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Research Article

Induced Maps on Matrices over Fields

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Suppose that $\mathbb{F}$ is a field and $m, n \geq 3$ are integers. Denote by $M_{mn}(\mathbb{F})$ the set of all $m \times n$ matrices over $\mathbb{F}$ and by $M_n(\mathbb{F})$ the set $M_{nn}(\mathbb{F})$. Let f_{ij} ($i \in [1, m]$, $j \in [1, n]$) be functions on $\mathbb{F}$, where $[1, n]$ stands for the set $\{1, \ldots, n\}$. We say that a map $f : M_{mn}(\mathbb{F}) \to M_{mn}(\mathbb{F})$ is induced by $\{f_{ij}\}$ if f is defined by $f : [a_{ij}] \mapsto [f_{ij}(a_{ij})]$. We say that a map f on $M_n(\mathbb{F})$ preserves similarity if $A \sim B \Rightarrow f(A) \sim f(B)$, where $A \sim B$ represents that A and B are similar. A map f on $M_n(\mathbb{F})$ preserving inverses of matrices means $f(A)f(A^{-1}) = I_n$ for every invertible $A \in M_n(\mathbb{F})$. In this paper, we characterize induced maps preserving similarity and inverses of matrices, respectively.

1. Introduction

Suppose that $\mathbb{F}$ is a field and $m, n \geq 3$ are integers. Denote by $M_{mn}(\mathbb{F})$ the set of all $m \times n$ matrices over $\mathbb{F}$ and by $M_n(\mathbb{F})$ the set $M_{nn}(\mathbb{F})$. Let f_{ij} ($i \in [1, m]$, $j \in [1, n]$) be functions on $\mathbb{F}$, where $[1, n]$ stands for the set $\{1, \ldots, n\}$. We say that map $f : M_{mn}(\mathbb{F}) \to M_{mn}(\mathbb{F})$ is induced by $\{f_{ij}\}$ if f is defined by

$$f : A = [a_{ij}] \longmapsto [f_{ij}(a_{ij})], \quad \text{for every } A \in M_{mn}(\mathbb{F}). \quad (1)$$

It is easy to see that induced map may not be linear or additive.

Example 1. Let $\mathbb{R}$ be real field, $f_{11}(x) = \sin x$, $f_{12}(x) = \exp(x)$, $f_{21}(x) = \tan(x)$, and $f_{22}(x) = x^2 + 1$, then $f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ induced by $\{f_{ij}\}$ is

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sin a & \exp b \\ \tan c & d^2 + 1 \end{bmatrix}, \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}). \quad (2)$$

Example 2. The transposition map $X \mapsto X^t$ is not an induced map on $M_n(\mathbb{F})$.

Example 3. Let $P \in M_n(\mathbb{F})$ be a matrix; then $f : X \mapsto PX$ is an induced map on $M_n(\mathbb{F})$ if and only if P is diagonal.

If f_{ij} is independent of the choices of i and j (i.e., $f_{ij} \equiv \varphi$, for every $i \in [1, m]$ and $j \in [1, n]$), then f is said to be induced by the function φ, and denote by $f(A) = A^{\varphi}(= [\varphi(a_{ij})])$. Denote by $\operatorname{rank} A$ the rank of matrix A. We say that an induced map f preserves rank-1 if $\operatorname{rank} f(A) = 1$ whenever $\operatorname{rank} A = 1$.

Preserver problem is a hot area in matrix and operator algebra; there are many results about this area. Kalinowski [1] showed that an induced map $f(\Theta) = \Theta^{\varphi}$, where φ is a monotonic and continuous function of real field R such that $f(0) = 0$, preserves ranks of matrices if and only if it is linear. Furthermore, in [2], Kalinowski generalized the results in [1] by removing any restrictions on the map φ. In [3], Liu and Zhang characterized the general form of all maps f induced by f_{ij} and preserving rank-1 matrices over a field. In particular, nonlinear maps preserving similarity were studied by Du et al. [4]. One can see [5–15] and their references for some background on preserver problems.

