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Research Article

Solving Generalized Mixed Equilibria, Variational Inequalities, and Constrained Convex Minimization

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We propose implicit and explicit iterative algorithms for finding a common element of the set of solutions of the minimization problem for a convex and continuously Fréchet differentiable functional, the set of solutions of a finite family of generalized mixed equilibrium problems, and the set of solutions of a finite family of variational inequalities for inverse strong monotone mappings in a real Hilbert space. We prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets under very mild conditions.

1. Introduction and Problems Formulation

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. Recall that a mapping $A : C \to H$ is said to be L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$, then A is called a nonexpansive mapping [1], and if $L \in [0, 1)$, then A is called a contraction.

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C; \tag{2}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{3}$$

(iii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{4}$$

It is obvious that if A is α-inverse strongly monotone, then A is monotone and $(1/\alpha)$-Lipschitz continuous.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The solution set of VIP (5) is denoted by $\mathrm{VI}(C, A)$.

The VIP (5) was first discussed by Lions [2] and is now well known. The VIP (5) has many potential applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and so on; see, for example, [3–5] and the references therein.

In 1976, Korpelevich [6] proposed an iterative algorithm for solving the VIP (5) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \tau Ax_n), \\ x_{n+1} &= P_C(x_n - \tau Ay_n), \quad \forall n \geq 0, \end{aligned} \tag{6}$$

with $\tau > 0$, a given number which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many researchers. See, for example, [7–16] and the references therein. In particular, motivated by the idea of Korpelevich's extragradient method [6], Nadezhkina and Takahashi [17] introduced an extragradient iterative scheme:

$$\begin{gathered} x_0 = x \in C \text{ chosen arbitrary}, \\ y_n = P_C\left(x_n - \lambda_n A x_n\right), \\ x_{n+1} = \alpha_n x_n + \left(1 - \alpha_n\right) S P_C\left(x_n - \lambda_n A y_n\right), \\ \forall n \geq 0, \end{gathered} \tag{7}$$

where $A : C \to H$ is a monotone, L-Lipschitz continuous mapping, $S : C \to C$ is a nonexpansive mapping, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$, and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. They proved the weak convergence of $\{x_n\}$ to an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, let $A : H \to H$ be a nonlinear mapping, and let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [18] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta\left(x, y\right) + \varphi\left(y\right) - \varphi\left(x\right) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{8}$$

We denote the set of solutions of GMEP (8) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The GMEP (8) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and Nash equilibrium problems in noncooperative games. The GMEP is further considered and studied. See, for example, [19, 20]. Some special cases of GMEP (8) are as follows.

If $\varphi = 0$, then GMEP (8) reduces to the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta\left(x, y\right) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{9}$$

It is introduced and studied by S. Takahashi and W. Takahashi [21]. The set of solutions of GEP is denoted by $\mathrm{GEP}(\Theta, A)$.

If $A = 0$, then GMEP (8) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta\left(x, y\right) + \varphi\left(y\right) - \varphi\left(x\right) \geq 0, \quad \forall y \in C. \tag{10}$$

It is considered and studied in [22]. The set of solutions of MEP is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

If $\varphi = 0$ and $A = 0$, then GMEP (8) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta\left(x, y\right) \geq 0, \quad \forall y \in C. \tag{11}$$

It is considered and studied in [23]. The set of solutions of EP is denoted by $\mathrm{EP}(\Theta)$.

Throughout this paper, it is assumed as in [18] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$, for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta\left(tz + (1 - t)x, y\right) \leq \Theta\left(x, y\right); \tag{12}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$\Theta\left(z, y_x\right) + \varphi\left(y_x\right) - \varphi\left(z\right) + \frac{1}{r}\left\langle y_x - z, z - x\right\rangle < 0; \tag{13}$$

(B2) C is a bounded set.

Next we list some known results for the MEP as follows.

