\lambda_0 T u_1)(t) + \mu_1 e(t) \\ &= \lambda_0 \int_0^1 G(t,s) q(s) f(s, u_1(s)) \, ds + \mu_1 \\ &\geq \gamma \lambda_0 \int_\alpha^{1-\alpha} g(s) q(s) f(s, u_1(s)) \, ds + \mu_1 \\ &\geq \gamma \lambda_0 \frac{1}{\lambda_0 \gamma^2 A} \int_\alpha^{1-\alpha} g(s) q(s) u_1(s) \, ds + \mu_1 \\ &\geq \frac{1}{A} \|u_1\| \int_0^1 g(s) q(s) \, ds + \mu_1 \\ &= \|u_1\| + \mu_1 = r + \mu_1; \end{aligned} \tag{47}$$

we get $r \geq r + \mu_1$, which is a contradiction. Thus, (46) holds. It follows from Lemma 5 that

$$i(\lambda_0 T, P \cap K_r, P) = 0. \tag{48}$$

By virtue of the additivity of the fixed point index, from (44) and (48), we have

$$\begin{aligned} &i\left(\lambda_0 T, P \cap \left(K_R \setminus \overline{K}_r\right), P\right) \\ &\quad = i(\lambda_0 T, P \cap K_R, P) - i(\lambda_0 T, P \cap K_r, P) = 1, \end{aligned} \tag{49}$$

which implies that the nonlinear operator $\lambda_0 T$ has one fixed point $u_0 \in P \cap (K_R \setminus \overline{K}_r)$. Therefore, $(\lambda_0, u_0) \in \Phi$. The proof is complete. □

Lemma 9. *Suppose (H_1)–(H_4) hold and $f_0 = f_\infty = \infty$. Then $0 < \lambda^* < \infty$.*

Proof. By Lemma 8, it is easy to see that $\lambda^* > 0$. It follows from (H_4) and $f_0 = f_\infty = \infty$ that there exists $C > 0$ such that $f(t,x) \geq Cx$ for all $x \geq 0$ and $t \in [0,1]$. Let $(\lambda, u) \in \Phi$; by the definition of cone P and Lemma 2, for $t \in [\alpha, 1-\alpha]$, we obtain that

$$\begin{aligned} u(t) = (\lambda T u)(t) &= \lambda \int_0^1 G(t,s) q(s) f(s, u(s)) \, ds \\ &\geq \lambda \gamma C \int_0^1 g(s) q(s) u(s) \, ds \\ &\geq \lambda \gamma^2 C \|u\| \int_\alpha^{1-\alpha} g(s) q(s) \, ds = \lambda \gamma^2 AC \|u\|. \end{aligned} \tag{50}$$

So, $\|u\| \geq \lambda \gamma^2 AC \|u\|$. We get $\lambda \leq (\gamma^2 AC)^{-1}$. This completes the proof of lemma. □

Lemma 10. *Suppose (H_1)–(H_4) hold and $f_0 = f_\infty = \infty$. Then $(0, \lambda^*) \subset \Lambda$. Moreover, for any $\lambda \in (0, \lambda^*)$, BVP (1) has at least two positive solutions.*

Proof. For any fixed $\lambda \in (0, \lambda^*)$, we prove that $\lambda \in \Lambda$. By the definition of λ^*, there exists $\lambda_2 \in \Lambda$, such that $\lambda < \lambda_2 \leq \lambda^*$ and $(\lambda_2, u_2) \in \Phi$. Let $R < \min_{t \in [0,1]} u_2(t)$ be fixed. From the proof of Lemma 8, we see that there exist $\lambda_1 < \lambda$, $r < R$, and $u_1(t) \in P \cap (K_R \setminus \overline{K}_r)$ such that $(\lambda_1, u_1) \in \Phi$. It is easy to see that $0 < u_1(t) < u_2(t)$ for all $t \in [0,1]$. Then, by (H_2), we have

$$\begin{aligned} -u_1''(t) - \beta^2 u_1(t) &= \lambda_1 q(t) f(t, u_1(t)), \quad t \in (0,1), \\ -u_2''(t) - \beta^2 u_2(t) &= \lambda_2 q(t) f(t, u_2(t)), \quad t \in (0,1). \end{aligned} \tag{51}$$

Consider now the modified BVP:

$$\begin{gathered} -u''(t) - \beta^2 u(t) = \lambda q(t) f_1(t, u(t)), \quad t \in (0,1) \\ u(0) = 0, \qquad u(1) = \delta u(\eta), \end{gathered} \tag{52}$$

where

$$f_1(t, u(t)) = \begin{cases} f(t, u_1(t)), & u(t) \leq u_1(t), \\ f(t, u(t)), & u_1(t) < u(t) < u_2(t), \\ f(t, u_2(t)), & u(t) \geq u_2(t). \end{cases} \tag{53}$$

