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Research Article

Positive Periodic Solution for the Generalized Neutral Differential Equation with Multiple Delays and Impulse

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By using a fixed point theorem of strict-set-contraction, which is different from Gaines and Mawhin's continuation theorem and abstract continuation theory for k-set contraction, we established some new criteria for the existence of positive periodic solution of the following generalized neutral delay functional differential equation with impulse: $x'(t) = x(t)[a(t) - f(t, x(t), x(t - \tau_1(t, x(t))), \ldots, x(t - \tau_n(t, x(t))), x'(t - \gamma_1(t, x(t))), \ldots, x'(t - \gamma_m(t, x(t))))]$, $t \neq t_k$, $k \in Z_+$; $x(t_k^+) = x(t_k^-) + \theta_k(x(t_k))$, $k \in Z_+$. As applications of our results, we also give some applications to several Lotka-Volterra models and new results are obtained.

1. Introduction

Many systems in physics, chemistry, biology, and information science have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. This complex dynamical behavior can be modeled by impulsive differential equations. Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics; see [1–8]. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs [9–11].

In this paper, we consider more general neutral delay functional differential equation with impulse:

$$\begin{aligned} x'(t) = x(t)\Big[a(t) - f\Big(t, x(t), x(t - \tau_1(t, x(t))), \ldots, \\ x(t - \tau_n(t, x(t))), \\ x'(t - \gamma_1(t, x(t))), \ldots, \\ x'(t - \gamma_m(t, x(t)))\Big)\Big], \\ t \neq t_k,\ k \in Z_+, \\ x(t_k^+) = x(t_k^-) + \theta_k(x(t_k)), \quad k \in Z_+, \end{aligned} \tag{1}$$

where $a \in C(R, R^+)$, $\tau_i(t), \gamma_j(t) \in C(R, R)$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ are ω-periodic functions and $f \in C(R^{2+n+m}, R)$ is ω-periodic function with respect to its first argument. Moreover, $x(t_k^+), x(t_k^-)$ represents the right, left limit of $x(t)$ at the point t_k, respectively. In this paper, it is assumed that x is left continuous at t_k; that is, x changes decreasingly suddenly at times t_k. $\theta_k \in C(R^+, R^+)$, $\omega > 0$ is a constant, $R = (-\infty, +\infty)$, and $R^+ = [0, +\infty)$, $Z_+ = \{1, 2, 3, \ldots\}$. We assume that there exists an integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $\theta_{k+q} = \theta_k$, where $0 < t_1 < t_2 < \cdots < t_q < \omega$. For the ecological justification of (1) and the similar types refer to [8, 12–17].

In 1993, Kuang in [12] proposed an open problem (open problem 9.2) to obtain sufficient conditions for the existence of a positive periodic solution of the following equation:

$$\begin{aligned} \frac{dN}{dt} = N(t)\Big[a(t) - \beta(t)N(t) - b(t)N(t - \tau(t)) \\ - c(t)N'(t - \tau(t))\Big]. \end{aligned} \tag{2}$$

In [13], Fang and Li studied model (2) and gave an answer to the open problem 9.2 of [12]. But paper [13] required that $b(t) \geq 0$, $c(t) \geq 0$ and $c_0'(t) > b(t)$, $\beta(t) \geq 0$ or $c_0'(t) \leq b(t)$, $\beta(t) \leq 0$ for $t \in [0, \omega]$, where $c_0(t) = c(t)/(1 - \tau'(t))$. In [14], Yang and Cao studied a general neutral delay model of single-species population growth:

$$\frac{dN}{dt} = N(t)\left[a(t) - \beta(t)N(t) - \sum_{i=1}^{n} b_i(t) N(t-\tau_i(t)) - \sum_{i=1}^{n} c_i(t) N'(t-\gamma_i(t))\right]. \quad (3)$$

They applied the theory of coincidence degree to obtain verifiable sufficient conditions of the existence of positive periodic solutions of system (3). In [15], Lu considered the following neutral functional differential equation:

$$\frac{dN}{dt} = N(t)\left[a(t) - \beta(t)N(t) - \sum_{i=1}^{n} b_i(t) N(t-\tau_i(t)) - \sum_{j=1}^{m} c_j(t) N'(t-\gamma_j(t))\right]. \quad (4)$$

He obtained some sufficient conditions for the existence of positive periodic solutions of model (4) by using the theory of abstract continuous theorem of k-set contractive operator and some analysis techniques. In [16], Yang and Cao used the theory of coincidence degree to investigate a complex neutral equation with several state-dependent delays as follows:

$$\frac{dN}{dt} = N(t)\left[a(t) - \beta(t)N(t) - \sum_{i=1}^{n} b_i(t) N(t-\tau_i(t, N(t))) - \sum_{i=1}^{n} c_i(t) N'(t-\gamma_i(t))\right]. \quad (5)$$

They also got some verifiable sufficient conditions of the existence of positive periodic solutions of system (5). In [17], Li and Kuang considered the periodic Lotka-Volterra equation with state-dependent delays:

$$\frac{dx}{dt} = x(t)\left[r(t) - a(t)x(t) + \sum_{i=1}^{n} b_i(t) x(t-\tau_i(t, x(t))) - \sum_{j=1}^{m} c_j(t) x'(t-\gamma_j(t, x(t)))\right]. \quad (6)$$

They used the continuation theorem of coincidence degree theory to obtain some sufficient and realistic conditions for the existence of positive periodic solutions of system (6). In [8], Wang and Dai investigated the following periodic neutral population model with delays and impulse:

$$\begin{aligned} \frac{dN}{dt} &= N(t)\left[a(t) - e(t)N(t) - \sum_{j=1}^{n} b_j(t) N(t-\sigma_j(t)) - \sum_{i=1}^{m} c_i(t) N'(t-\tau_i(t))\right], \quad t \neq t_k, \\ N(t^+) &= (1+\theta_k) N(t_k), \quad k = 1, 2, \ldots. \end{aligned} \quad (7)$$

They obtained some sufficient conditions for the existence of positive periodic solutions of model (7) by using the theory of abstract continuous theorem of k-set contractive operator and some analysis techniques.

The main purpose of this paper is to establish new criteria to guarantee the existence of positive periodic solutions of the system (1) by using a fixed point theorem of strict-set-contraction [18–20].

For convenience, we introduce the notation

$$\begin{gathered} h^M = \max_{t\in[0,\omega]} \{h(t)\}, \qquad h^L = \min_{t\in[0,\omega]} \{h(t)\}, \\ \delta = \lim_{u\to 0} \sup \sum_{t\le t_k \le t+\omega} \frac{\theta_k(u)}{u}, \qquad \sigma = e^{-\int_0^{\omega} a(t)dt}, \\ B_1 = \int_0^{\omega}\left[\sigma\beta(t) + \sigma\sum_{i=1}^{n} b_i(t) - \sum_{j=1}^{m} c_j(t)\right]dt, \\ B_2 = \int_0^{\omega}\left[\beta(t) + \sum_{i=1}^{n} b_i(t) + \sum_{j=1}^{m} c_j(t)\right]dt, \end{gathered} \quad (8)$$

where $h(t)$ is a continuous ω-periodic function.

