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Research Article

Oscillation of Certain Emden-Fowler Dynamic Equations on Time Scales

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We deal with the oscillation of a generalized Emden-Fowler dynamic equation in the form $(r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t))^{\Delta} + f(t, x(\delta(t))) = 0$. We establish some new oscillation criteria for the equation, which improve some of the main results of (H. Liu and P. Liu, 2013). Some examples are given to illustrate the new results.

1. Introduction

The theory of time scales has attracted a great deal of attention since it was first introduced by Hilger [1] in order to unify continuous and discrete analysis. For completeness, we recall the following concepts related to the notion of time scales; see [2, 3] for more details. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In this paper, since we shall be concerned with the oscillatory behavior of solutions, we shall also assume that $\sup \mathbb{T} = \infty$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. The forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},\qquad \rho(t) := \sup\{s \in \mathbb{T}, s < t\}, \tag{1}$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$; here $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ and $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for the time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$ and left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s} \tag{2}$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \tag{3}$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is rd-continuous is denoted by $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

In this paper, we consider the oscillatory behavior of the nontrivial solutions of the second-order Emden-Fowler dynamic equation of the form

$$\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1}Z^{\Delta}(t)\right)^{\Delta} + f(t, x(\delta(t))) = 0,\qquad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4}$$

on an arbitrary time scale $\mathbb{T}$, with $\sup \mathbb{T} = \infty$, where $Z(t) = x(t) + p(t)x(\tau(t))$, and $\alpha > 0$ is a constant. Throughout this paper, we always assume that

- $(A1)$ $r \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ with $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t = \infty$;
- $(A2)$ $p \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ with $0 \le p(t) < 1$;
- $(A3)$ $\tau, \delta \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \le t$, $\delta(t) \le t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$;
- $(A4)$ $f(t,u) \in C([t_0,\infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ is a continuous function such that $uf(t,u) > 0$, for all $u \ne 0$ and there exists a positive right-dense continuous function $q(t)$ defined on $[t_0,\infty)_{\mathbb{T}}$ such that $|f(t,u)| \ge q(t)|u|^{\beta}$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and for all $u \in \mathbb{R}$, where $\beta > 0$ is a constant.

By a solution of (4), we mean a nontrivial real-valued function $x \in C^1_{\mathrm{rd}}([T_x,\infty),\mathbb{R})$, $T_x \ge t_0$ which has the property that $r(t)(Z^{\Delta}(t))^{\alpha} \in C^1_{\mathrm{rd}}([T_x,\infty),\mathbb{R})$ and satisfies (4) that holds on $[T_x,\infty)$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (4) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. The equation itself is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been an increasing interest in studying the oscillation behavior of second-order dynamic equations on time scales; see for example [4–10] and the references contained therein. In [5], the authors presented some criteria for the oscillation and asymptotic behavior of (4) in the case where

$$\alpha \ge \beta > 0, \qquad \delta^{\Delta}(t) > 0. \tag{5}$$

Also, we note further that in the proof of [5], the authors used the chain rule in the form

$$\begin{gathered} (x(\delta(t)))^{\Delta} = x^{\Delta}(\delta(t))\,\delta^{\Delta}(t), \\ \delta(\sigma(t)) = \sigma(\delta(t)). \end{gathered} \tag{6}$$

So the natural question which arises is can we find some new oscillation conditions for (4) which do not require (5) and (6) and, in addition, improve the main results in [5]?

The purpose of this paper is to give an affirmative answer to this question. That is, we shall establish some new criteria for the oscillation of (4) which improve the main results in [5]. We also demonstrate that our results cover certain cases which were not covered in [5]. Finally, we give two examples to illustrate the main results.

2. Main Results

For notational simplicity, define

$$\begin{gathered} R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s; \\ \theta(t,u) := \left(\int_{u}^{t} r^{-1/\alpha}(s)\,\Delta s\right)^{-1} \int_{u}^{\delta(t)} r^{-1/\alpha}(s)\,\Delta s, \\ t > u \ge t_0. \end{gathered} \tag{7}$$

We begin with the following lemmas.