Proposition 1 (see [22]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned} T_r^{(\Theta,\varphi)}\left(x\right) = \Big\{ & z \in C : \Theta\left(z, y\right) + \varphi\left(y\right) - \varphi\left(z\right) \\ & + \frac{1}{r}\left\langle y - z, z - x\right\rangle \geq 0,\ \forall y \in C \Big\}, \end{aligned} \tag{14}$$

for all $x \in H$. Then the following conditions hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\| T_r^{(\Theta,\varphi)} x - T_r^{(\Theta,\varphi)} y \right\|^2 \leq \langle T_r^{(\Theta,\varphi)} x - T_r^{(\Theta,\varphi)} y, x - y\rangle; \tag{15}$$

(iii) *$\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = MEP(\Theta, \varphi)$;*

(iv) *$MEP(\Theta, \varphi)$ is closed and convex;*

(v) *$\|T_s^{(\Theta,\varphi)} x - T_t^{(\Theta,\varphi)} x\|^2 \leq ((s - t)/s)\langle T_s^{(\Theta,\varphi)} x - T_t^{(\Theta,\varphi)} x, T_s^{(\Theta,\varphi)} x - x\rangle$, for all $s, t > 0$ and $x \in H$.*

Let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0,1]$, $n \geq 1$. Given the nonexpansive mappings $S_1, S_2, \ldots, S_N$ on H, for each $n \geq 1$, the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ are defined by

$$\begin{aligned} U_{n,1} &= \lambda_{n,1} S_1 + (1-\lambda_{n,1}) I, \\ U_{n,2} &= \lambda_{n,2} S_n U_{n,1} + (1-\lambda_{n,2}) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1} S_{N-1} U_{n,N-2} + (1-\lambda_{n,N-1}) I, \\ W_n := U_{n,N} &= \lambda_{n,N} S_N U_{n,N-1} + (1-\lambda_{n,N}) I. \end{aligned} \tag{16}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that the nonexpansivity of S_i implies the nonexpansivity of W_n.

In 2012, combining the hybrid steepest-descent method in [24] and hybrid viscosity approximation method in [25], Ceng et al. [20] proposed and analyzed the following hybrid iterative method for finding a common element of the set of solutions of GMEP (8) and the set of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^N$.

Theorem CGY (see [20, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse strongly monotone, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H such that $\bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A) \neq \emptyset$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $V : H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$, and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a,b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Given $x_1 \in H$ arbitrarily, suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n \rangle \\ &\quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \alpha_n \gamma V x_n + \beta_n x_n \\ &\quad + ((1-\beta_n) I - \alpha_n \mu F) W_n u_n, \quad \forall n \geq 1, \end{aligned} \tag{17}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty} (r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty} (\lambda_{n+1,i} - \lambda_{n,i}) = 0$, *for all* $i = 1, 2, \ldots, N$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A)$, where $x^* = P_{\bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta,\varphi,A)} (I - \mu F + \gamma f) x^*$ is a unique solution of the variational inequality problem (VIP):

$$\begin{aligned} &\langle (\mu F - \gamma V) x^*, x^* - x \rangle \leq 0, \\ &\forall x \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A). \end{aligned} \tag{18}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C:

$$\min_{x \in C} f(x) \tag{19}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (19). It is well known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{20}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{21}$$

where, in both (20) and (21), the initial guess x_0 is taken from C arbitrarily and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (20) and (21) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that, if ∇f is α-strongly monotone and L-Lipschitz continuous, then, for $0 < \lambda < 2\alpha/L^2$, the operator

$$S := P_C(I - \lambda \nabla f) \tag{22}$$

is a contraction. Hence, the sequence $\{x_n\}$ defined by the GPA (20) converges in norm to the unique solution of CMP (19). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{L^2}, \tag{23}$$

then the sequence $\{x_n\}$ defined by the GPA (21) converges in norm to the unique minimizer of CMP (19). If the gradient ∇f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite dimensional (a counterexample is given in Section 5 of Xu [26]).

Since the Lipschitz continuity of the gradient ∇f implies that it is actually $(1/L)$-inverse strongly monotone (ism) [27], its complement can be an averaged mapping (i.e., it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping). Consequently, the GPA can be rewritten as the composite of a projection and an averaged mapping, which is again an averaged mapping. This shows that averaged mappings play an important role in the

GPA. Recently, Xu [26] used averaged mappings to study the convergence analysis of the GPA, which is hence an operator-oriented approach.