Clearly, the function λf_1 is bounded for $t \in [0,1]$ and $u \in P$ and is continuous in u. Define the operator $T_1 : E \to E$ by

$$\begin{gathered} (T_1 u)(t) = \int_0^1 G(t,s) q(s) f_1(s, u(s)) \, ds, \\ u \in E, \ t \in [0,1]. \end{gathered} \tag{54}$$

Then $T_1 : P \to P$ is completely continuous and all the fixed points of operator λT_1 are the solutions for BVP (52). It is easy to see that there exists $r_0 > \|u_2\|$ such that $\|\lambda T_1 u\| < r_0$ for any $u \in P$. From Lemma 7, we have

$$i\left(\lambda T_1, P \cap K_{r_0}, P\right) = 1. \tag{55}$$

Let

$$U = \{u \in P : u_1(t) < u(t) < u_2(t), \forall t \in [0,1]\}. \tag{56}$$

We claim that if $u \in P$ is a fixed point of operator λT_1, then $u \in U$. In fact, if $u = \lambda T_1 u$, then

$$\begin{aligned} u(t) = (\lambda T_1 u)(t) &= \lambda \int_0^1 G(t,s) q(s) f_1(s, u(s)) \, ds \\ &< \lambda_2 \int_0^1 G(t,s) q(s) f(s, u_2(s)) \, ds \\ &= (\lambda_2 T u_2)(t) = u_2(t), \\ u(t) = (\lambda T_1 u)(t) &= \lambda \int_0^1 G(t,s) q(s) f_1(s, u(s)) \, ds \\ &> \lambda_1 \int_0^1 G(t,s) q(s) f(s, u_1(s)) \, ds \\ &= (\lambda_1 T u_1)(t) = u_1(t). \end{aligned} \tag{57}$$

From the excision property of the fixed point index and (55), we obtain that

$$i(\lambda T_1, U, P) = i\left(\lambda T_1, P \cap K_{r_0}, P\right) = 1. \quad (58)$$

From the definition of T_1, we know that $T_1 = T$ on $\overline{U}$. Then,

$$i(\lambda T, U, P) = 1. \quad (59)$$

Hence, the nonlinear operator λT has at least fixed point $v_1 \in U$. Then v_1 is one positive solution of BVP (1). This gives $\lambda \in \Lambda$, $(\lambda, v_1) \in \Phi$ and $(0, \lambda) \subset \Lambda$.

We now find the second positive solution of BVP (1). By $f_\infty = \infty$ and the continuity of $f(t,x)$ with respect to x, there exists $C > 0$ such that

$$f(t,x) \geq \frac{2x}{\lambda\gamma^2 A} - \frac{C}{\gamma A}, \quad \forall x \geq 0,\ t \in [0,1]. \quad (60)$$

For $e(t) \equiv 1$, let

$$\Omega = \{u \in P : \text{there exists } \tau \geq 0 \text{ such that } u = \lambda T u + \tau e\}. \quad (61)$$

We claim that Ω is bounded in E. In fact, for any $u \in \Omega$, it follows from Lemma 3 and (60) that

$$\begin{aligned} u(t) &= (\lambda T u)(t) + \tau e(t) = (\lambda T u)(t) + \tau \\ &\geq \lambda \int_0^1 G(t,s) q(s) f(s, u(s))\, ds \\ &\geq \lambda\gamma \int_\alpha^{1-\alpha} g(s) q(s) \left[\frac{2u(s)}{\lambda\gamma^2 A} - \frac{C}{\gamma A}\right] ds \\ &\geq \lambda\gamma \int_\alpha^{1-\alpha} g(s) q(s) \left[\frac{2\gamma\|u\|}{\lambda\gamma^2 A} - \frac{C}{\gamma A}\right] ds \\ &= 2\|u\| - \lambda C, \quad t \in [\alpha, 1-\alpha]. \end{aligned} \quad (62)$$

This implies $\|u\| \leq \lambda C$. Thus Ω is bounded in E. Therefore there exists $R_1 > \|u_2\|$ such that

$$u \neq \lambda T u + \tau e, \quad \forall u \in P \cap \partial K_{R_1},\ \tau \geq 0. \quad (63)$$

By Lemma 5, we get that

$$i\left(\lambda T, P \cap K_{R_1}, P\right) = 0. \quad (64)$$

Using a similar argument as in deriving (48), we have that

$$i\left(\lambda T, P \cap K_{r_1}, P\right) = 0, \quad (65)$$

where $0 < r_1 < \min_{t \in [0,1]} u_1(t)$. According to the additivity of the fixed point index and by (59), (64), and (65), we have

$$\begin{aligned} & i\left(\lambda T, P \cap \left(K_{R_1} \setminus \left(\overline{U} \cup \overline{K}_{r_1}\right), P\right)\right) \\ &= i\left(\lambda T, P \cap K_{R_1}, P\right) \\ &\quad - i(\lambda T, U, P) - i\left(\lambda T, P \cap K_{r_1}, P\right) = -1, \end{aligned} \quad (66)$$

which implies that the nonlinear operator λT has at least one fixed point $v_2 \in P \cap \left(K_{R_1} \setminus (\overline{U} \cup \overline{K}_{r_1})\right)$. Thus, BVP (1) has another positive solution. The proof is complete. □

Lemma 11. *Suppose* (H_1)–(H_4) *hold and* $f_0 = f_\infty = \infty$. *Then* $\Lambda = (0, \lambda^*]$.