Throughout this paper, we assume the following.

(A_1) $a, \tau_i, \gamma_j \in C(R, R)$ are ω-periodic functions. In addition, $a(t) \ge 0$, $t \in [0, \omega]$, and $\sigma = e^{-\int_0^{\omega} a(\xi)d\xi} < 1$.

(A_2) $f \in C(R^{2+n+m}, R)$ is ω-periodic function with $f(t + \omega, \cdot) = f(t, \cdot)$, $f(t, 0, \ldots, 0) = 0$.

(A_3) There exist ω-periodic functions $\beta(t), b_i(t) \in C(R, R^+)$, $c_j(t) \in C^1(R, R^+)$, such that

$$\begin{aligned} &\sigma\beta(t) + \sigma\sum_{i=1}^{n} b_i(t) - \sum_{j=1}^{m} c_j(t) > 0, \\ &|f(t, x_0, x_1, \ldots, x_n, y_1, \ldots, y_m) - f(t, x_0^*, x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*)| \\ &\quad \le \beta(t)|x_0 - x_0^*| + \sum_{i=1}^{n} b_i(t)|x_i - x_i^*| + \sum_{j=1}^{m} c_j(t)|y_j - y_j^*|, \end{aligned}$$

$$f(t, x_0, x_1, \ldots, x_n, y_1, \ldots, y_m) \geq \beta(t) x_0 + \sum_{i=1}^{n} b_i(t) x_i - \sum_{j=1}^{m} c_j(t) y_j, \tag{9}$$

where $t \in [0, \omega]$, $\sigma|y_j| \leq x_i$, $\sigma|y_j^*| \leq x_i^*$, and $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

(A_4) We assume that $(1 + a^L)\sigma^2 B_1/(1 - \sigma) \geq \max_{t\in[0,\omega]}\{\beta(t) + \sum_{i=1}^{n} b_i(t) + \sum_{j=1}^{m} c_j(t)\}$.

(A_5) We assume that $(a^M - 1)B_2/\sigma(1 - \sigma) \leq \min_{t\in[0,\omega]}\{\sigma\beta(t) + \sigma \sum_{i=1}^{n} b_i(t) - \sum_{j=1}^{m} c_j(t)\}$.

(A_6) We assume that $((1 - \sigma)/\sigma^2 B_1) \sum_{j=1}^{m} c_j^M < 1$.

The paper is organized as follows. In the next section, we give some definitions and lemmas to prove the main results of this paper. In Section 3, we established some criteria to guarantee the existence of at least one positive periodic solution of system (1) by using a fixed point theorem of strict-set-contraction. As applications in Section 4, we study some particular cases of system (1) which have been investigated extensively in the references mentioned previously.

2. Preliminaries

In order to obtain the existence of a periodic solution of system (1), we first introduce some definitions and lemmas.

Definition 1 (see [17]). A function $x : R \to (0, +\infty)$ is said to be a positive solution of (1), if the following conditions are satisfied:

(a) $x(t)$ is absolutely continuous on each (t_k, t_{k+1});

(b) for each $k \in Z_+$, $x(t_k^+)$ and $x(t_k^-)$ exist, and $x(t_k^-) = x(t_k)$;

(c) $x(t)$ satisfies the first equation of (1) for almost everywhere in R and $x(t_k)$ satisfies the second equation of (1) at impulsive point t_k, $k \in Z_+$.

Definition 2 (see [18]). Let X be a real Banach space and E a closed, nonempty subset of X. E is a cone provided that

(i) $\alpha x + \beta y \in E$ for all $x, y \in E$ and all $\alpha, \beta \geq 0$;

(ii) $x, -x \in E$ imply $x = 0$.

Definition 3 (see [18]). Let A be a bounded subset in X. Define $\alpha_X(A) = \inf\{\delta > 0$: there is a finite number of subsets $A_i \subset A$ such that $A = \bigcup_i A_i$ and $\operatorname{diam}(A_i) \leq \delta\}$, where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i; obviously, $0 \leq \alpha_X(A) < \infty$. So $\alpha_X(A)$ is called the Kuratowski measure of noncompactness of X.

Definition 4 (see [18]). Let X, Y be two Banach spaces and $D \subset X$; a continuous and bounded map $T : D \to Y$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_Y(T(S)) \leq k\alpha_X(S). \tag{10}$$

T is called strict-set-contractive if it is k-set contractive for some $0 \leq k < 1$.

Definition 5 (see [19]). The set $F \in PC_\omega$ is said to be quasiequicontinuous in $[0, \omega]$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in F$, $k \in N^+$, $t_1, t_2 \in (t_{k-1}, t_k) \cap [0, \omega]$, and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

Lemma 6 (see [19]). *The set $F \subset PC_\omega$ is relatively compact if and only if*

(1) *F is bounded, that is, $\|x\| \leq M$, for each $x \in F$, and some $M > 0$;*

(2) *F is quasiequicontinuous in $[0, \omega]$.*

Lemma 7. *$x(t)$ is an ω-periodic solution of (1) is equivalent to $x(t)$ is an ω-periodic solution of the following equation:*

$$\begin{aligned} x(t) = \int_t^{t+\omega} \Big[G(t, s) x(s) f\Big(t, x(s),\ & x(s - \tau_1(s, x(s))), \ldots, \\ & x(s - \tau_n(s, x(s))), \\ & x'(s - \gamma_1(s, x(s))), \ldots, \\ & x'(s - \gamma_m(s, x(s)))\Big)\Big] ds \\ + \sum_{t \leq t_k < t+\omega} G(t, t_k)\, \theta_k(x(t_k)), & \end{aligned} \tag{11}$$

where

$$G(t, s) = \frac{e^{-\int_t^s a(\xi) d\xi}}{1 - e^{-\int_0^\omega a(\xi) d\xi}}, \quad s \in [t, t + \omega]. \tag{12}$$

Proof. Assume that $x(t) \in X$ is a periodic solution of (1). Then, we have

$$\begin{aligned} \frac{d}{dt}\Big[x(t) e^{-\int_0^t a(\xi) d\xi}\Big] = e^{-\int_0^t a(\xi) d\xi} x(t) f\Big(t, x(t),\ & x(t - \tau_1(t, x(t))), \ldots, \\ & x(t - \tau_n(t, x(t))), \\ & x'(t - \gamma_1(t, x(t))), \ldots, \\ & x'(t - \gamma_m(t, x(t)))\Big), \quad t \neq t_k. \end{aligned} \tag{13}$$