In 2011, combining the hybrid steepest-descent method in [24], viscosity approximation method, and averaged mapping approach to the GPA in [26], Ceng et al. [28] introduced and analyzed the following implicit and explicit iterative algorithms:

$$x_\lambda = P_C\left[s\gamma V x_\lambda + (I - s\mu F) T_\lambda x_\lambda\right], \quad \lambda \in \left(0, \frac{2}{L}\right), \tag{24}$$

$$x_{n+1} = P_C\left[s_n\gamma V x_n + (I - s_n\mu F) T_n x_n\right], \quad \forall n \geq 0, \tag{25}$$

where $V : C \to H$ is l-Lipschitzian mapping with constant $l \geq 0$ and $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$. Assume that $0 < \mu < 2\eta/\kappa^2$, $0 \leq \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$, $s := s(\lambda) = (2 - \lambda L)/4$ for each $\lambda \in (0, 2/L)$, $P_C(I - \lambda\nabla f) = sI + (1 - s)T_\lambda$ for each $\lambda \in (0, 2/L)$, $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4$ with $\{\lambda_n\} \subset (0, 2/L)$ and $\lambda_n \to 2/L$, and $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$. The authors proved that the net $\{x_\lambda\}$ defined by (24) converges strongly to some $q \in \Gamma$, which is a unique solution of the variational inequality problem (VIP):

$$\langle(\mu F - \gamma V) q, p - q\rangle \geq 0, \quad \forall p \in \Gamma. \tag{26}$$

Furthermore, utilizing control conditions (i) $s_n \to 0$, (ii) $\sum_{n=0}^{\infty} s_n = \infty$, and (iii) either $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$ or $\lim_{n\to\infty} s_{n+1}/s_n = 1$, the authors also proved that the sequence $\{x_n\}$ generated by (25) converges strongly to some $q \in \Gamma$, which is a unique solution of the VIP (26).

Motivated and inspired by the above facts, in this paper we introduce implicit and explicit iterative algorithms for finding a common element of the set of solutions of the CMP (19) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of a finite family of GMEPs, and the set of solutions of a finite family of VIPs for inverse strong monotone mappings in a real Hilbert space. Under very mild control conditions, we prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets. Our iterative algorithms are based on Korpelevich's extragradient method, hybrid steepest-descent method in [24], viscosity approximation method, and averaged mapping approach to the GPA in [26]. The results obtained in this paper improve and extend the corresponding results announced by many others.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\right\}. \tag{27}$$

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C). \tag{28}$$

Some important properties of projections are listed in the following proposition.

Proposition 2. *For given $x \in H$ and $z \in C$,*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$, *for all* $y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, *for all* $y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, *for all* $y \in H$.

Consequently, P_C is nonexpansive and monotone. If A is an α-inverse strongly monotone mapping of C into H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A) u - (I - \lambda A) v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle \\ &\quad + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.\end{aligned} \tag{29}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 3. A mapping $T : H \to H$ is said to be

(a) nonexpansive [1] if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \tag{30}$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or, equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H; \tag{31}$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{32}$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as cocoercive) operators have been applied widely in solving practical problems in various fields.

Definition 4. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1 - \alpha) I + \alpha S, \tag{33}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $(1/2)$-averaged mappings.

Proposition 5 (see [29]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then, for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I-T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Proposition 6 (see [29]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1 \cdots T_N). \tag{34}$$

The notation $\operatorname{Fix}(T)$ *denotes the set of all fixed points of the mapping T; that is,* $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 7. *Let X be a real inner product space. Then there holds the following inequality:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{35}$$

Lemma 8. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2(i)) implies*

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \text{for some } \lambda > 0. \tag{36}$$

Lemma 9 (see [30, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with* $\operatorname{Fix}(T) \ne \emptyset$. *Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 10 (see [31]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \le (1 - \alpha_n) s_n + \alpha_n\beta_n, \quad \forall n \ge 1, \tag{37}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) *$\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, or, equivalently,*

$$\prod_{n=1}^{\infty} (1 - \alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1 - \alpha_k) = 0; \tag{38}$$

(ii) *$\limsup_{n\to\infty}\beta_n \le 0$, or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$. Then $\lim_{n\to\infty}s_n = 0$.*

Lemma 11 (see [32]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1. \tag{39}$$

Suppose that $x_{n+1} = (1 - \beta_n)z_n + \beta_nx_n$ for each $n \ge 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{40}$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

The following lemma can be easily proven and, therefore, we omit the proof.