Lemma 1. *Assume that (4) has a positive solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then for sufficiently large T, one has*

$$\begin{gathered} Z(t) > 0, \qquad Z^{\Delta}(t) > 0, \\ \left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \le 0, \\ t \in [T,\infty)_{\mathbb{T}}. \end{gathered} \tag{8}$$

Proof. Assume that (4) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that there exists a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ on $[T,\infty)_{\mathbb{T}}$. Then it follows that $Z(t) \ge x(t) > 0$. From (4), we have

$$\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} = -q(t)\,x^{\beta}(\delta(t)) \le 0. \tag{9}$$

Hence, $r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t)$ is decreasing on $[T,\infty)_{\mathbb{T}}$. We now claim that $Z^{\Delta}(t) > 0$ on $t \in [T,\infty)_{\mathbb{T}}$. If not, then there exists a $t_1 \in [T,\infty)_{\mathbb{T}}$ such that $Z^{\Delta}(t_1) < 0$. Therefore,

$$\begin{aligned} & r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t) \\ &\quad \le r(t_1)\left|Z^{\Delta}(t_1)\right|^{\alpha-1} Z^{\Delta}(t_1) := -c < 0, \quad t \ge t_1, \end{aligned} \tag{10}$$

that is,

$$Z^{\Delta}(t) \le -\left[\frac{c}{r(t)}\right]^{1/\alpha}. \tag{11}$$

Integrating (11) from t_1 to t, we find from $(A1)$ that

$$Z(t) \le Z(t_1) - c^{1/\alpha}\int_{t_1}^{t}\frac{1}{r^{1/\alpha}(s)}\Delta s \longrightarrow -\infty \quad \text{as } t \to \infty, \tag{12}$$

which implies that $Z(t)$ is eventually negative. This contradicts the fact that $Z(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. Thus, $Z^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. This completes the proof. □

Lemma 2. *Assume that (4) has a positive solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then for sufficiently large t_1,*

$$\frac{Z(\delta(t))}{Z(t)} \ge \theta(t,t_1), \quad t \ge t_1. \tag{13}$$

Proof. As in the proof of Lemma 1, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so that

$$\begin{gathered} Z(t) > 0, \qquad Z^{\Delta}(t) > 0, \\ \left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \le 0, \\ t \in [t_1,\infty)_{\mathbb{T}}. \end{gathered} \tag{14}$$

Since $r(t)(Z^{\Delta}(t))^{\alpha}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, we can choose $t_2 > t_1$ so that $\delta(t) \geq t_1$, for $t \geq t_2$. Then

$$Z(t) - Z(\delta(t)) = \int_{\delta(t)}^{t} \frac{1}{r^{1/\alpha}(s)} \left[r(s)\left(Z^{\Delta}(s)\right)^{\alpha}\right]^{1/\alpha} \Delta s \leq \left[r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}\right]^{1/\alpha} \int_{\delta(t)}^{t} \frac{1}{r^{1/\alpha}(s)} \Delta s, \tag{15}$$

consequently,

$$\frac{Z(t)}{Z(\delta(t))} \leq 1 + \frac{\left[r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}\right]^{1/\alpha}}{Z(\delta(t))} \int_{\delta(t)}^{t} \frac{1}{r^{1/\alpha}(s)} \Delta s. \tag{16}$$

Also, we have, for $t \geq t_2$

$$\begin{aligned} Z(\delta(t)) &> Z(\delta(t)) - Z(t_1) \\ &= \int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \left[r(s)\left(Z^{\Delta}(s)\right)^{\alpha}\right]^{1/\alpha} \Delta s \\ &\geq \left[r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}\right]^{1/\alpha} \int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s, \end{aligned} \tag{17}$$

hence,

$$\frac{\left[r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}\right]^{1/\alpha}}{Z(\delta(t))} \leq \left(\int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s\right)^{-1}. \tag{18}$$

Therefore, by combining inequalities (16) and (18) we have

$$\frac{Z(t)}{Z(\delta(t))} \leq \left(\int_{t_1}^{t} \frac{1}{r^{1/\alpha}(s)} \Delta s\right)\left(\int_{t_1}^{\delta(t)} \frac{1}{r^{1/\alpha}(s)} \Delta s\right)^{-1}, \tag{19}$$

from which we have

$$\frac{Z(\delta(t))}{Z(t)} \geq \theta(t, t_1). \tag{20}$$

This completes the proof. □

Lemma 3 (see [11]). *Let $\phi(u) = au - bu^{(\lambda+1)/\lambda}$, where $a \geq 0$, $b > 0$, $\lambda > 0$, and $u \in [0,\infty)$. Then*

$$\phi(u) \leq \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}} \frac{a^{\lambda+1}}{b^{\lambda}}. \tag{21}$$