Integrating the above equation over $[t, t + \omega]$, we can have

$$x(s) e^{-\int_0^s a(\xi) d\xi}\Big|_t^{t_{m_1}+n\omega} + x(s) e^{-\int_0^s a(\xi) d\xi}\Big|_{t_{m_1}+nw}^{t_{m_2}+n\omega} + \cdots + x(s) e^{-\int_0^s a(\xi) d\xi}\Big|_{t_{m_q}+n\omega}^{t+\omega}$$

$$= \int_t^{t+\omega} \Big[x(s) e^{-\int_0^s a(\xi)d\xi} \times f\Big(t, x(s), x(s-\tau_1(s, x(s))), \ldots, x(s-\tau_n(s, x(s))), x'(s-\gamma_1(s, x(s))), \ldots, x'(s-\gamma_m(s, x(s)))\Big)\Big] ds, \tag{14}$$

where $t_{m_k} + n\omega \in (t, t+\omega)$, $m_k \in \{1, 2, \ldots, q\}$, $k = 1, 2, \ldots, q$, $n \in Z_+$. Therefore,

$$x(t) e^{-\int_0^t a(\xi)d\xi} \Big[1 - e^{-\int_t^{t+\omega} a(\xi)d\xi}\Big] + \sum_{t \le t_k < t+\omega} \Delta x(t_{m_k}) e^{-\int_0^{t_{m_k}+n\omega} a(\xi)d\xi} = \int_t^{t+\omega} \Big[x(s) e^{-\int_0^s a(\xi)d\xi} \times f\Big(t, x(s), x(s-\tau_1(s, x(s))), \ldots, x(s-\tau_n(s, x(s))), x'(s-\gamma_1(s, x(s))), \ldots, x'(s-\gamma_m(s, x(s)))\Big)\Big] ds, \tag{15}$$

which can be transformed into

$$x(t) = \int_t^{t+\omega} \Big[G(t,s) x(s) \times f\big(t, x(s), x(s-\tau_1(s, x(s))), \ldots, x(s-\tau_n(s, x(s))), x'(s-\gamma_1(s, x(s))), \ldots, x'(s-\gamma_m(s, x(s)))\big)\Big] ds + \sum_{t \le t_k < t+\omega} G(t, t_k)\theta_k(x(t_k)). \tag{16}$$

Thus, x is a periodic solution for (11).

If $x(t) \in E$ is a periodic solution of (11), for any $t = t_k$, from (11) we have

$$x'(t) = \frac{d}{dt}\Big\{\int_t^{t+\omega} \big[G(t,s) x(s) \times f\big(t, x(s), x(s-\tau_1(s, x(s))), \ldots, x(s-\tau_n(s, x(s))), x'(s-\gamma_1(s, x(s))), \ldots, x'(s-\gamma_m(s, x(s)))\big)\big] ds\Big\}$$
$$= \big[G(t, t+\omega) x(t+\omega) \times f\big(t+\omega, x(t+\omega), x(t+\omega-\tau_1(t+\omega, x(t+\omega))), \ldots, x(t+\omega-\tau_n(t+\omega, x(t+\omega))), x'(t+\omega-\gamma_1(t+\omega, x(t+\omega))), \ldots, x'(t+\omega-\gamma_m(t+\omega, x(t+\omega)))\big)$$
$$- G(t,t) x(t) f\big(t, x(t), x(t-\tau_1(t, x(t))), \ldots, x(t-\tau_n(t, x(t))), x'(t-\gamma_1(t, x(t))), \ldots, x'(t-\gamma_m(t, x(t)))\big)\big] + a(t)x(t)$$
$$= x(t)\big[a(t) - f\big(t, x(t), x(t-\tau_1(t, x(t))), \ldots, x(t-\tau_n(t, x(t))), x'(t-\gamma_1(t, x(t))), \ldots, x'(t-\gamma_m(t, x(t)))\big)\big]. \tag{17}$$

For any $t = t_j$, $j \in Z_+$, we have from (11) that

$$x(t_j^+) - x(t_j) = \int_{t_j}^{t_j+\omega} \big[G(t_j^+, s) - G(t_j, s)\big] x(s) \times f\big(t, x(s), x(s-\tau_1(s, x(s))), \ldots, x(s-\tau_n(s, x(s))), x'(s-\gamma_1(s, x(s))), \ldots, x'(s-\gamma_m(s, x(s)))\big) ds + \sum_{t_j^+ \le t_k < t_j+\omega} G(t_j^+, t_k)\theta_k(x(t_k)) - \sum_{t_j \le t_k < t_j+\omega} G(t_j, t_k)\theta_k(x(t_k)) = \theta_k(x(t_k)). \tag{18}$$

Hence $x(t)$ is a positive ω-periodic solution of (1). Thus we complete the proof of Lemma 7. □

Lemma 8 (see [18–20]). *Let E be a cone of the real Banach space X and $E_{r,R} = \{x \in E : r \le \|x\| \le R\}$ with $0 < r < R$. Assume that $A : E_{r,R} \to E$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(a) *$Ax \nleq x$, $\forall x \in E$, $\|x\| = r$ and $Ax \ngeq x$, $\forall x \in E$, $\|x\| = R$;*

(b) $Ax \ngeq x$, $\forall x \in E$, $\|x\| = r$ *and* $Ax \nleq x$, $\forall x \in E$, $\|x\| = R$.

Then A *has at least one fixed point in* $E_{r,R}$.

In order to apply Lemma 8 to system (1), we set

$$PC(R) = \{x : R \longrightarrow R \mid x \in C((t_k, t_{k+1}), R), \exists x(t_k^-) = x(t_k), x(t_k^+), k \in Z_+, t \in R\},$$
$$PC^1(R) = \left\{x : R \longrightarrow R \mid x \in C^1((t_k, t_{k+1}), R), \exists x'(t_k^-) = x'(t_k), x(t_k^+), k \in Z_+, t \in R\right\}. \tag{19}$$

Define

$$X = \{x : x \in PC(R) \mid x(t+\omega) = x(t)\} \tag{20}$$

with the norm defined by $\|x\| = \max_{t\in[0,\omega]}\{|x(t)|\}$ and

$$Y = \left\{x : x \in PC^1(R) \mid x(t+\omega) = x(t), t \in R\right\} \tag{21}$$

with the norm defined by $\|x\|_1 = \max\{\|x\|, \|x'\|\}$. Then X and Y are both Banach spaces. Define the cone E in Y by

$$E = \left\{x : x \in PC^1(R) \mid x(t) \geq \sigma\|x\|_1, t \in [0,\omega]\right\}. \tag{22}$$

Let the map ϕ be defined by

$$\begin{aligned}(\phi x)(t) = \int_t^{t+\omega} \Big[G(t,s)x(s) &\times f\big(s, x(s), x(s-\tau_1(s,x(s))), \ldots, x(s-\tau_n(s,x(s))),\\ &x'(s-\gamma_1(s,x(s))), \ldots, x'(s-\gamma_m(s,x(s)))\big)\Big] ds\\ &+ \sum_{t\leq t_k<t+\omega} G(t,t_k)\theta_k(x(t_k)),\end{aligned} \tag{23}$$

where $x \in E$, $t \in R$, and $G(t,s)$ is defined by (12). It is obvious to see that $G(t+\omega, s+\omega) = G(t,s)$, $\partial G(t,s)/\partial t = a(t)G(t,s)$, $G(t,t+\omega) - G(t,t) = -1$, and

$$\frac{\sigma}{1-\sigma} \leq G(t,s) \leq \frac{1}{1-\sigma}, \quad s \in [t, t+\omega]. \tag{24}$$

In what follows, we will give some lemmas concerning E and ϕ defined by (22) and (23), respectively.