We say that a map f on $M_n(\mathbb{F})$ preserves similarity if

$$A \sim B \Longrightarrow f(A) \sim f(B) \quad \forall A, B \in M_n(\mathbb{F}), \quad (3)$$

where $A \sim B$ represents that A and B are similar. A map f on $M_n(\mathbb{F})$ preserving inverses of matrices means $f(A)f(A^{-1}) = I_n$ for every invertible $A \in M_n(\mathbb{F})$. In this paper, we describe the forms of induced map preserving similarity and inverses of matrices, respectively.

We end this section by introducing some notations which will be used in the following sections. Let $\operatorname{diag}(a_1, a_2, \ldots, a_n)$

be the diagonal matrix of order n. E_{ij} is the matrix with 1 in the (i, j)th entry and 0 elsewhere and I_n is the identity matrix of order n. Denote by $\oplus$ the usual direct sum of matrices.

2. Induced Map Preserving Similarity of Matrices

In this section, we use the form of induced rank-1 preserver to describe forms of induced similarity preservers. Firstly, we need the following theorem from [3].

Lemma 4 (see [3, Corollary 1]). *Suppose that $\mathbb{F}$ is any field and $n \geq 2$ are integers. Suppose that f on $M_n(\mathbb{F})$ is induced by $\{f_{ij}\}$ such that $f(0) = 0$. Then f preserves rank-1 if and only if there exist invertible and diagonal $P, Q \in M_n(\mathbb{F})$ and a multiplicative map δ on $\mathbb{F}$ satifying $\delta(x) = 0 \Leftrightarrow x = 0$ such that*

$$f(A) = PA^{\delta}Q, \quad \forall A \in M_n(\mathbb{F}). \tag{4}$$

Lemma 5. *Suppose that $\mathbb{F}$ is any field, and n is an integer with $n \geq 2$. If $A \in M_n(\mathbb{F})$ satisfies $A^2 = 0$ and $\operatorname{rank} A = r$, then there exists an invertible matrix $P \in M_n(\mathbb{F})$ such that*

$$P^{-1}AP = \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} \oplus 0. \tag{5}$$

Proof. It is easy to see that there exists an invertible matrix $P_1 \in M_n(\mathbb{F})$ such that

$$A = P_1 \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} P_1^{-1}, \tag{6}$$

where $A_1 \in M_r(\mathbb{F})$ and $A_2 \in M_{r(n-r)}(\mathbb{F})$ satisfy $\operatorname{rank}[A_1 \ \ A_2] = r$. From $A^2 = 0$, we have

$$A_1 [A_1 \ \ A_2] = 0, \tag{7}$$

so that $A_1 = 0$. Thus, $\operatorname{rank} A_2 = r$, which implies that

$$A_2 Q = [I_r \ \ 0], \tag{8}$$

for some invertible $Q \in M_{n-r}(\mathbb{F})$. Let $P = P_1(I_r \oplus Q)$; then (6) turns into

$$P^{-1}AP = \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix} \oplus 0. \tag{9}$$

This completes the proof. □

Theorem 6. *Let $\mathbb{F}$ be a field and let $m, n \geq 3$ be positive integers. Suppose that f is a map on $M_n(\mathbb{F})$ induced by $\{f_{ij}\}$ such that $f(0) = 0$. Then f preserves similarity if and only if there exist an invertible and diagonal $P \in M_n(\mathbb{F})$, $q \in \mathbb{F}$, and an injective endomorphism δ of $\mathbb{F}$ such that*

$$f(A) = qPA^{\delta}P^{-1}, \quad \forall A \in M_n(\mathbb{F}). \tag{10}$$

Proof. The sufficiency is obvious. We will prove the necessary part by the following four steps.

Step 1. If there exists some $i \neq j$ and $a \neq 0$ such that $f_{ij}(a) = 0$, then $f = 0$.

