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Research Article

A New Auto-Bäcklund Transformation of the KdV Equation with General Variable Coefficients and Its Application

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First, by improving some key steps in the homogeneous balance method, a new auto-Bäcklund transformation (BT) to the KdV equation with general variable coefficients is derived. The new auto-BT in this paper does not require the coefficients of the equation to be linearly dependent. Then, based on the new auto-BT in which there is only one quadratic homogeneity equation to be solved, an exact soliton-like solution containing 2-solitary wave is given.

1. Introduction

The KdV equation with general variable coefficients reads

$$u_t + f(t)\,uu_x + g(t)\,u_{xxx} = 0, \tag{1}$$

where $f(t)$ and $g(t)$ are arbitrary analytic functions of t, which was originally proposed in [1]. Equation (1) is well known as a model equation describing the propagation of weakly nonlinear and weakly dispersive waves in inhomogeneous media. In recent decades, much progress has been made in the studies of (1) obtaining its auto-BT and exact solutions [2–6]. In [3], by using the truncate Painlevé expansion [7], Hong and Jung obtained an auto-BT which includes 5 Painlevé-Bäcklund equations to be solved. In [4], by using the homogeneous balance principle (HBP) [8–11], M. Wang and Y. Wang derived an auto-BT of (1) as follows:

$$\begin{gathered}
u(x,t) = 12k(\ln\varphi)_{xx} + V(x,t),\\
\varphi\left(\varphi_{xt} + g\varphi_{xxxx} + Vf\varphi_{xx}\right) - \varphi_x\left(\varphi_t + g\varphi_{xxx} + Vf\varphi_x\right) + 3g\left(\varphi_{xx}^2 - \varphi_x\varphi_{xxx}\right) = 0,\\
V_t + f(t)\,VV_x + g(t)\,V_{xxx} = 0,\\
g(t) = kf(t), \quad k = \text{const} \neq 0.
\end{gathered} \tag{2}$$

Then, based on auto-BT (2), by using the ε-expansion method, 2-soliton solution of (1) is given and discussed in detail. In [5], Fan obtained an auto-BT of (1) in the following form:

$$\begin{gathered}
u(x,t) = 12k(\ln\varphi)_{xx} + V(x,t),\\
\varphi_x\varphi_t + 4g\varphi_x\varphi_{xxx} - 3g\varphi_{xx}^2 + Vf\varphi_x^2 = 0,\\
\varphi_{xt} + g\varphi_{xxxx} + Vf\varphi_{xx} = 0,\\
V_t + f(t)\,VV_x + g(t)\,V_{xxx} = 0,\\
g(t) = kf(t), \quad k = \text{const} \neq 0.
\end{gathered} \tag{3}$$

It is easy to see that auto-BT (3) is a special one of auto-BT (2).

The aim of the present paper is to derive a new auto-BT for (1) by improving some key steps in the homogeneous balance method. One can see that the new auto-BT in this paper does not require the coefficients of the equation to be linearly dependent. Then, based on the new auto-BT, an exact soliton-like solution containing 2-solitary wave is given.

2. Derivation of a New Auto-BT for (1)

According to the idea of HBP, considering homogeneous balance between $f(t)uu_x$ and $g(t)u_{xxx}$ in (1), the first crucial step is the assumption that we seek for the auto-BT of (1) in the following form:

$$u(x,t) = H(t)\frac{\partial^2}{\partial x^2}\left[F(\varphi(x,t))\right] + v_0(x,t) = H(t)\left[F''\varphi_x^2 + F'\varphi_{xx}\right] + v_0(x,t), \tag{4}$$

where $F = F(\varphi)$ is a function of one argument only, $F' = dF/d\varphi$, $F^{(k)} = d^kF/d\varphi^k$, the subscripts denote the partial derivatives, $v_0(x,t)$ is a given solution of (1), $F(\varphi)$, $\varphi(x,t)$, and $H(t)$ are to be determined later. By using (4), we have

$$u_t + f(t)uu_x + g(t)u_{xxx} = \left[g(t)H(t)F^{(5)} + f(t)H^2(t)F''F'''\right]\varphi_x^5 + \cdots, \tag{5}$$

where the unwritten part in (5) is a polynomial of various partial derivatives of $\varphi(x,t)$, the degree of which is lower than 5. Setting the coefficient of φ_x^5 to zero yields an ordinary differential equation for $F(\varphi)$ as follows:

$$g(t)H(t)F^{(5)} + f(t)H^2(t)F''F''' = 0, \tag{6}$$

which has a solution as follows:

$$F(\varphi) = 12\ln\varphi, \qquad H(t) = \frac{g(t)}{f(t)}; \tag{7}$$

thereby

$$F''^2 = -2F^{(4)}, \qquad F'F''' = -4F^{(4)}, \qquad F'F'' = -6F''', \qquad F'^2 = -12F''. \tag{8}$$