Lemma 12. *Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$, and let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Then for $0 \le \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma V) x - (\mu F - \gamma V) y, x - y\rangle \ge (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in H. \tag{41}$$

That is, $\mu F - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{42}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the following conditions:

$$\|Fx - Fy\| \le \kappa \|x - y\|,$$
$$\langle Fx - Fy, x - y\rangle \ge \eta \|x - y\|^2, \tag{43}$$

for all $x, y \in H$.

Lemma 13 (see [31, Lemma 3.1]). *T^λ is a contraction provided $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\left\|T^\lambda x - T^\lambda y\right\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{44}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Remark 14. (i) Since F is κ-Lipschitzian and η-strongly monotone on H, we get $0 < \eta \le \kappa$. Hence, whenever $0 < \mu < 2\eta/\kappa^2$, we have

$$\begin{aligned} 0 &\le (1 - \mu\eta)^2 \\ &= 1 - 2\mu\eta + \mu^2\eta^2 \\ &\le 1 - 2\mu\eta + \mu^2\kappa^2 \\ &< 1 - 2\mu\eta + \frac{2\eta}{\kappa^2}\mu\kappa^2 = 1, \end{aligned} \tag{45}$$

which implies

$$0 < 1 - \sqrt{1 - 2\mu\eta + \mu^2\kappa^2} \le 1. \tag{46}$$

So, $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

(ii) In Lemma 13, put $F = (1/2)I$ and $\mu = 2$. Then we know that $\kappa = \eta = 1/2, 0 < \mu = 2 < 2\eta/\kappa^2 = 4$, and

$$\begin{aligned} \tau &= 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\ &= 1 - \sqrt{1 - 2\left(2 \times \frac{1}{2} - 2 \times \left(\frac{1}{2}\right)^2\right)} = 1. \end{aligned} \tag{47}$$

Finally, recall that a set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is, $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0, \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{48}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$; see [33].

3. Implicit Iterative Algorithm and Its Convergence Criteria

We now state and prove the first main result of this paper.

Theorem 15. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M and N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \bigcap_{k=1}^M GMEP(\Theta_k, \varphi_k, B_k) \cap \bigcap_{i=1}^N VI(C, A_i) \cap \Gamma \ne \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}B_M) T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})}(I - r_{M-1,n}B_{M-1}) \\ &\quad \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ v_n &= P_C(I - \lambda_{N,n}A_N)P_C(I - \lambda_{N-1,n}A_{N-1}) \\ &\quad \cdots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ x_n &= s_n\gamma V x_n + (I - s_n\mu F)T_n v_n, \quad \forall n \ge 1, \end{aligned} \tag{49}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);*

(ii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \ldots, N\}$;*

(iii) *$\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $k \in \{1, 2, \ldots, M\}$.*

Then $\{x_n\}$ converges strongly as $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) to a point $q \in \Omega$, which is a unique solution of the VIP:

$$\langle(\mu F - \gamma V)q, p - q\rangle \ge 0, \quad \forall p \in \Omega. \tag{50}$$

Equivalently, $q = P_\Omega(I - \mu F + \gamma V)q$.

Proof. First of all, let us show that the sequence $\{x_n\}$ is well defined. Indeed, since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism; see [34]. By Proposition 5(ii) we know that, for $\lambda > 0$, $\lambda\nabla f$ is $(1/\lambda L)$-ism. So by Proposition 5(iii) we deduce that $I - \lambda\nabla f$ is $(\lambda L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, it is easy to see from Proposition 6(iv) that the composite $P_C(I - \lambda\nabla f)$ is $((2 + \lambda L)/4)$-averaged for $\lambda \in (0, 2/L)$. Hence we obtain that for each $n \ge 1$, $P_C(I - \lambda_n\nabla f)$ is

$((2+\lambda_n L)/4)$-averaged for each $\lambda_n \in (0, 2/L)$. Therefore, we can write

$$P_C(I-\lambda_n \nabla f) = \frac{2-\lambda_n L}{4} I + \frac{2+\lambda_n L}{4} T_n = s_n I + (1-s_n) T_n, \tag{51}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{52}$$