Lemma 9. *Assume that* (A_1)*–*(A_4) *hold.*

(i) *If* $a^M \leq 1$*, then* $\phi : E \to E$ *is well defined.*

(ii) *If* (A_5) *holds and* $a^M > 1$*, then* $\phi : E \to E$ *is well defined.*

Proof. For any $x \in E$, it is clear that $\phi x \in PC^1(R)$. From (23), for $t \in [0,\omega]$, we have

$$\begin{aligned}(\phi x)(t+\omega) &= \int_{t+\omega}^{t+2\omega} \Big[G(t+\omega,s)x(s) \times f\big(s, x(s), x(s-\tau_1(s,x(s))), \ldots, x(s-\tau_n(s,x(s))),\\ &\qquad x'(s-\gamma_1(s,x(s))), \ldots, x'(s-\gamma_m(s,x(s)))\big)\Big] ds + \sum_{t+\omega\leq t_k<t+2\omega} G(t+\omega,t_k)\theta_k(x(t_k))\\ &= \int_t^{t+\omega} \Big[G(t+\omega,u+\omega)x(u+\omega) \times f\big(u+\omega, x(u+\omega), x(u+\omega-\tau_1(u+\omega,x(u+\omega))), \ldots,\\ &\qquad x(u+\omega-\tau_n(u+\omega,x(u+\omega))), x'(u+\omega-\gamma_1(u+\omega,x(u+\omega))), \ldots,\\ &\qquad x'(u+\omega-\gamma_m(u+\omega,x(u+\omega)))\big)\Big] du + \sum_{t\leq t_k<t+\omega} G(t,t_k)\theta_k(x(t_k))\\ &= \int_t^{t+\omega} \Big[G(t,s)x(s) \times f\big(s, x(s), x(s-\tau_1(s,x(s))), \ldots, x(s-\tau_n(s,x(s))),\\ &\qquad x'(s-\gamma_1(s,x(s))), \ldots, x'(s-\gamma_m(s,x(s)))\big)\Big] ds + \sum_{t\leq t_k<t+\omega} G(t,t_k)\theta_k(x(t_k)) = (\phi x)(t).\end{aligned} \tag{25}$$

That is, $(\phi x)(t+\omega) = (\phi x)(t)$, $t \in [0,\omega]$. So $\phi x \in Y$. In view of (A_3), for $x \in E$, $t \in [0,\omega]$, we have

$$f\big(t, x(t), x(t-\tau_1(t,x(t))), \ldots, x(t-\tau_n(t,x(t))), x'(t-\gamma_1(t,x(t))), \ldots, x'(t-\gamma_m(t,x(t)))\big)$$

$$\begin{aligned}&\geq \beta(t)x(t)+\sum_{i=1}^{n}b_i(t)x(t-\tau_i(t,x(t)))\\&\quad-\sum_{j=1}^{n}c_j(t)x'(t-\gamma_j(t,x(t)))\\&\geq \sigma\beta(t)\|x'\|+\sum_{i=1}^{n}b_i(t)\sigma\|x'\|-\sum_{j=1}^{n}c_j(t)\|x'\|\\&=\|x'\|\left[\sigma\beta(t)+\sum_{i=1}^{n}b_i(t)\sigma-\sum_{j=1}^{n}c_j(t)\right]>0,\end{aligned}\tag{26}$$

$$\begin{aligned}&f\Big(t,x(t),x(t-\tau_1(t,x(t))),\dots,\\&\quad x(t-\tau_n(t,x(t))),x'(t-\gamma_1(t,x(t))),\dots,\\&\quad x'(t-\gamma_m(t,x(t)))\Big)\\&\quad\leq\Big|f\Big(t,x(t),x(t-\tau_1(t,x(t))),\dots,\\&\qquad x(t-\tau_n(t,x(t))),\\&\qquad x'(t-\gamma_1(t,x(t))),\dots,\\&\qquad x'(t-\gamma_m(t,x(t)))\Big)-f(t,0,\dots,0)\Big|\\&\quad\leq\beta(t)x(t)+\sum_{i=1}^{n}b_i(t)x(t-\tau_i(t,x(t)))\\&\qquad+\sum_{j=1}^{n}c_j(t)x'(t-\gamma_j(t,x(t))).\end{aligned}\tag{27}$$

Therefore, for $x\in E$, $t\in[0,\omega]$, we find

$$\begin{aligned}\|\phi x\|&=\max_{t\in[0,\omega]}\{|\phi x(t)|\}\\&=\max_{t\in[0,\omega]}\Bigg\{\int_t^{t+\omega}\Big[G(t,s)x(s)\\&\qquad\times f\Big(s,x(s),\\&\qquad\quad x(s-\tau_1(s,x(s))),\dots,\\&\qquad\quad x(s-\tau_n(s,x(s))),\\&\qquad\quad x'(s-\gamma_1(s,x(s))),\dots,\\&\qquad\quad x'(s-\gamma_m(s,x(s)))\Big)\Big]ds\\&\qquad+\sum_{t\leq t_k<t+\omega}G(t,t_k)\theta_k(x(t_k))\Bigg\}\\&\leq\frac{1}{1-\sigma}\Bigg\{\int_0^{\omega}\Big[x(s)f\Big(s,x(s),\\&\qquad x(s-\tau_1(s,x(s))),\dots,\\&\qquad x(s-\tau_n(s,x(s))),\\&\qquad x'(s-\gamma_1(s,x(s))),\dots,\\&\qquad x'(s-\gamma_m(s,x(s)))\Big)\Big]ds\\&\qquad+\sum_{t\leq t_k<t+\omega}\theta_k(x(t_k))\Bigg\}.\end{aligned}\tag{28}$$

Furthermore, for $x\in E$, $t\in[0,\omega]$, we have

$$\begin{aligned}&(\phi x)(t)\\&\geq\frac{\sigma}{1-\sigma}\Bigg\{\int_t^{t+\omega}\Big[x(s)f\Big(s,x(s),\\&\qquad x(s-\tau_1(s,x(s))),\dots,\\&\qquad x(s-\tau_n(s,x(s))),\\&\qquad x'(s-\gamma_1(s,x(s))),\dots,\\&\qquad x'(s-\gamma_m(s,x(s)))\Big)\Big]ds\\&\qquad+\sum_{t\leq t_k<t+\omega}\theta_k(x(t_k))\Bigg\}\\&=\frac{\sigma}{1-\sigma}\Bigg\{\int_0^{\omega}\Big[x(s)f\Big(s,x(s),\\&\qquad x(s-\tau_1(s,x(s))),\dots,\\&\qquad x(s-\tau_n(s,x(s))),\\&\qquad x'(s-\gamma_1(s,x(s))),\dots,\\&\qquad x'(s-\gamma_m(s,x(s)))\Big)\Big]ds\\&\qquad+\sum_{t\leq t_k<t+\omega}\theta_k(x(t_k))\Bigg\}\geq\sigma\|\phi x\|.\end{aligned}\tag{29}$$