Substituting (4) with (7) into (1) and using (8), we get

$$\begin{aligned} u_t &+ f(t)uu_x + g(t)u_{xxx} \\ &= \Big\{H(t)\varphi_t\varphi_x^2 + f(t)H^2(t) \times \left[4\varphi_x^2\varphi_{xxx} - 3\varphi_x\varphi_{xx}^2\right] + f(t)H(t)v_0\varphi_x^3\Big\}F''' \\ &+ \Big\{2H(t)\varphi_x\varphi_{xt} + H(t)\varphi_{xx}\varphi_t + f(t)H^2(t) \times \left[5\varphi_x\varphi_{xxxx} - 2\varphi_{xx}\varphi_{xxx}\right] + H'(t)\varphi_x^2 \\ &\quad + 3f(t)H(t)v_0\varphi_x\varphi_{xx} + f(t)H(t)v_{0x}\varphi_x^2\Big\}F'' \\ &+ \Big\{H(t)\varphi_{xxt} + f(t)H^2(t)\varphi_{xxxxx} + H'(t)\varphi_{xx} + f(t)H(t)\left[v_0\varphi_{xxx} + v_{0x}\varphi_{xx}\right]\Big\}F' \\ &+ v_{0t} + f(t)v_0v_{0x} + g(t)v_{0xxx} = 0. \end{aligned} \tag{9}$$

Decompose the coefficient of F''' as

$$\Big\{H(t)\varphi_t\varphi_x + f(t)H^2(t)\left[4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right] + f(t)H(t)v_0\varphi_x^2\Big\}\varphi_x. \tag{10}$$

Denote

$$A = H(t)\varphi_t\varphi_x + f(t)H^2(t)\left[4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right] + f(t)H(t)v_0\varphi_x^2. \tag{11}$$

Integrate the coefficient of F' with respect to x once and let the constant of integration be zero. Denote

$$B = H(t)\varphi_{xt} + f(t)H^2(t)\varphi_{xxxx} + H'(t)\varphi_x + f(t)H(t)v_0\varphi_{xx}. \tag{12}$$

Rewriting the coefficient of F'', we have

$$\text{Coeff. of } F'' = A_x + B\varphi_x. \tag{13}$$

Substituting (11), (12), and (13) into (9) gives

$$A\varphi_xF''' + \left(A_x + B\varphi_x\right)F'' + B_xF' = 0; \tag{14}$$

that is,

$$\frac{24A\varphi_x}{\varphi^3} - \frac{12\left(A_x + B\varphi_x\right)}{\varphi^2} + \frac{12B_x}{\varphi} = 0. \tag{15}$$

By direct calculating, we can write (15) as follows:

$$12\left(\frac{B\varphi - A}{\varphi^2}\right)_x = 0. \tag{16}$$

Note the following:

$$\begin{aligned} B\varphi - A = &\Big\{H(t)\varphi_{xt} + f(t)H^2(t)\varphi_{xxxx} + H'(t)\varphi_x + f(t)H(t)v_0\varphi_{xx}\Big\}\varphi \\ &- \Big\{H(t)\varphi_t\varphi_x + f(t)H^2(t) \times \left[4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right] + f(t)H(t)v_0\varphi_x^2\Big\}; \end{aligned} \tag{17}$$

thus we have a new auto-Bäcklund transformation of (1) as follows:

$$u(x,t) = 12H(t)\frac{\partial^2}{\partial x^2}\left[\ln\varphi(x,t)\right] + v_0(x,t), \qquad H(t) = \frac{g(t)}{f(t)}, \tag{18}$$

$$\begin{aligned} &\Big\{H(t)\varphi_{xt} + f(t)H^2(t)\varphi_{xxxx} + H'(t)\varphi_x + f(t)H(t)v_0\varphi_{xx}\Big\}\varphi \\ &- \Big\{H(t)\varphi_t\varphi_x + f(t)H^2(t)\left[4\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right] + f(t)H(t)v_0\varphi_x^2\Big\} = 0. \end{aligned} \tag{19}$$

The meaning of the new auto-BT is that if $v_0 = v_0(x,t)$ is a given solution of (1) and $\varphi = \varphi(x,t)$ is a solution of (19), then expression (18) is another solution of (1).