Proof of Step 1. For any $b \neq 0$, $k \neq l$, since $\operatorname{rank} bE_{lk} = 1$ and $(bE_{lk})^2 = 0$, by Lemma 5, we have

$$bE_{lk} \sim E_{12} \sim aE_{ij}. \tag{11}$$

Since f preserves similarity, we derive

$$f_{kl}(b) E_{kl} \sim f_{ij}(a) E_{ij} = 0, \tag{12}$$

and thus,

$$f_{kl}(b) = 0, \quad \forall k \neq l \in [1, n], \ b \in \mathbb{F}. \tag{13}$$

Because of $\operatorname{rank}(bE_{kk} - bE_{ll} - bE_{kl} + bE_{lk}) = 1$ and $(bE_{kk} - bE_{ll} - bE_{kl} + bE_{lk})^2 = 0$, by Lemma 5, we have

$$(bE_{kk} - bE_{ll} - bE_{kl} + bE_{lk}) \sim E_{12} \sim aE_{ij}. \tag{14}$$

Using f preserves similarity and (13), one can obtain that

$$f_{kk}(b) E_{kk} + f_{ll}(-b) E_{ll} = 0, \tag{15}$$

hence,

$$f_{kk}(b) = 0, \quad \forall k \in [1, n], \ b \in \mathbb{F}. \tag{16}$$

It follows from (13) and (16) that $f_{ij} = 0$, that is, $f = 0$.

Step 2. If there exist some i and $a \neq 0$ such that $f_{ii}(a) = 0$, then $f = 0$.

Proof of Step 2. For $j \neq i$, it follows from $aE_{ii} \sim aE_{jj}$ that $f_{ii}(a)E_{ii} \sim f_{jj}(a)E_{jj}$. Thus,

$$f_{jj}(a) = 0, \quad \forall j \in [1, n], \ a \in \mathbb{F}. \tag{17}$$

Because of

$$\left(aE_{ii} - aE_{jj} - aE_{ij} + aE_{ji}\right) \sim E_{12}, \tag{18}$$

one can obtain by using (17) that $(f_{ij}(a)E_{ij} + f_{ji}(-a)E_{ij}) \sim f_{12}(1)E_{12}$, and hence,

$$\operatorname{rank}\left(f_{ij}(a) E_{ij} + f_{ji}(-a) E_{ij}\right) = \operatorname{rank} f_{12}(1) E_{12} \leq 1. \tag{19}$$

Thus, $f_{ij}(a) = 0$ or $f_{ji}(-a) = 0$. We complete the proof of this step by using the result of Step 1.

Step 3. If $f \neq 0$, then f preserves rank-1.

Proof of Step 3. For any rank-1 matrix A we have

$$A \sim \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{20}$$

and hence,

$$f(A) \sim \begin{bmatrix} f_{11}(a_1) & f_{12}(a_2) & \cdots & f_{1n}(a_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{21}$$

Thus, $\operatorname{rank} f(A) \leq 1$; it follows from $f \neq 0$ that $\operatorname{rank} f(A) = 1$.

Step 4. If $f \neq 0$, then there exist an invertible and diagonal $P \in M_n(\mathbb{F})$, $0 \neq q \in \mathbb{F}$, and an injective endomorphism δ of $\mathbb{F}$ such that

$$f(A) = qPA^{\delta}P^{-1}, \quad \forall A \in M_n(\mathbb{F}). \tag{22}$$

Proof of Step 4. Since $f \neq 0$, by Step 3 and Lemma 4, there exist invertible and diagonal $P, Q \in M_n(\mathbb{F})$ and a multiplicative map δ on $\mathbb{F}$ satisfying $\delta(x) = 0 \Leftrightarrow x = 0$ such that

$$f(A) = PA^{\delta}(QP)P^{-1}, \quad \forall A \in M_n(\mathbb{F}). \tag{23}$$

Let

$$A = \begin{bmatrix} 1 & 1 & x+y \\ 0 & 1 & x \\ 1 & 0 & y \end{bmatrix} \oplus 0, \qquad B = \begin{bmatrix} 1 & 1-x-y & x+y \\ 0 & 1-x & x \\ 1 & 1-x-y & x+y \end{bmatrix} \oplus 0. \tag{24}$$