Now, we show that $(\phi x)(t)\geq\sigma\|\phi x\|$, $t\in[0,\omega]$.
On the other hand, from (23), we obtain

$$\begin{aligned}(\phi x)'(t)&=G(t,t+\omega)x(t+\omega)\\&\quad\times f\Big(t+\omega,x(t+\omega),\\&\qquad x(t+\omega-\tau_1(t+\omega,x(t+\omega))),\dots,\\&\qquad x(t+\omega-\tau_n(t+\omega,x(t+\omega))),\end{aligned}$$

$$\begin{aligned}
&x'\left(t+\omega-\gamma_1\left(t+\omega,x\left(t+\omega\right)\right)\right),\ldots,\\
&x'\left(t+\omega-\gamma_m\left(t+\omega,x\left(t+\omega\right)\right)\right)\Big)\\
&-G\left(t,t\right)x\left(t\right)f\Big(t,x\left(t\right),x\left(t-\tau_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x\left(t-\tau_n\left(t,x\left(t\right)\right)\right),\\
&x'\left(t-\gamma_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x'\left(t-\gamma_m\left(t,x\left(t\right)\right)\right)\Big)\\
&+a\left(t\right)\left(\phi x\right)\left(t\right)\\
&=a\left(t\right)\left(\phi x\right)\left(t\right)\\
&-x\left(t\right)f\Big(t,x\left(t\right),\\
&x\left(t-\tau_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x\left(t-\tau_n\left(t,x\left(t\right)\right)\right),\\
&x'\left(t-\gamma_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x'\left(t-\gamma_m\left(t,x\left(t\right)\right)\right)\Big).
\end{aligned} \tag{30}$$

It follows from (29) and (30) that if $(\phi x)'(t)\geq 0$, then

$$\left(\phi x\right)'\left(t\right)\leq a\left(t\right)\left(\phi x\right)\left(t\right)\leq a^M\left(\phi x\right)\left(t\right)\leq\left(\phi x\right)\left(t\right). \tag{31}$$

On the other hand, from (30) and (A_4), if $(\phi x)'(t)<0$, then

$$\begin{aligned}
-\left(\phi x\right)'\left(t\right)=&-a\left(t\right)\left(\phi x\right)\left(t\right)\\
&+x\left(t\right)f\Big(t,x\left(t\right),\\
&x\left(t-\tau_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x\left(t-\tau_n\left(t,x\left(t\right)\right)\right),\\
&x'\left(t-\gamma_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&x'\left(t-\gamma_m\left(t,x\left(t\right)\right)\right)\Big)\\
\leq&\|x\|_1^2\left[\beta\left(t\right)+\sum_{i=1}^{n}b_i\left(t\right)+\sum_{j=1}^{n}c_j\left(t\right)\right]-a^L\left(\phi x\right)\left(t\right)\\
\leq&\left(1+a^L\right)\frac{\sigma^2}{1-\sigma}\|x\|_1^2\\
&\times\int_t^{t+\omega}\left[\beta\left(s\right)+\sum_{i=1}^{n}b_i\left(s\right)-\sum_{j=1}^{n}c_j\left(s\right)\right]ds\\
&-a^L\left(\phi x\right)\left(t\right)\\
=&\left(1+a^L\right)\int_t^{t+\omega}\frac{\sigma}{1-\sigma}\sigma\|x\|_1\\
&\times\left[\beta\left(s\right)\|x\|_1+\sum_{i=1}^{n}b_i\left(s\right)\|x\|_1-\sum_{j=1}^{n}c_j\left(s\right)\|x\|_1\right]ds\\
&-a^L\left(\phi x\right)\left(t\right)\\
\leq&\left(1+a^L\right)\int_t^{t+\omega}G\left(t,s\right)x\left(s\right)\\
&\times\left[\beta\left(t\right)x\left(s\right)+\sum_{i=1}^{n}b_i\left(t\right)x\left(s-\tau_i\left(s\right)\right)-\sum_{j=1}^{n}c_j\left(t\right)\ x'\left(s-\gamma_j\left(s\right)\right)\right]ds\\
&-a^L\left(\phi x\right)\left(t\right)\\
=&\left(1+a^L\right)\int_t^{t+\omega}G\left(t,s\right)x\left(s\right)\\
&\times f\left(s,x\left(s\right),x\left(s-\tau_1\left(s,x\left(s\right)\right)\right),\ldots,\right.\\
&x\left(s-\tau_n\left(s,x\left(s\right)\right)\right),\\
&x'\left(s-\gamma_1\left(s,x\left(s\right)\right)\right),\ldots,\\
&\left.x'\left(s-\gamma_m\left(s,x\left(s\right)\right)\right)\right)ds\\
&-a^L\left(\phi x\right)\left(t\right)\\
=&\left(1+a^L\right)\left(\phi x\right)\left(t\right)-a^L\left(\phi x\right)\left(t\right)=\left(\phi x\right)\left(t\right).
\end{aligned} \tag{32}$$

It follows from (31) and (32) that $\|(\phi x)'\|\leq\|\phi x\|$. So $\|\phi x\|_1=\|\phi x\|_0$. By (29) we have $(\phi x)(t)\geq\sigma\|\phi x\|_1$. Hence, $\phi x\in E$. This completes the proof of (i).

(ii) In view of the proof of (i), we only need to prove that $(\phi x)'(t)\geq 0$ implies $(\phi x)'(t)\leq(\phi x)(t)$. From (23), (26), (A_3), and (A_5), we have

$$\begin{aligned}
\left(\phi x\right)'\left(t\right)=&\ a\left(t\right)\left(\phi x\right)\left(t\right)-x\left(t\right)\\
&\times f\left(t,x\left(t\right),x\left(t-\tau_1\left(t,x\left(t\right)\right)\right),\ldots,\right.\\
&x\left(t-\tau_n\left(t,x\left(t\right)\right)\right),\\
&x'\left(t-\gamma_1\left(t,x\left(t\right)\right)\right),\ldots,\\
&\left.x'\left(t-\gamma_m\left(t,x\left(t\right)\right)\right)\right)\\
\leq&\ a\left(t\right)\left(\phi x\right)\left(t\right)-\sigma\|x\|_1
\end{aligned}$$