Put

$$\begin{aligned}\Delta_n^k = {} & T_{r_{k,n}}^{(\Theta_k, \varphi_k)} (I - r_{k,n} B_k) T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})} (I - r_{k-1,n} B_{k-1}) \\ & \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} (I - r_{1,n} B_1) x_n,\end{aligned} \tag{53}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned}\Lambda_n^i = {} & P_C (I - \lambda_{i,n} A_i) P_C (I - \lambda_{i-1,n} A_{i-1}) \\ & \cdots P_C (I - \lambda_{2,n} A_2) P_C (I - \lambda_{1,n} A_1),\end{aligned} \tag{54}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

Consider the following mapping G_n on H defined by

$$\begin{aligned}G_n x = s_n \gamma V x + (I - s_n \mu F) T_n \Lambda_n^N \Delta_n^M x, \\ \forall x \in H, \quad n \geq 1,\end{aligned} \tag{55}$$

where $s_n = (2-\lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. By Proposition 1(ii) and Lemma 13 we obtain from (29) that, for all $x, y \in H$,

$$\begin{aligned}
&\|G_n x - G_n y\| \\
&\quad \leq s_n \gamma \|Vx - Vy\| \\
&\qquad + \|(I - s_n \mu F) T_n \Lambda_n^N \Delta_n^M x \\
&\qquad\quad - (I - s_n \mu F) T_n \Lambda_n^N \Delta_n^M y\| \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Lambda_n^N \Delta_n^M x - \Lambda_n^N \Delta_n^M y\| \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|(I - \lambda_{N,n} A_N) \Lambda_n^{N-1} \Delta_n^M x \\
&\qquad\quad - (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} \Delta_n^M y\| \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Lambda_n^{N-1} \Delta_n^M x - \Lambda_n^{N-1} \Delta_n^M y\| \\
&\quad \vdots \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Lambda_n^0 \Delta_n^M x - \Lambda_n^0 \Delta_n^M y\| \\
&\quad = s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Delta_n^M x - \Delta_n^M y\| \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|(I - r_{M,n} B_M) \Delta_n^{M-1} x \\
&\qquad\quad - (I - r_{M,n} B_M) \Delta_n^{M-1} y\| \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Delta_n^{M-1} x - \Delta_n^{M-1} y\| \\
&\quad \vdots \\
&\quad \leq s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|\Delta_n^0 x - \Delta_n^0 y\| \\
&\quad = s_n \gamma l \|x - y\| \\
&\qquad + (1 - s_n \tau) \|x - y\| \\
&\quad = (1 - s_n (\tau - \gamma l)) \|x - y\|.
\end{aligned} \tag{56}$$

Since $0 < 1 - s_n(\tau - \gamma l) < 1$, $G_n : H \to H$ is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n \in H$, which uniquely solves the fixed point equation

$$x_n = s_n \gamma V x_n + (I - s_n \mu F) T_n \Lambda_n^N \Delta_n^M x_n. \tag{57}$$

This shows that the sequence $\{x_n\}$ is defined well.

Note that $0 \leq \gamma l < \tau$ and $\mu\eta \geq \tau \Leftrightarrow \kappa \geq \eta$. Hence by Lemma 12 we know that

$$\begin{aligned}\langle (\mu F - \gamma V) x - (\mu F - \gamma V) y, x - y \rangle \\ \geq (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in H.\end{aligned} \tag{58}$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Moreover, it is clear that $\mu F - \gamma V$ is Lipschitz continuous. So the VIP (50) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the VIP (50).

Now, let us show that $\{x_n\}$ is bounded. In fact, take $p \in \Omega$ arbitrarily. Then from (29) and Proposition 1(ii) we have

$$\begin{aligned}
&\|u_n - p\| \\
&\quad = \left\| T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}B_M) \Delta_n^{M-1} x_n \right. \\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n}B_M) \Delta_n^{M-1} p \right\| \\
&\quad \le \left\| (I - r_{M,n}B_M) \Delta_n^{M-1} x_n \right. \\
&\qquad \left. - (I - r_{M,n}B_M) \Delta_n^{M-1} p \right\| \\
&\quad \le \left\| \Delta_n^{M-1} x_n - \Delta_n^{M-1} p \right\| \\
&\quad \vdots \\
&\quad \le \left\| \Delta_n^0 x_n - \Delta_n^0 p \right\| \\
&\quad = \|x_n - p\| .
\end{aligned} \tag{59}$$