For the positive solution $x(t)$ of (4), it follows from $Z(t)$ and Lemma 1 that, for $t \geq T$,

$$\begin{aligned} x(t) &= Z(t) - p(t)x(\tau(t)) \geq Z(t) - p(t)Z(\tau(t)) \\ &\geq (1 - p(t))Z(t), \end{aligned} \tag{22}$$

which implies

$$x^{\beta}(\delta(t)) \geq (1 - p(\delta(t)))^{\beta} Z^{\beta}(\delta(t)). \tag{23}$$

Combining (23) ($A4$), (4) one obtains

$$\begin{aligned} \left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} &\leq -q(t)(1 - p(\delta(t)))^{\beta} Z^{\beta}(\delta(t)) \\ &= -\overline{p}(t) Z^{\beta}(\delta(t)), \end{aligned} \tag{24}$$

where $\overline{p}(t) := q(t)(1 - p(\delta(t)))^{\beta}$.

One may now state and prove the main results. In these, one shall consider the two cases $\alpha \geq \beta$ and $\alpha < \beta$.

Theorem 4. *Let $\alpha \geq \beta$. Assume that there exist a positive rd-continuous differentiable function $\xi(t)$ and a constant $M > 0$ such that, for some $T \geq t_0$,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\xi(s)\overline{p}(s)\theta^{\beta}(s,T) - \frac{M\alpha^{\alpha} r(s)(R(\sigma(s)))^{\alpha-\beta}\left(\xi_{+}^{\Delta}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)} \right) \Delta s = \infty, \tag{25}$$

where $\xi_{+}^{\Delta}(s) := \max\{\xi^{\Delta}(s), 0\}$. Then (4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution $x(t)$ of (4) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that there exists a $T \in [t_0,\infty)_{\mathbb{T}}$ (sufficiently large) such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ on $[T,\infty)_{\mathbb{T}}$, and $Z(t)$ satisfies the conclusions of Lemmas 1 and 2 on $[T,\infty)_{\mathbb{T}}$. Consider the Riccati substitution

$$w(t) = \xi(t) \frac{r(t)\left(Z^{\Delta}(t)\right)^{\alpha}}{Z^{\beta}(t)}, \quad t \geq T. \tag{26}$$

Then $w(t) > 0$. By [2, Theorem 1.20], Lemma 2, and (24), we have

$$\begin{aligned} w^{\Delta}(t) &= \left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \frac{\xi(t)}{Z^{\beta}(t)} + \left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\sigma} \left(\frac{\xi(t)}{Z^{\beta}(t)}\right)^{\Delta} \\ &\leq -\xi(t)\overline{p}(t)\left(\frac{Z(\delta(t))}{Z(t)}\right)^{\beta} + \left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\sigma} \times \frac{\xi^{\Delta}(t)Z^{\beta}(t) - \xi(t)\left(Z^{\beta}(t)\right)^{\Delta}}{Z^{\beta}(t)\left(Z^{\beta}(t)\right)^{\sigma}} \\ &\leq -\xi(t)\overline{p}(t)\theta^{\beta}(t,T) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w^{\sigma}(t) - \frac{\xi(t)}{\xi(\sigma(t))} w^{\sigma}(t) \frac{\left(Z^{\beta}(t)\right)^{\Delta}}{Z^{\beta}(t)}. \end{aligned} \tag{27}$$

By the Pötzsche chain rule [2, Theorem 1.87],

$$\begin{aligned}\left(Z^{\beta}(t)\right)^{\Delta} &= \beta\left\{\int_0^1 [(1-h)Z(t)+hZ(\sigma(t))]^{\beta-1}\mathrm{d}h\right\}Z^{\Delta}(t)\\ &\geq \begin{cases}\beta(Z(t))^{\beta-1}Z^{\Delta}(t), & \beta>1,\\ \beta(Z(\sigma(t)))^{\beta-1}Z^{\Delta}(t), & 0<\beta\leq 1.\end{cases}\end{aligned} \tag{28}$$

Thus,

$$\frac{\left(Z^{\beta}(t)\right)^{\Delta}}{Z^{\beta}(t)} \geq \begin{cases}\beta\dfrac{Z^{\Delta}(t)}{Z(t)}, & \beta>1,\\ \beta\dfrac{(Z(\sigma(t)))^{\beta-1}}{Z^{\beta}(t)}Z^{\Delta}(t), & 0<\beta\leq 1.\end{cases} \tag{29}$$