$$\begin{aligned}
&\times f\Big(t, x(t), x(t-\tau_1(t,x(t))), \ldots, \\
&\qquad x(t-\tau_n(t,x(t))), \\
&\qquad x'(t-\gamma_1(t,x(t))), \ldots, \\
&\qquad x'(t-\gamma_m(t,x(t)))\Big) \\
&\le a^M(\phi x)(t) - \sigma\|x\|_1^2 \\
&\quad \times \left[\sigma\beta(t) + \sigma\sum_{i=1}^{n} b_i(t) - \sum_{j=1}^{n} c_j(t)\right] \\
&\le a^M(\phi x)(t) - \sigma\|x\|_1^2 \frac{a^M - 1}{\sigma(1-\sigma)} \\
&\quad \times \int_0^{\omega}\left[\beta(s) + \sum_{i=1}^{n} b_i(s) + \sum_{j=1}^{n} c_j(s)\right] ds \\
&= a^M(\phi x)(t) - \left(a^M - 1\right) \\
&\quad \times \int_t^{t+\omega} \frac{1}{1-\sigma}\|x\|_1 \left[\beta(s)\|x\|_1 + \sum_{i=1}^{n} b_i(s)\|x\|_1 \right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n} c_j(s)\|x\|_1\right] ds \\
&\le a^M(\phi x)(t) - \left(a^M - 1\right) \\
&\quad \times \int_t^{t+\omega} G(t,s)\, x(s) \\
&\qquad \times \left[\beta(s)x(s) + \sum_{i=1}^{n} b_i(s)\, x(s-\tau_i(s,x(s))) \right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n} c_j(s)\, x\left(s-\gamma_j(s,x(s))\right)\right] ds \\
&\le a^M(\phi x)(t) - \left(a^M - 1\right) \\
&\quad \times \left\{\int_t^{t+\omega} G(t,s)\, x(s)\right. \\
&\qquad \times f\big(s, x(s), x(s-\tau_1(s,x(s))), \ldots, \\
&\qquad\qquad x(s-\tau_n(s,x(s))), \\
&\qquad\qquad x'(s-\gamma_1(s,x(s))), \ldots, \\
&\qquad\qquad x'(s-\gamma_m(s,x(s)))\big)\, ds \\
&\qquad \left. + \sum_{t \le t_k < t+\omega} G(t,t_k)\,\theta_k(x(t_k))\right\} \\
&= a^M(\phi x)(t) - \left(a^M - 1\right)(\phi x)(t) = (\phi x)(t).
\end{aligned} \tag{33}$$

The proof of (ii) is complete. Thus we complete the proof of Lemma 9. □

Lemma 10. *Assume that* (A_1)*–*(A_4) *hold and* $R\sum_{j=1}^{n} c_j^M < 1$.

(i) *If* $a^M \le 1$, *then* $\phi : E \bigcap \overline{\Omega_R} \to E$ *is strict-set-contractive.*

(ii) *If* (A_5) *holds and* $a^M > 1$, *then* $\phi : E \bigcap \overline{\Omega_R} \to E$ *is strict-set-contractive,*

where $\Omega_R = \{x \in Y : |x|_1 < R\}$.

Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that ϕ is continuous and bounded. Now we prove that a $\alpha_Y(\phi(S)) \le R\sum_{j=1}^{n} c_j^M \alpha_Y(S)$ for any bounded set $S \in \overline{\Omega}_R$. Let $\eta = \alpha_Y(S)$; then, for any positive number $\epsilon < R\sum_{j=1}^{n} c_j^M \eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\mathrm{diam}(S_i) \le \eta + \epsilon$. Therefore,

$$\left|x - y\right|_1 \le \eta + \epsilon, \quad \text{for any } x, y \in S_i. \tag{34}$$

As S and S_i are precompact in X, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i = \bigcup_j S_{ij}$ and

$$\left|x - y\right|_0 \le \epsilon, \quad \text{for any } x, y \in S_{ij}. \tag{35}$$

In addition, for any $x \in S$ and $t \in [0, \omega]$, we have

$$\begin{aligned}
(\phi x)(t) = &\int_t^{t+\omega}\Big[G(t,s)\, x(s) \\
&\qquad \times f\Big(s, x(s), x(s-\tau_1(s,x(s))), \ldots, \\
&\qquad\qquad x(s-\tau_n(s,x(s))), \\
&\qquad\qquad x'(s-\gamma_1(s,x(s))), \ldots, \\
&\qquad\qquad x'(s-\gamma_m(s,x(s)))\Big)\Big] ds \\
&+ \sum_{t \le t_k < t+\omega} G(t,t_k)\,\theta_k(x(t_k)) \\
\le\ &\frac{R^2}{1-\sigma}\int_0^{\omega}\left[\beta(s) + \sum_{j=1}^{n} b_i(s) + \sum_{j=1}^{n} c_j(s)\right] ds \\
&+ \frac{1}{1-\sigma}\sum_{t \le t_k < t+\omega} \theta_k(x(t_k)) := \Delta,
\end{aligned} \tag{36}$$

$$\begin{aligned}\left|(\phi x)'(t)\right| = & \left|a(t)(\phi x)(t) - x(t)\right. \\ & \times f\left(t, x(t),\ x(t-\tau_1(t,x(t))),\ldots,\right. \\ & x(t-\tau_n(t,x(t))), \\ & x'(t-\gamma_1(t,x(t))),\ldots, \\ & \left.\left. x'(t-\gamma_m(t,x(t)))\right)\right| \\ \le & a^M\Delta + R^2\left(\beta^M + \sum_{i=1}^{n} b_i^M + \sum_{j=1}^{m} c_j^M\right).\end{aligned} \tag{37}$$

Hence,

$$\begin{gathered}\|(\phi x)\| \le \Delta, \\ \|(\phi x)'\| \le a^M\Delta + R^2\left(\beta^M + \sum_{i=1}^{n} b_i^M + \sum_{j=1}^{m} c_j^M\right).\end{gathered} \tag{38}$$

Applying the Arzela-Ascoli theorem, we know that $\phi(S)$ is precompact in X. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|\phi x - \phi y|_0 \le \epsilon, \quad \text{for any } x, y \in S_{ijk}. \tag{39}$$