Similarly, we have

$$\begin{aligned}
&\|v_n - p\| \\
&\quad = \left\| P_C (I - \lambda_{N,n}A_N) \Lambda_n^{N-1} u_n \right. \\
&\qquad \left. - P_C (I - \lambda_{N,n}A_N) \Lambda_n^{N-1} p \right\| \\
&\quad \le \left\| (I - \lambda_{N,n}A_N) \Lambda_n^{N-1} u_n \right. \\
&\qquad \left. - (I - \lambda_{N,n}A_N) \Lambda_n^{N-1} p \right\| \\
&\quad \le \left\| \Lambda_n^{N-1} u_n - \Lambda_n^{N-1} p \right\| \\
&\quad \vdots \\
&\quad \le \left\| \Lambda_n^0 u_n - \Lambda_n^0 p \right\| \\
&\quad = \|u_n - p\| .
\end{aligned} \tag{60}$$

Combining (59) and (60), we have

$$\|v_n - p\| \le \|x_n - p\| . \tag{61}$$

Since

$$p = P_C (I - \lambda_n \nabla f) p = s_n p + (1 - s_n) T_n p, \quad \forall \lambda_n \in \left(0, \frac{2}{L}\right), \tag{62}$$

where $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$. It is clear that $T_n p = p$ for each $\lambda_n \in (0, 2/L)$. Thus, utilizing Lemma 13 and the nonexpansivity of T_n, we obtain from (61) that

$$\begin{aligned}
&\|x_n - p\| \\
&\quad = \| s_n (\gamma V x_n - \mu F p) + (I - s_n \mu F) T_n v_n \\
&\qquad - (I - s_n \mu F) T_n p \| \\
&\quad \le \| (I - s_n \mu F) T_n v_n - (I - s_n \mu F) T_n p \| \\
&\qquad + s_n \| \gamma V x_n - \mu F p \| \\
&\quad \le (1 - s_n \tau) \| v_n - p \| \\
&\qquad + s_n (\gamma \| V x_n - V p \| + \| \gamma V p - \mu F p \|) \\
&\quad \le (1 - s_n \tau) \| x_n - p \| \\
&\qquad + s_n (\gamma l \| x_n - p \| + \| \gamma V p - \mu F p \|) \\
&\quad = (1 - s_n (\tau - \gamma l)) \| x_n - p \| \\
&\qquad + s_n \| \gamma V p - \mu F p \| .
\end{aligned} \tag{63}$$

This implies that $\|x_n - p\| \le \|\gamma V p - \mu F p\| / (\tau - \gamma l)$. Hence $\{x_n\}$ is bounded. So, according to (59) and (61) we know that $\{u_n\}, \{v_n\}, \{T_n v_n\}, \{V x_n\}$, and $\{F T_n v_n\}$ are bounded.

Next let us show that $\|u_n - x_n\| \to 0$, $\|v_n - u_n\| \to 0$, and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

Indeed, from (29) it follows that, for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$,

$$\begin{aligned}
&\|v_n - p\|^2 \\
&\quad = \left\| \Lambda_n^N u_n - p \right\|^2 \\
&\quad \le \left\| \Lambda_n^i u_n - p \right\|^2 \\
&\quad = \left\| P_C (I - \lambda_{i,n} A_i) \Lambda_n^{i-1} u_n - P_C (I - \lambda_{i,n} A_i) p \right\|^2 \\
&\quad \le \left\| (I - \lambda_{i,n} A_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n} A_i) p \right\|^2 \\
&\quad \le \left\| \Lambda_n^{i-1} u_n - p \right\|^2 \\
&\qquad + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\|^2 \\
&\quad \le \|u_n - p\|^2 \\
&\qquad + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\|^2 \\
&\quad \le \|x_n - p\|^2 \\
&\qquad + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\|^2 ,
\end{aligned}$$

$$\begin{aligned}
&\|u_n - p\|^2 \\
&\quad = \left\|\Delta_n^M x_n - p\right\|^2 \\
&\quad \leq \left\|\Delta_n^k x_n - p\right\|^2 \\
&\quad = \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}B_k)\,\Delta_n^{k-1} x_n \right.\\
&\qquad \left. - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}B_k)\, p\right\|^2 \\
&\quad \leq \left\|(I - r_{k,n}B_k)\Delta_n^{k-1} x_n - (I - r_{k,n}B_k)p\right\|^2 \\
&\quad \leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 \\
&\qquad + r_{k,n}(r_{k,n} - 2\mu_k)\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \\
&\quad \leq \|x_n - p\|^2 \\
&\qquad + r_{k,n}(r_{k,n} - 2\mu_k)\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2.
\end{aligned} \tag{64}$$