It is easy to see that $B = (I_n + E_{32})A(I_n + E_{32})^{-1}$. Since f preserves similarity, we have that $f(A)$ and $f(B)$ are similar; further, $\operatorname{rank} f(A) = \operatorname{rank} f(B)$. It follows from (23) that $\operatorname{rank} f(B) \leq 2$, thus,

$$\operatorname{rank} \begin{bmatrix} \delta(1) & \delta(1) & \delta(x+y) \\ 0 & \delta(1) & \delta(x) \\ \delta(1) & 0 & \delta(y) \end{bmatrix} \oplus 0 \leq 2. \tag{25}$$

This implies $\delta(x+y) = \delta(x) + \delta(y)$, hence, δ is an injective endomorphism of $\mathbb{F}$.

Set $QP = \operatorname{diag}(q_1, \ldots, q_n)$. Since $E_{ii} \sim E_{jj}$, one obtains by using (23) that

$$q_i E_{ii} \sim q_j E_{jj}. \tag{26}$$

Thus, $q_i = q_j$. Letting $q = q_1$, then $QP = qI_n$ and $f(A) = qPA^{\delta}P^{-1}$.

This completes the proof of Theorem. □

3. Induced Map Preserving Inverses of Matrices

Theorem 7. *Let $\mathbb{F}$ be a field and let $m, n \geq 3$ be positive integers. Suppose that f is a map on $M_n(\mathbb{F})$ induced by $\{f_{ij}\}$ such that $f(0) = 0$. Then f preserves inverses of matrices if and only if there exist an invertible and diagonal $P \in M_n(\mathbb{F})$, $c \in \{-1, 1\}$, and an injective endomorphism φ of $\mathbb{F}$ such that*

$$f(A) = cPA^{\varphi}P^{-1}, \quad \forall A \in M_n(\mathbb{F}). \tag{27}$$

Proof. The sufficiency is obvious. We will prove the necessary part. For any $i \neq j \in [1, n]$, $a \in \mathbb{F}$, and $b \in \mathbb{F}^*$, since

$$\begin{aligned} &\left[E_{ii} - aE_{ij} + E_{ji} + \left(b^{-1} - a\right)E_{jj} + \Sigma_{k \neq i,j}E_{kk}\right]^{-1} \\ &\quad = (1-ab)E_{ii} + abE_{ij} - bE_{ji} + bE_{jj} + \Sigma_{k \neq i,j}E_{kk}, \end{aligned} \tag{28}$$

by f preserving inverses of matrices, we have

$$\begin{bmatrix} f_{ii}(1) & f_{ij}(-a) \\ f_{ji}(1) & f_{jj}\left(b^{-1} - a\right) \end{bmatrix} \begin{bmatrix} f_{ii}(1-ab) & f_{ij}(ab) \\ f_{ji}(-b) & f_{jj}(b) \end{bmatrix} = I_2, \tag{29}$$

so that

$$f_{ii}(1) f_{ij}(ab) + f_{ij}(-a) f_{jj}(b) = 0, \tag{30}$$

$$f_{ii}(1) f_{ii}(1-ab) + f_{ij}(-a) f_{ji}(-b) = 1. \tag{31}$$

Let $b = 1$; then (30) turns into

$$f_{ii}(1) f_{ij}(a) + f_{ij}(-a) f_{jj}(1) = 0. \tag{32}$$

Replacing a by ab, then the above turns into

$$f_{ii}(1) f_{ij}(ab) + f_{ij}(-ab) f_{jj}(1) = 0. \tag{33}$$

It follows from (30) and (33) that

$$f_{ij}(-ab) f_{jj}(1) = f_{ij}(-a) f_{jj}(b). \tag{34}$$

Replacing $-a$ by a, then the above turns into

$$f_{ij}(ab) f_{jj}(1) = f_{ij}(a) f_{jj}(b). \tag{35}$$

By $[aE_{ii} + \Sigma_{k \neq i}E_{kk}]^{-1} = a^{-1}E_{ii} + \Sigma_{k \neq i}E_{kk}$, we have