From (34)–(39) and (A_3), for any $x, y \in S_{ijk}$, we have

$$\begin{aligned}&\|(\phi x)' - (\phi y)'\| \\ &= \max_{t\in[0,\omega]}\left\{\left|a(t)(\phi x)(t) - a(t)(\phi y)(t) - x(t)\right.\right. \\ &\quad \times f\left(t, x(t), x(t-\tau_1(t,x(t))),\ldots,\right. \\ &\quad x(t-\tau_n(t,x(t))), \\ &\quad x'(t-\gamma_1(t,x(t))),\ldots, \\ &\quad \left. x'(t-\gamma_m(t,x(t)))\right) \\ &\quad + y(t) f\left(t, y(t), y(t-\tau_1(t,y(t))),\ldots,\right. \\ &\quad y(t-\tau_n(t,y(t))), \\ &\quad y'(t-\gamma_1(t,y(t))),\ldots, \\ &\quad \left.\left.\left. y'(t-\gamma_m(t,y(t)))\right)\right|\right\} \\ &\le \max_{t\in[0,\omega]}\{|a(t)((\phi x)(t) - (\phi y)(t))|\} \\ &\quad + \max_{t\in[0,\omega]}\left\{\left|x(t) f\left(t, x(t), x(t-\tau_1(t,x(t))),\ldots,\right.\right.\right. \\ &\quad x(t-\tau_n(t,x(t))), \\ &\quad x'(t-\gamma_1(t,x(t))),\ldots, \\ &\quad \left. x'(t-\gamma_m(t,x(t)))\right) \\ &\quad - y(t) f\left(t, y(t), y(t-\tau_1(t,y(t))),\ldots,\right. \\ &\quad y(t-\tau_n(t,y(t))), \\ &\quad y'(t-\gamma_1(t,y(t,y(t)))),\ldots, \\ &\quad \left.\left.\left. y'(t-\gamma_m(t,y(t)))\right)\right|\right\} \\ &\le a^M\|(\phi x) - (\phi y)\| \\ &\quad + \max_{t\in[0,\omega]}\left\{\left|x(t)\left[f\left(t, x(t), x(t-\tau_1(t,x(t))),\ldots,\right.\right.\right.\right. \\ &\quad x(t-\tau_n(t,x(t))), \\ &\quad x'(t-\gamma_1(t,x(t))),\ldots, \\ &\quad \left. x'(t-\gamma_m(t,x(t)))\right) \\ &\quad - f\left(t, y(t), y(t-\tau_1(t,y(t))),\ldots,\right. \\ &\quad y(t-\tau_n(t,y(t))), \\ &\quad y'(t-\gamma_1(t,y(t,y(t)))),\ldots, \\ &\quad \left.\left.\left.\left. y'(t-\gamma_m(t,y(t)))\right)\right]\right|\right\} \\ &\quad + \max_{t\in[0,\omega]}\left\{\left|[x(t) - y(t)]\right.\right. \\ &\quad \times f\left(t, y(t), y(t-\tau_1(t,y(t))),\ldots,\right. \\ &\quad y(t-\tau_n(t,y(t))), \\ &\quad y'(t-\gamma_1(t,y(t,y(t)))),\ldots, \\ &\quad \left.\left.\left. y'(t-\gamma_m(t,y(t)))\right)\right]\right|\Big\} \\ &\le a^M\|(\phi x) - (\phi y)\| \\ &\quad + R\max_{t\in[0,\omega]}\Big\{\gamma(t)|x(t) - y(t)| \\ &\quad + \sum_{i=1}^{n} b_i(t)\left|x(t-\tau_i(t,x(t)))\right. \\ &\quad \left. - y(t-\tau_i(t,y(t)))\right|\end{aligned}$$

$$+\sum_{j=1}^{m} c_j(t)\left|x'\left(t-\gamma_j(t,x(t))\right)\right.$$
$$\left.\left.-y'\left(t-\gamma_j(t,y(t))\right)\right|\right\}$$

$$\epsilon \max_{t\in[0,\omega]}\left\{\beta(t)y(t)+\sum_{i=1}^{n} b_i(t)y\left(t-\tau_i(t,y(t))\right)\right.$$
$$\left.+\sum_{j=1}^{m} c_j(t)\left|y'\left(t-\gamma_j(t,y(t))\right)\right|\right\}$$
$$\le a^M\epsilon + R\epsilon\left(\beta^M+\sum_{i=1}^{n} b_i^M\right)+R\sum_{j=1}^{m} c_j^M(\epsilon+\eta)$$
$$+R\epsilon\sum\left(\beta^M+\sum_{i=1}^{n} b_i^M+\sum_{j=1}^{m} c_j^M\right)$$
$$=\eta\left(\sum_{j=1}^{m} c_j^M R\right)+\Gamma\epsilon, \tag{40}$$

where

$$\Gamma := a^M + 2R\left(\beta^M+\sum_{i=1}^{n} b_i^M+\sum_{j=1}^{m} c_j^M\right). \tag{41}$$

From (40) we obtain

$$\left\|\phi x-\phi y\right\|_1 \le \eta\left(\sum_{j=1}^{m} c_j^M R\right)+\Gamma\epsilon, \quad \text{for any } x, y \in S_{ijk}. \tag{42}$$

As ϵ is arbitrary small, it follows that

$$\alpha_Y(\phi(S)) \le \left(R\sum_{j=1}^{n} c_j^M\right)\alpha_Y(S). \tag{43}$$

Therefore, ϕ is strict-set-contractive. The proof of Lemma 10 is complete. □

Lemma 11. *Assume that* (A_1)–(A_4) *hold.*

(i) *If* $a^M \le 1$, *then* x *is a positive* ω*-periodic solution of model* (1), *where* x *is a nonzero fixed point of the operator* ϕ *on* E.

(ii) *If* (A_5) *holds and* $a^M > 1$, *then* x *is a positive* ω*-periodic solution of model* (1), *where* x *is a nonzero fixed point of the operator* ϕ *on* E.

3. Main Results

In this section, we will study the existence of positive ω-periodic solutions of system (1).

Theorem 12. *Assume that* (A_1)–(A_4), *and* (A_6) *hold.*

(i) *If* $a^M \le 1$, *then system* (1) *has at least one positive* ω*-periodic solution.*

(ii) *If* (A_5) *holds and* $a^M > 1$, *then system* (1) *has at least one positive* ω*-periodic solution.*

Proof. We only need to prove (i), since the proof of (ii) is similar. Let

$$R=\frac{1-\sigma}{\sigma^2 B_1}, \qquad 0<r<\frac{\sigma(1-\sigma)-\xi}{B_2}. \tag{44}$$

Then it is easy to see that $0<r<R$. From Lemmas 9 and 10, we know that ϕ is strict-set-contractive on $E_{r,R}$. By Lemma 11, we see that if there exists $x^* \in E$ such that $\phi x^* = x^*$, then x^* is one positive ω-periodic solution of system (1). Now, we will prove that condition (b) of Lemma 8 holds.

First, we prove that $Tx \ngeq x$, $\forall x \in E$, $\|x\|_1 < r$. Otherwise, there exist $x\in E$, $\|x\|_1<r$, such that $Tx \ge x$. So $\|x\|>0$ and $\phi x - x \ge 0$, which implies that

$$(\phi x)(t)-x(t) \ge \sigma\left\|\phi x - x\right\|_1 \ge 0, \quad \text{for any } t\in[0,\omega]. \tag{45}$$

Moreover, for $t\in[0,\omega]$, we have

$$(\phi x)(t) = \int_t^{t+\omega}\left[G(t,s)x(s)\right.$$
$$\times f\left(s, x(s),\ x\left(s-\tau_1(s,x(s))\right),\dots,\right.$$
$$x\left(s-\tau_n(s,x(s))\right),$$
$$x'\left(s-\gamma_1(s,x(s))\right),\dots,$$
$$\left.\left.x'\left(s-\gamma_m(s,x(s))\right)\right)\right]ds$$
$$+\sum_{t\le t_k<t+\omega} G(t,t_k)\theta_k(x(t_k))$$
$$\le \frac{r}{1-\sigma}\|x\|$$
$$\times\left\{\int_0^{\omega}\left[\beta(s)+\sum_{i=1}^{n} b_i(s)+\sum_{j=1}^{m} c_j(s)\right]ds+\xi\right\}$$
$$\le \frac{B_2 r+\xi}{1-\sigma}\|x\| \le \sigma\|x\|. \tag{46}$$

In view of (45) and (46), we obtain

$$\|x\| \le \|\phi x\| \le \sigma\|x\| < \|x\|, \tag{47}$$

which is a contradiction.