The result shows that the new auto-BT does not require the coefficients of the equation to be linearly dependent; in fact, there are no constraints between $f(t)$ and $g(t)$.

3. Exact Soliton-Like Solutions of (1)

Now, we use the auto-BT that consisted of (18) and (19) to find soliton-like solutions of (1).

First, notice that, under the condition $H(t) = \widehat{H}(t) = a_0 \int^t f(s)ds + b_0$, $(a_0^2 + b_0^2 \neq 0)$, (1) admits a special solution as follows:

$$v(x,t) = \widehat{v}_0 = \frac{a_0 x + c_0}{a_0 \int^t f(s)\,ds + b_0}, \tag{20}$$

where a_0, b_0, and c_0 are arbitrary constants. After a direct calculation, we find the following:

$$\begin{aligned} \widehat{H}'(t)\varphi_x + f(t)\widehat{H}(t)\widehat{v}_0\varphi_{xx} &= \left[(a_0 x + c_0) f(t)\varphi_x\right]_x \\ &= \left[f(t)\widehat{H}(t)\widehat{v}_0\varphi_x\right]_x. \end{aligned} \tag{21}$$

Then, (19) becomes the following form:

$$\begin{aligned} &\varphi\left\{\widehat{H}(t)\varphi_t + f(t)\widehat{H}^2(t)\varphi_{xxx} + (a_0 x + c_0) f(t)\varphi_x\right\}_x \\ &\quad - \varphi_x\left\{\widehat{H}(t)\varphi_t + f(t)\widehat{H}^2(t)\varphi_{xxx} + (a_0 x + c_0) f(t)\varphi_x\right\} \\ &\quad - 3f(t)\widehat{H}^2(t)\left[\varphi_x\varphi_{xxx} - \varphi_{xx}^2\right] = 0. \end{aligned} \tag{22}$$

Denote

$$\begin{aligned} P &= \widehat{H}(t)\partial_t + f(t)\widehat{H}^2(t)\partial_x^3 + (a_0 x + c_0) f(t)\partial_x, \\ P[\varphi] &= \widehat{H}(t)\varphi_t + f(t)\widehat{H}^2(t)\varphi_{xxx} + (a_0 x + c_0) f(t)\varphi_x; \end{aligned} \tag{23}$$

thus (22) can be rewritten as follows:

$$\varphi\{P[\varphi]\}_x - \varphi_x P[\varphi] - 3f(t)\widehat{H}^2(t)\left[\varphi_x\varphi_{xxx} - \varphi_{xx}^2\right] = 0. \tag{24}$$

It is not difficult to find that linear equation $P[\varphi] = 0$ admits an exponential solution as follows:

$$\varphi(x,t) = 1 + \exp\eta, \qquad \eta = p(t)x + q(t) + \eta_0, \tag{25}$$

where

$$\begin{aligned} p(t) &= \frac{\beta}{a_0 \int^t f(s)\,ds + b_0}, \\ q(t) &= -\int^t \frac{\left[\beta^3 + c_0\beta\right] f(\tau)}{\left[a_0 \int^\tau f(s)\,ds + b_0\right]^2} d\tau, \end{aligned} \tag{26}$$

and η_0 and β are arbitrary constants. In fact, substituting (25) into $P[\varphi] = 0$ gives

$$\begin{aligned} &\left\{\widehat{H}(t)\left[p'(t)x + q'(t)\right] + f(t)\widehat{H}^2(t)p^3(t) \right. \\ &\quad \left. + (a_0 x + c_0) f(t)p(t)\right\}\exp\eta = 0. \end{aligned} \tag{27}$$