$$f_{ii}(a) f_{ii}\left(a^{-1}\right) = 1. \tag{36}$$

In particular,

$$f_{ii}(1)^2 = 1. \tag{37}$$

From $[aE_{ij} + aE_{ji} + \Sigma_{k \neq i,j}E_{kk}]^{-1} = a^{-1}E_{ij} + a^{-1}E_{ji} + \Sigma_{k \neq i,j}E_{kk}$, we have

$$f_{ij}(a) f_{ji}\left(a^{-1}\right) = 1. \tag{38}$$

In particular,

$$f_{ij}(1) f_{ji}(1) = 1. \tag{39}$$

Multiplying $f_{ij}(-b^{-1})$ by (31), we obtain by using (38) that

$$f_{ii}(1) f_{ii}(1-ab) f_{ij}\left(-b^{-1}\right) + f_{ij}(-a) = f_{ij}\left(-b^{-1}\right). \tag{40}$$

It follows from (35) and (38) that

$$f_{ii}(1) f_{ij}\left((1-ab)\left(-b^{-1}\right)\right) = f_{ii}(1-ab) f_{ij}\left(-b^{-1}\right). \tag{41}$$

This, together with (40), implies that

$$f_{ii}(1) f_{ii}(1) f_{ij}\left(-b^{-1} + a\right) + f_{ij}(-a) = f_{ij}\left(-b^{-1}\right). \tag{42}$$

Hence, it follows from $f_{ii}(1)^2 = 1$ that

$$f_{ij}\left(-b^{-1} + a\right) + f_{ij}(-a) = f_{ij}\left(-b^{-1}\right). \tag{43}$$

Let $x = -b^{-1}$ and $y = -a$; we have

$$f_{ij}(x-y) = f_{ij}(x) - f_{ij}(y), \quad \forall x \in \mathbb{F}^*, \ y \in \mathbb{F}. \tag{44}$$

From

$$\left[E_{ii} + E_{ij} - E_{jj} + \Sigma_{k \neq i,j} E_{kk}\right]^{-1} = E_{ii} + E_{ij} - E_{jj} + \Sigma_{k \neq i,j} E_{kk} \tag{45}$$

we have

$$\begin{bmatrix} f_{ii}(1) & f_{ij}(1) \\ 0 & f_{jj}(-1) \end{bmatrix} \begin{bmatrix} f_{ii}(1) & f_{ij}(1) \\ 0 & f_{jj}(-1) \end{bmatrix} = I_2, \tag{46}$$

so that

$$f_{ii}(1) f_{ij}(1) = -f_{ij}(1) f_{jj}(-1). \tag{47}$$

This, together with (39), implies

$$f_{ii}(1) = -f_{jj}(-1). \tag{48}$$

Similarly, $f_{kk}(1) = -f_{jj}(-1)$; hence,

$$f_{ii}(1) = f_{kk}(1), \quad \forall i,k \in [1,n]. \tag{49}$$

It follows from

$$\begin{aligned} &\left[E_{ii} + xE_{ij} + E_{jj} + \Sigma_{k \neq i,j} E_{kk}\right]^{-1} \\ &\quad = E_{ii} - xE_{ij} + E_{jj} + \Sigma_{k \neq i,j} E_{kk}, \quad \forall x \in \mathbb{F} \end{aligned} \tag{50}$$

that

$$\begin{bmatrix} f_{ii}(1) & f_{ij}(x) \\ 0 & f_{jj}(1) \end{bmatrix} \begin{bmatrix} f_{ii}(1) & f_{ij}(-x) \\ 0 & f_{jj}(1) \end{bmatrix} = I_2. \tag{51}$$