Finally, we prove that $\phi x \nleq x$, $\forall x \in E$, $\|x\|_1 = R$. For this case, for the sake of contradiction, suppose that there exist $x \in E$, $\|x\|_1 = R$ such that $\phi x \le x$. Furthermore, for any $t\in[0,\omega]$, we have

$$x(t)-\phi x(t) \ge \sigma\left\|x-\phi x\right\| \ge 0, \quad \text{for any } t\in[0,\omega]. \tag{48}$$

In addition, for any $t \in [0,\omega]$, we find

$$\begin{aligned}(\phi x)(t) = & \int_t^{t+\omega}\Big[G(t,s)\,x(s) \\ & \times f\Big(s, x(s),\ x(s-\tau_1(s,x(s))),\dots, \\ & x(s-\tau_n(s,x(s))), \\ & x'(s-\gamma_1(s,x(s))),\dots, \\ & x'(s-\gamma_m(s,x(s)))\Big)\Big]\,ds \\ & + \sum_{t\le t_k<t+\omega} G(t,t_k)\,\theta_k(x(t_k)) \\ > & \frac{\sigma^2}{1-\sigma}\|x\|^2 \\ & \times \int_0^{\omega}\left[\sigma\beta(s)+\sigma\sum_{i=1}^{n} b_i(s)-\sum_{j=1}^{m} c_j(s)\right]ds \\ = & \frac{\sigma^2}{1-\sigma}B_1R^2 = R,\end{aligned} \tag{49}$$

which is a contradiction. Therefore, condition (b) of Lemma 8 holds. By Lemma 8, we see that ϕ has at least one nonzero fixed point in E. Thus, the system (11) has at least one positive ω-periodic solution. Therefore, it follows from Lemma 7 that system (1) has a positive ω-periodic solution. The proof of Theorem 12 is complete. □

4. Applications

In this section, we apply the result obtained in the previous section to some periodic population models with impulses which are mentioned in the first section.

First, we consider a general neutral delay model of single-species population growth with impulse:

$$\begin{aligned}\frac{dN}{dt} = & N(t)\Big[a(t)-\beta(t)N(t)-\sum_{i=1}^{n} b_i(t)\,N(t-\tau_i(t)) \\ & -\sum_{i=1}^{n} c_i(t)\,N'(t-\gamma_i(t))\Big],\quad t\neq t_k,\ k\in Z_+, \\ N(t_k^+) = & N(t_k^-)+\theta_k(N(t_k)),\quad k\in Z_+,\end{aligned} \tag{50}$$

and we investigate a complex neutral equation with several state-dependent delays and impulse:

$$\begin{aligned}\frac{dN}{dt} = & N(t)\Big[a(t)-\beta(t)N(t) \\ & -\sum_{i=1}^{n} b_i(t)\,N(t-\tau_i(t,N(t))) \\ & -\sum_{i=1}^{n} c_i(t)\,N'(t-\gamma_i(t,N(t)))\Big], \\ & t\neq t_k,\ k\in Z_+, \\ N(t_k^+) = & N(t_k^-)+\theta_k(N(t_k)),\quad k\in Z_+.\end{aligned} \tag{51}$$

For convenience, we list several assumptions:

(A_1^*), (A_2^*), (A_3^*), and (A_4^*) are the same as (A_1), (A_4), (A_5), and (A_6), respectively;

(A_5^*) $\beta(t), b_i(t), c_i(t) \in C(R,R^+)$ $(i=1,2,\dots,n)$ are ω-periodic functions and

$$\sigma\beta(t)+\sigma\sum_{i=1}^{n} b_i(t)-\sum_{i=1}^{n} c_i(t) > 0,\quad t\in[0,\omega]. \tag{52}$$

Theorem 13. *Assume (A_1^*)–(A_3^*), (A_5^*) hold.*

(i) *If $a^M \le 1$, then systems* (50) *and* (51) *have at least one positive ω-periodic solution.*

(ii) *If (A_5^*) holds and $a^M > 1$, then systems* (50) *and* (51) *have at least one positive ω-periodic solution.*

Proof. The proof is similar to that of Theorem 12; we omit the details here.

Second, we consider a general neutral delay model of single-species population growth with impulse:

$$\begin{aligned}\frac{dN}{dt} = & N(t)\Big[a(t)-\beta(t)N(t) \\ & -\sum_{i=1}^{n} b_i(t)\,N(t-\tau_i(t)) \\ & -\sum_{j=1}^{m} c_j(t)\,N'(t-\gamma_j(t))\Big], \\ & t\neq t_k,\ k\in Z_+, \\ N(t_k^+) = & N(t_k^-)+\theta_k(N(t_k)),\quad k\in Z_+,\end{aligned} \tag{53}$$

and we investigate a periodic Lotka-Volterra equation with state-dependent delays and impulse:

$$\begin{aligned}\frac{dN}{dt} = & N(t)\Big[r(t)-a(t)N(t) \\ & -\sum_{i=1}^{n} b_i(t)\,N(t-\tau_i(t,N(t))) \\ & -\sum_{j=1}^{m} c_j(t)\,N'(t-\gamma_j(t,N(t)))\Big], \\ & t\neq t_k,\ k\in Z_+, \\ N(t_k^+) = & N(t_k^-)+\theta_k(N(t_k)),\quad k\in Z_+.\end{aligned} \tag{54}$$

For convenience, we list several assumptions:

(H_1), (H_2), (H_3), and (H_4) are the same as (A_1), (A_4), (A_5), and (A_6), respectively;

(H_5) $\beta(t), b_i(t), c_j(t) \in C(R, R^+)$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ are ω-periodic functions and

$$\sigma\beta(t) + \sigma\sum_{i=1}^{n} b_i(t) - \sum_{j=1}^{m} c_j(t) > 0, \quad t \in [0, \omega]. \tag{55}$$

Then we can obtain the following theorem. □

Theorem 14. *Assume* (H_1)–(H_4) *hold.*

(i) *If* $a^M \le 1$, *then systems* (53) *and* (54) *have at least one positive* ω*-periodic solution.*

(ii) *If* (H_5) *holds and* $a^M > 1$, *then systems* (53) *and* (54) *have at least one positive* ω*-periodic solution.*

Proof. The proof is similar to that of Theorem 12; we omit the details here. □

Remark 15. We apply the main result obtained in the previous section to study some examples which have some biological implications. A very basic and important ecological problem associated with the study of population is that of the existence of a positive periodic solution which plays the role played by the equilibrium of the autonomous models and means that the species is in an equilibrium state. From Theorems 13 and 14, we see that, under the appropriate conditions, the impulsive perturbations do not affect the existence of periodic solution of systems. Therefore, our result generalizes and improves the corresponding results in [12–17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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