Noting that $Z(t)$ is increasing on $[T,\infty)_{\mathbb{T}}$, we get $Z(t) \leq Z(\sigma(t))$ for $t\in[T,\infty)_{\mathbb{T}}$. Thus,

$$\frac{\left(Z^{\beta}(t)\right)^{\Delta}}{Z^{\beta}(t)} \geq \beta\frac{Z^{\Delta}(t)}{Z(\sigma(t))}. \tag{30}$$

Substituting (30) into (27), we obtain

$$\begin{aligned}w^{\Delta}(t) \leq & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w^{\sigma}(t)\\ & -\frac{\beta\xi(t)}{\xi(\sigma(t))}w^{\sigma}(t)\frac{Z^{\Delta}(t)}{Z(\sigma(t))}, \quad t\geq T.\end{aligned} \tag{31}$$

Noting that $r^{1/\alpha}(t)Z^{\Delta}(t)$ is decreasing, we have $r^{1/\alpha}(t)Z^{\Delta}(t) \geq (r^{1/\alpha}(t)Z^{\Delta}(t))^{\sigma}$. It follows from the definition of $w(t)$ that

$$Z^{\Delta}(t) \geq \frac{1}{(r(t)\xi(\sigma(t)))^{1/\alpha}}w^{1/\alpha}(\sigma(t))Z^{\beta/\alpha}(\sigma(t)). \tag{32}$$

Substituting (32) into (31), we obtain

$$\begin{aligned}w^{\Delta}(t) \leq & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w^{\sigma}(t)\\ & -\frac{\beta\xi(t)w^{(\alpha+1)/\alpha}(\sigma(t))}{r^{1/\alpha}(t)\xi^{(\alpha+1)/\alpha}(\sigma(t))Z^{(\alpha-\beta)/\alpha}(\sigma(t))}, \quad t\geq T.\end{aligned} \tag{33}$$

Since $r^{1/\alpha}(t)Z^{\Delta}(t)$ is decreasing, there exists a constant $M_1 > 0$ such that $r^{1/\alpha}(t)Z^{\Delta}(t) \leq M_1$ for $t \geq T$, which implies

$$Z^{\Delta}(t) \leq \frac{M_1}{r^{1/\alpha}(t)}, \quad t\geq T. \tag{34}$$

Integrating both sides of (34) from T to t, we get

$$\begin{aligned}Z(t) &\leq Z(T)+M_1(R(t)-R(T))\\ &= R(t)\left(M_1+\frac{Z(T)-M_1R(T)}{R(t)}\right).\end{aligned} \tag{35}$$

Hence, there exists a $T_1 \geq T$ such that $Z(t) \leq (M_1+1)R(t)$ for $t\geq T_1$. Then,

$$\begin{aligned}Z^{(\alpha-\beta)/\alpha}(\sigma(t)) &\leq \left(M_1+1\right)^{(\alpha-\beta)/\alpha}(R(\sigma(t)))^{(\alpha-\beta)/\alpha}\\ &= M_2(R(\sigma(t)))^{(\alpha-\beta)/\alpha}, \quad t\geq T_1,\end{aligned} \tag{36}$$

where $M_2 = (M_1+1)^{(\alpha-\beta)/\alpha}$. Substituting (36) into (33), we get

$$\begin{aligned}w^{\Delta}(t) \leq & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w^{\sigma}(t)\\ & -\frac{\beta\xi(t)w^{(\alpha+1)/\alpha}(\sigma(t))}{M_2r^{1/\alpha}(t)\xi^{(\alpha+1)/\alpha}(\sigma(t))(R(\sigma(t)))^{(\alpha-\beta)/\alpha}}\\ = & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w^{\sigma}(t)\\ & -\Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t))\\ \leq & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)+\frac{\xi_+^{\Delta}(t)}{\xi(\sigma(t))}w^{\sigma}(t)\\ & -\Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t)), \quad t\geq T_1,\end{aligned} \tag{37}$$

where

$$\Psi(t) := \frac{\beta\xi(t)}{M_2r^{1/\alpha}(t)\xi^{(\alpha+1)/\alpha}(\sigma(t))(R(\sigma(t)))^{(\alpha-\beta)/\alpha}}. \tag{38}$$