Hence,

$$f_{ii}(1) f_{ij}(-x) = -f_{ij}(x) f_{jj}(1). \tag{52}$$

This, together with (39) and (49), implies

$$f_{ij}(-x) = -f_{ij}(x), \quad \forall x \in \mathbb{F}. \tag{53}$$

Since

$$\begin{aligned} &\left[aE_{ii} + E_{ij} - E_{ji} + \Sigma_{k \neq i,j} E_{kk}\right]^{-1} \\ &\quad = -E_{ij} + E_{ji} + aE_{jj} + \Sigma_{k \neq i,j} E_{kk}, \end{aligned} \tag{54}$$

we have

$$\begin{bmatrix} f_{ii}(a) & f_{ij}(1) \\ f_{ji}(-1) & 0 \end{bmatrix} \begin{bmatrix} 0 & f_{ij}(-1) \\ f_{ji}(1) & f_{jj}(a) \end{bmatrix} = I_2, \tag{55}$$

so that

$$f_{ii}(a) f_{ij}(-1) = -f_{ij}(1) f_{jj}(a). \tag{56}$$

Hence,

$$f_{ii}(a) = f_{jj}(a). \tag{57}$$

For distinct $i, j, k \in [1,n]$ and $x, y \in \mathbb{F}$, since

$$\begin{aligned} &\left[E_{ii} + xE_{ij} + xE_{ik} + E_{jj} + E_{jk} + E_{kk} + \Sigma_{l \neq i,j,k} E_{ll}\right]^{-1} \\ &\quad = E_{ii} - xE_{ij} + E_{jj} - E_{jk} + E_{kk} + \Sigma_{l \neq i,j,k} E_{ll}, \end{aligned} \tag{58}$$

we have

$$\begin{bmatrix} f_{ii}(1) & f_{ij}(x) & f_{ik}(xy) \\ 0 & f_{jj}(1) & f_{jk}(y) \\ 0 & 0 & f_{kk}(1) \end{bmatrix} \begin{bmatrix} f_{ii}(1) & f_{ij}(-x) & 0 \\ 0 & f_{jj}(1) & f_{jk}(-y) \\ 0 & 0 & f_{kk}(1) \end{bmatrix} = I_3, \tag{59}$$

so that

$$f_{ij}(x) f_{jk}(-y) + f_{ik}(xy) f_{kk}(1) = 0. \tag{60}$$

This, together with (53), implies that

$$f_{ij}(x) = f_{ik}(x) f_{kk}(1) f_{jk}(1)^{-1}. \tag{61}$$

It follows from (35) and (61) that

$$\begin{aligned} &f_{ij}(x) \\ &\quad = f_{ik}(x) f_{kk}(1) f_{jk}(1)^{-1} \\ &\quad = f_{ik}(x) f_{11}(1) f_{jk}(1)^{-1} \\ &\quad = \left(f_{kk}(1)^{-1} f_{i1}(1) f_{11}(x)\right) \\ &\qquad \times f_{11}(1) \left(f_{kk}(1)^{-1} f_{j1}(1) f_{11}(1)\right)^{-1} \\ &\quad = f_{i1}(1) f_{11}(x) f_{j1}(1)^{-1}, \quad \forall i \neq j \in [1,n]. \end{aligned} \tag{62}$$

Using this, together with (57), we obtain

$$f(A) = P\left[f_{11}\left(a_{ij}\right)\right] P^{-1}, \tag{63}$$

where $P = \operatorname{diag}(f_{11}(1), \ldots, f_{n1}(1))$. Let $c = f_{11}(1) \in \{-1, 1\}$; $\varphi(x) = f_{11}(1) f_{11}(x)$ and then

$$f(A) = cP\left[\varphi\left(a_{ij}\right)\right] P^{-1} = cPA^{\varphi} P^{-1}. \tag{64}$$

Let $\phi(A) = [\varphi(a_{ij})]$; since f preserves inverses of matrices, one can see that ϕ also preserves inverses of matrices. By $\varphi(1) = f_{11}(1)^2 = 1$, we obtain by using similar method to (35), (37), (44), and (53) that for any $a, b \in \mathbb{F}$

$$\begin{aligned} \varphi(ab) = \varphi(1)\varphi(ab) = \varphi(a)\varphi(b), \\ \varphi(a+b) = \varphi(a) + \varphi(b). \end{aligned} \tag{65}$$

This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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