Proof. In view of Lemma 10, it suffices to prove that $\lambda^* \in \Lambda$. By the definition of λ^*, we can choose $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \geq (\lambda^*/2)(n = 1, 2, \ldots)$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. By the definition of Λ, there exists $\{u_n\} \subset P \setminus \{\theta\}$ such that $(\lambda_n, u_n) \in \Phi$. We now show that $\{u_n\}$ is bounded. Supposing the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from $\{u_n\} \subset P \setminus \{\theta\}$ that $u_n \geq \gamma\|u_n\|$ for all $t \in [0,1]$. Choose sufficiently large τ such that

$$\frac{\lambda^* \gamma^2 A \tau}{2} > 1. \quad (67)$$

By $f_\infty = \infty$, there exists $R > 0$ such that $f(t,u) \geq \tau u$ for all $u > \gamma R$ and $t \in [0,1]$. Since $\|u_n\| \to \infty$ as $n \to \infty$, there exists sufficiently large n_0 such that $\|u_{n_0}\| \geq R$. Thus, for $t \in [\alpha, 1-\alpha]$, we have

$$\begin{aligned} u_{n_0}(t) &= \left(\lambda_{n_0} T u_{n_0}\right)(t) = \lambda_{n_0} \int_0^1 G(t,s) q(s) f\left(s, u_{n_0}(s)\right) ds \\ &\geq \frac{\lambda^*}{2} \gamma\tau \int_\alpha^{1-\alpha} g(s) q(s) u_{n_0}(s)\, ds, \\ &\geq \frac{\lambda^*}{2} \gamma^2 \tau \left\|u_{n_0}\right\| \int_\alpha^{1-\alpha} g(s) q(s)\, ds = \frac{\lambda^*}{2} \gamma^2 \tau A \left\|u_{n_0}\right\|. \end{aligned} \quad (68)$$

This gives

$$\frac{\lambda^* \gamma^2 A \tau}{2} \leq 1, \quad (69)$$

which contradicts the choice of τ. Hence, $\{u_n\}$ is bounded. It follows from the completely continuity of T that $\{Tu_n\}$ is equicontinuous; that is, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\begin{aligned} \left|u_n(t_1) - u_n(t_2)\right| &= \lambda_n \left|(T u_n)(t_1) - (T u_n)(t_2)\right| \\ &< \lambda_n \varepsilon \leq \lambda^* \varepsilon, \end{aligned} \quad (70)$$

where $n = 1, 2, \ldots, t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$. Then $\{u_n\}$ is equicontinuous. According to the Ascoli-Arzela theorem, $\{u_n\}$ is relatively compact. Hence, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u^* \in P$ such that $u_n \to u^*$ as $n \to \infty$. By $u_n = \lambda_n T u_n$, letting $n \to \infty$, we obtain that $u^* = \lambda^* T u^*$. If $u^* = \theta$, using a similar argument as in deriving (69) and by $f_0 = \infty$, we also get a contradiction. Then $u^* \in P \setminus \{\theta\}$, and so $\lambda^* \in \Lambda$. This completes the proof. □

Proof of Theorem 1. Theorem 1 readily follows from Lemmas 8, 9, 10, and 11. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] V. A. Il'in and E. I. Moiseev, "Non-local boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects," *Differential Equations*, vol. 23, pp. 803–810, 1987.

[2] C. P. Gupta, "Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation," *Journal of Mathematical Analysis and Applications*, vol. 168, no. 2, pp. 540–551, 1992.

[3] B. Liu, L. Liu, and Y. Wu, "Positive solutions for a singular second-order three-point boundary value problem," *Applied Mathematics and Computation*, vol. 196, no. 2, pp. 532–541, 2008.

[4] R. Ma, "Multiplicity of positive solutions for second-order three-point boundary value problems," *Computers & Mathematics with Applications*, vol. 40, no. 2-3, pp. 193–204, 2000.

[5] R. Ma and H. Wang, "Positive solutions of nonlinear three-point boundary-value problems," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 1, pp. 216–227, 2003.

[6] H. B. Thompson and C. Tisdell, "Three-point boundary value problems for second-order, ordinary, differential equations," *Mathematical and Computer Modelling*, vol. 34, no. 3-4, pp. 311–318, 2001.

[7] J. Webb, "Positive solutions of some three point boundary valueproblems via fixed point index theory," *Nonlinear Analysis*, vol. 47, pp. 4319–4332, 2001.

[8] X. Han, "Positive solutions for a three-point boundary value problem at resonance," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 556–568, 2007.

[9] X. Han, "Positive solutions for a three-point boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 3, pp. 679–688, 2007.

[10] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[11] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, New York, NY, USA, 1988.