Let

$$\begin{aligned} &\widehat{H}(t)p'(t) + a_0 f(t)p(t) = 0, \\ &\widehat{H}(t)q'(t) + f(t)\widehat{H}^2(t)p^3(t) + c_0 f(t)p(t) = 0, \end{aligned} \tag{28}$$

and solving (28) yields (26). In addition, noting that the third part of (24) vanishes if it is substituted with (25), we immediately find that (25) with (26) is a solution of (24). Substituting (25) with (26) and $\widehat{v}_0$ into (18) yields a new exact solution containing a single solitary wave of (1) as

$$\begin{aligned} u(x,t) &= \left(\frac{a_0 x + c_0}{a_0 \int^t f(s)\,ds + b_0}\right) + \left(\frac{3\beta^2}{a_0 \int^t f(s)\,ds + b_0}\right) \\ &\quad \times \operatorname{sech}^2\frac{1}{2}\left[p(t)x + q(t) + \eta_0\right], \end{aligned} \tag{29}$$

where $p(t)$ and $q(t)$ are expressed by (26).

From (29), it is seen that the amplitude of single solitary wave is $3\beta^2[a_0 \int^t f(s)ds + b_0]^{-1}$; and the propagating speed of single solitary wave is given by

$$\frac{dx}{dt} = \frac{\beta^2 f(t) + (a_0 x + c_0) f(t)}{a_0 \int^t f(s)\,ds + b_0}, \tag{30}$$

which depends upon not only time t but also the spatial variable x.

In order to obtain more general solutions of (1), according to auto-BT that consisted of (18) and (19), we need to seek more general solutions of homogeneity equation (24). Take

$$\begin{aligned} \overset{\star}{\varphi}(x,t) &= 1 + \exp\eta_1 + \exp\eta_2 + a_{12}\exp(\eta_1 + \eta_2), \\ \eta_i &= p_i(t)x + q_i(t) + \eta_{i0}, \quad (i = 1,2), \end{aligned} \tag{31}$$

where

$$\begin{aligned} p_i(t) &= \frac{\beta_i}{a_0 \int^t f(s)\,ds + b_0}, \\ q_i(t) &= -\int^t \frac{\left[\beta_i^3 + c_0\beta_i\right] f(\tau)}{\left[a_0 \int^\tau f(s)\,ds + b_0\right]^2} d\tau, \end{aligned} \tag{32}$$

η_{i0} and β_i $(i = 1,2)$ are arbitrary constants, and a_{12} is an undetermined constant. Because P is a linear operator and $P[\exp\eta_i] = 0$, $(i = 1,2)$,

$$\begin{aligned} P\left[\overset{\star}{\varphi}\right] &= P\left[1 + \exp\eta_1 + \exp\eta_2 + a_{12}\exp\left(\eta_1 + \eta_2\right)\right] \\ &= a_{12}P\left[\exp\left(\eta_1 + \eta_2\right)\right] \\ &= a_{12}\left\{\widehat{H}(t)\left\{\left[p_1'(t) + p_2'(t)\right]x + q_1'(t) + q_2'(t)\right\}\right. \\ &\quad + f(t)\widehat{H}^2(t)\left[p_1(t) + p_2(t)\right]^3 \\ &\quad \left. + \left(a_0x + c_0\right)f(t)\left[p_1(t) + p_2(t)\right]\right\} \\ &\quad \times \exp\left(\eta_1 + \eta_2\right) \\ &= 3a_{12}f(t)\widehat{H}^2(t)p_1(t)p_2(t) \\ &\quad \times \left[p_1(t) + p_2(t)\right]\exp\left(\eta_1 + \eta_2\right). \end{aligned} \tag{33}$$

Thus,

$$\begin{aligned} \left\{P\left[\overset{\star}{\varphi}\right]\right\}_x &= 3a_{12}f(t)\widehat{H}^2(t)p_1(t)p_2(t) \\ &\quad \times \left[p_1(t) + p_2(t)\right]^2\exp\left(\eta_1 + \eta_2\right). \end{aligned} \tag{34}$$