Taking $a = \xi_+^{\Delta}(t)/\xi(\sigma(t))$, $b = \Psi(t)$, from Lemma 3 and (37), we obtain

$$\begin{aligned}w^{\Delta}(t) \leq & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)\\ & +\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}\Psi^{\alpha}(t)}\left(\frac{\xi_+^{\Delta}(t)}{\xi(\sigma(t))}\right)^{\alpha+1}\\ = & -\xi(t)\overline{p}(t)\theta^{\beta}(t,T)\\ & +\frac{M\alpha^{\alpha}r(t)(R(\sigma(t)))^{\alpha-\beta}\left(\xi_+^{\Delta}(t)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(t)},\\ & t\geq T_1,\end{aligned} \tag{39}$$

where $M = M_2^{\alpha}$. Integrating both sides of (39) from T_1 to t, we have

$$\begin{aligned}&\int_{T_1}^{t}\left(\xi(s)\overline{p}(s)\theta^{\beta}(s,T) -\frac{M\alpha^{\alpha}r(s)(R(\sigma(s)))^{\alpha-\beta}\left(\xi_+^{\Delta}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)}\right)\Delta s\\ &\quad\leq w\left(T_1\right)-w(t) < w\left(T_1\right).\end{aligned} \tag{40}$$

Taking lim sup of both sides of this last inequality as $t\to\infty$, we get a contradiction to (25). This completes the proof. □

Theorem 5. *Let $\alpha < \beta$. Assume that there exist a positive rd-continuous differentiable function $\xi(t)$ and a constant $K > 0$ such that, for some $T \geq t_0$,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\xi(s)\,\overline{p}(s)\,\theta^{\beta}(s,T) - \frac{\alpha^{\alpha} r(s)\left(\xi_{+}^{\Delta}(s)\right)^{\alpha+1}}{K(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)} \right)\Delta s = \infty, \tag{41}$$

where $\xi_{+}^{\Delta}(s)$ is defined as Theorem 4. Then (4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of (4). Proceeding as in the proof of Theorem 4 we get that (33) holds, that is,

$$w^{\Delta}(t) \leq -\xi(t)\,\overline{p}(t)\,\theta^{\beta}(t,T) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w^{\sigma}(t) - \frac{\beta\xi(t)\,w^{(\alpha+1)/\alpha}(\sigma(t))}{r^{1/\alpha}(t)\,\xi^{(\alpha+1)/\alpha}(\sigma(t))} Z^{(\beta-\alpha)/\alpha}(\sigma(t)), \quad t \geq T. \tag{42}$$

Since $\beta > \alpha$ and $Z(t)$ is increasing on $[T,\infty)_{\mathbb{T}}$, then there exist a $T_2 \geq T$ and a positive constant c_1 such that $Z^{(\beta-\alpha)/\alpha}(\sigma(t)) \geq c_1$ for $t \geq T_2$. Consequently,

$$w^{\Delta}(t) \leq -\xi(t)\,\overline{p}(t)\,\theta^{\beta}(t,T) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w^{\sigma}(t) - \frac{c_1\beta\xi(t)\,w^{(\alpha+1)/\alpha}(\sigma(t))}{r^{1/\alpha}(t)\,\xi^{(\alpha+1)/\alpha}(\sigma(t))}, \quad t \geq T_2. \tag{43}$$

Let

$$\overline{\Psi}(t) := \frac{c_1\beta\xi(t)}{r^{1/\alpha}(t)\,\xi^{(\alpha+1)/\alpha}(\sigma(t))}, \tag{44}$$

then $\overline{\Psi}(t) > 0$, and

$$w^{\Delta}(t) \leq -\xi(t)\,\overline{p}(t)\,\theta^{\beta}(t,T) + \frac{\xi_{+}^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) - \overline{\Psi}(t)\,w^{(\alpha+1)/\alpha}(\sigma(t)), \quad t \geq T_2. \tag{45}$$

The remainder of the proof is similar to that of Theorem 4 and is therefore omitted. This completes the proof for the case $\alpha < \beta$. □

Remark 6. Theorems 4 and 5 remove the Conditions (5) and (6). Moreover, the authors in [5] established oscillation theorems for (4) only for the case $\alpha \geq \beta > 0$. Our results here hold without this assumption, so our results improve the main results [5].

Remark 7. The results established here are valid for general time scales, with no additional restrictions, for example, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, and $\mathbb{T} = \mathbb{N}_0^2$; see [2, 3].

3. Some Examples

In this section, we give two examples to illustrate our main results.

Example 1. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ ($\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$), $\alpha = 3$, $\beta = 2$. Consider the neutral nonlinear dynamic equation

$$\Delta_2\left(\Delta_2 Z\left(2^k\right)\right)^3 + \frac{\delta^2\left(2^k\right)}{\theta^2\left(2^k,1\right)}\left|x\left(\delta\left(2^k\right)\right)\right| x\left(\delta\left(2^k\right)\right) = 0, \quad k_0 = 0, \tag{46}$$

where $\tau(2^k)$ satisfies $(A3)$, and $Z(2^k) = x(2^k) + (2^k - 1)/2^k x(\tau(2^k))$.