Substituting (31) and (32) into the first two parts on the left-hand side of (24), we get

$$\begin{aligned} &\overset{\star}{\varphi}\left\{P\left[\overset{\star}{\varphi}\right]\right\}_x - \overset{\star}{\varphi}_xP\left[\overset{\star}{\varphi}\right] \\ &\quad = 3a_{12}f(t)\widehat{H}^2(t)p_1(t)p_2(t) \\ &\qquad \times \left[p_1(t) + p_2(t)\right]^2\exp\left(\eta_1 + \eta_2\right) \\ &\qquad + 3a_{12}f(t)\widehat{H}^2(t)p_1^2(t)p_2(t) \\ &\qquad \times \left[p_1(t) + p_2(t)\right]\exp\left(\eta_1 + 2\eta_2\right) \\ &\qquad + 3a_{12}f(t)\widehat{H}^2(t)p_1(t)p_2^2(t) \\ &\qquad \times \left[p_1(t) + p_2(t)\right]\exp\left(2\eta_1 + \eta_2\right). \end{aligned} \tag{35}$$

In addition, by direct calculating, we obtain

$$\begin{aligned} &\overset{\star}{\varphi}_x\overset{\star}{\varphi}_{xxx} - \left[\overset{\star}{\varphi}_{xx}\right]^2 \\ &\quad = p_1(t)p_2(t)\left[p_1(t) - p_2(t)\right]^2\exp\left(\eta_1 + \eta_2\right) \\ &\qquad + a_{12}p_1^2(t)p_2(t)\left[p_1(t) + p_2(t)\right]\exp\left(\eta_1 + 2\eta_2\right) \\ &\qquad + a_{12}p_1(t)p_2^2(t)\left[p_1(t) + p_2(t)\right]\exp\left(2\eta_1 + \eta_2\right). \end{aligned} \tag{36}$$

Substituting (35) and (36) into (24) gives

$$\begin{aligned} &3f(t)\widehat{H}^2(t)p_1(t)p_2(t) \\ &\quad \times \left\{a_{12}\left[p_1(t) + p_2(t)\right]^2\right. \\ &\qquad \left. - \left[p_1(t) - p_2(t)\right]^2\right\}\exp\left(\eta_1 + \eta_2\right) = 0. \end{aligned} \tag{37}$$

If

$$a_{12} = \frac{\left[p_1(t) - p_2(t)\right]^2}{\left[p_1(t) + p_2(t)\right]^2} = \left(\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}\right)^2, \tag{38}$$

then $\overset{\star}{\varphi}(x,t)$ is a solution of (24). Therefore, a new exact soliton-like solution containing 2-solitary wave for (1) is obtained from

$$\begin{aligned} u(x,t) &= \left(\frac{a_0x + c_0}{a_0\int^t f(s)\,ds + b_0}\right) \\ &\quad + 12\left[a_0\int^t f(s)\,ds + b_0\right]\frac{\partial^2}{\partial x^2}\left[\ln\overset{\star}{\varphi}(x,t)\right], \end{aligned} \tag{39}$$

where

$$\begin{aligned} \overset{\star}{\varphi}(x,t) &= 1 + \exp\eta_1 + \exp\eta_2 + \left(\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}\right)^2\exp\left(\eta_1 + \eta_2\right), \\ \eta_i &= p_i(t)x + q_i(t) + \eta_{i0}, \quad (i = 1, 2), \\ p_i(t) &= \frac{\beta_i}{a_0\int^t f(s)\,ds + b_0}, \\ q_i(t) &= -\int^t\frac{\left[\beta_i^3 + c_0\beta_i\right]f(\tau)}{\left[a_0\int^\tau f(s)\,ds + b_0\right]^2}d\tau. \end{aligned} \tag{40}$$

To our knowledge, solutions (29) and (39) have not been shown in other literatures.

4. Conclusions

In summary, we have presented a method to derive a new auto-BT that consisted of (18) and (19) of (1). The new auto-BT also only involves one quadratic homogeneity equation to be solved. Comparing with auto BT (2), the new auto-BT in this paper does not require the coefficients of the equation to be linearly dependent; in fact, there are no restrictions between these coefficients. The crucial step involves making the assumption that the solution of (1) is of form (4). The key idea of the method is to remove the assumption, which is setting the coefficients of F', F'', and F''' to zero, in the homogeneous balance method. It is worthwhile to point out that this method is universal and may be applicable to other nonlinear evolution equations to construct their auto-BT and multisoliton solutions [12, 13]. Based on the new auto-BT, we obtain a new exact soliton-like solution containing 2-solitary wave under constraint $g(t) = [a_0\int^t f(s)ds + b_0]f(t)$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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