Here,

$$r\left(2^k\right) = 1, \qquad p\left(2^k\right) = \frac{2^k - 1}{2^k}, \qquad q\left(2^k\right) = \frac{\delta^2\left(2^k\right)}{\theta^2\left(2^k,1\right)}. \tag{47}$$

It is clear that $(A1)$ holds, and $\overline{p}(2^k) = q(2^k)(1 - p(\delta(2^k)))^{\beta} = 1/\theta^2(2^k,1)$, $R(\sigma(2^k)) = 2^{k+1} - 1$.

Let $\xi(2^k) = 2^k$. Noting that $\sum_{k=0}^{\infty} r^{-1/\alpha}(2^k)2^k = \infty$ implies $\lim_{k\to\infty}\theta(2^k,2^{k_T})/\theta(2^k,1) = 1$ for $k_T \geq 1$, we get

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_0}^{t}\left(\xi(s)\,\overline{p}(s)\,\theta^{\beta}(s,T) - \frac{M\alpha^{\alpha} r(s)\,(R(\sigma(s)))^{\alpha-\beta}\left(\xi_{+}^{\Delta}(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)}\right)\Delta s \\
&= \limsup_{k\to\infty}\sum_{i=0}^{k-1}\left(2^i\frac{\theta^2\left(2^i,2^{k_T}\right)}{\theta^2\left(2^i,1\right)} - \frac{3^3 M\left(2^{i+1}-1\right)}{4^4 2^3 2^{2i}}\right)2^i \\
&\geq \limsup_{k\to\infty}\sum_{i=0}^{k-1}\left(2^i - \frac{M\left(2^{i+1}-1\right)}{2^{2i}}\right)2^i = \infty.
\end{aligned} \tag{48}$$

Thus, by Theorem 4, (46) is oscillatory.

Example 2. Consider the neutral dynamic equation

$$\left(\frac{1}{\sigma^{1+\alpha}(t)}\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} + \frac{(1+\delta(t))^{\beta}}{\delta^{\beta}(t)\,\theta^{\beta}(t,t_0)}\left|x(\delta(t))\right|^{\beta-1} x(\delta(t)) = 0, \quad t_0 > 0, \tag{49}$$

where $\beta > \alpha > 0$ are constants, $\tau(t)$ satisfies $(A3)$, and $Z(t) = x(t) + 1/(t+1)x(\tau(t))$.

For (4), we let

$$r(t) = \frac{1}{\sigma^{1+\alpha}(t)}, \qquad p(t) = \frac{1}{t+1}, \qquad q(t) = \frac{(1+\delta(t))^{\beta}}{\delta^{\beta}(t)\,\theta^{\beta}(t,t_0)}. \tag{50}$$

Since

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \Delta s = \int_{t_0}^{\infty} \sigma^{(1+\alpha)/\alpha}(s)\,\Delta s = \infty, \tag{51}$$

then $(A1)$ holds and $\overline{p}(t) = q(t)(1 - p(\delta(t)))^{\beta} = 1/\theta^2(t,t_0)$.

Let $\xi(t) = t$. Noting that $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\Delta t = \infty$ implies $\lim_{t\to\infty}\theta(t,T)/\theta(t,t_0) = 1$ for $T \ge t_0$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^{t} \left(\xi(s)\,\overline{p}(s)\,\theta^{\beta}(s,T) - \frac{\alpha^{\alpha} r(s) \left(\xi_+^{\Delta}(s)\right)^{\alpha+1}}{K(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)} \right) \Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^{t} \left(s\frac{\theta^2(s,T)}{\theta^2(s,t_0)} - \frac{\alpha^{\alpha}}{K(\alpha+1)^{\alpha+1}\beta^{\alpha}\sigma^{1+\alpha}(s)\,s^{\alpha}} \right) \Delta s \\
&\ge \limsup_{t\to\infty} \int_{t_0}^{t} \left(s - \frac{1}{K\beta^{\alpha} s^{1+2\alpha}} \right) \Delta s \\
&\ge \limsup_{t\to\infty} \frac{1}{2} \int_{t_0}^{t} s\Delta s = \infty.
\end{aligned} \tag{52}$$

Thus, by Theorem 5, (49) is oscillatory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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