Thus, utilizing Lemma 7, from (49) and (64) we have

$$\begin{aligned}
&\|x_n - p\|^2 \\
&\quad = \|s_n(\gamma V x_n - \mu F p) \\
&\qquad + (I - s_n\mu F)\,T_n v_n \\
&\qquad - (I - s_n\mu F)\,T_n p\|^2 \\
&\quad \leq \|(I - s_n\mu F)\,T_n v_n - (I - s_n\mu F)\,T_n p\|^2 \\
&\qquad + 2s_n\langle \gamma V x_n - \mu F p, x_n - p\rangle \\
&\quad \leq (1 - s_n\tau)^2\|v_n - p\|^2 \\
&\qquad + 2s_n\langle \gamma V x_n - \mu F p, x_n - p\rangle \\
&\quad = (1 - s_n\tau)^2\|v_n - p\|^2 \\
&\qquad + 2s_n\gamma\langle V x_n - V p, x_n - p\rangle \\
&\qquad + 2s_n\langle \gamma V p - \mu F p, x_n - p\rangle \\
&\quad \leq (1 - s_n\tau)^2\|v_n - p\|^2 \\
&\qquad + 2s_n\gamma l\|x_n - p\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\quad \leq (1 - s_n\tau)^2\left[\|u_n - p\|^2 \right.\\
&\qquad\qquad + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i) \\
&\qquad\qquad \left.\times\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2\right] \\
&\qquad + 2s_n\gamma l\|x_n - p\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\quad \leq (1 - s_n\tau)^2\left[\|x_n - p\|^2 \right.\\
&\qquad\qquad + r_{k,n}(r_{k,n} - 2\mu_k) \\
&\qquad\qquad \times\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \\
&\qquad\qquad + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i) \\
&\qquad\qquad \left.\times\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2\right] \\
&\qquad + 2s_n\gamma l\|x_n - p\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\quad = \left[1 - 2s_n(\tau - \gamma l) + s_n^2\tau^2\right]\|x_n - p\|^2 \\
&\qquad - (1 - s_n\tau)^2 r_{k,n}(2\mu_k - r_{k,n}) \\
&\qquad \times\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \\
&\qquad + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\quad \leq \|x_n - p\|^2 + s_n^2\tau^2\|x_n - p\|^2 \\
&\qquad - (1 - s_n\tau)^2\left[r_{k,n}(2\mu_k - r_{k,n}) \right.\\
&\qquad\qquad \times\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \\
&\qquad\qquad + \lambda_{i,n}(2\eta_i - \lambda_{i,n}) \\
&\qquad\qquad \left.\times\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2\right] \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\|,
\end{aligned} \tag{65}$$

which implies that

$$\begin{aligned}
&(1 - s_n\tau)^2\left[r_{k,n}(2\mu_k - r_{k,n})\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \right.\\
&\qquad\qquad \left. + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2\right] \\
&\quad \leq s_n^2\tau^2\|x_n - p\|^2 \\
&\qquad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\|.
\end{aligned} \tag{66}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, from $s_n \to 0$ we conclude immediately that

$$\begin{aligned}
&\lim_{n\to\infty}\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\| = 0, \\
&\lim_{n\to\infty}\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\| = 0,
\end{aligned} \tag{67}$$

for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$.

Furthermore, by Proposition 1(ii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&\quad = \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}B_k) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}B_k) p\right\|^2 \\
&\quad \le \left\langle (I - r_{k,n}B_k) \Delta_n^{k-1} x_n - (I - r_{k,n}B_k) p, \Delta_n^k x_n - p \right\rangle \\
&\quad = \frac{1}{2} \Big(\left\|(I - r_{k,n}B_k) \Delta_n^{k-1} x_n - (I - r_{k,n}B_k) p\right\|^2 \\
&\qquad + \left\|\Delta_n^k x_n - p\right\|^2 \\
&\qquad - \left\|(I - r_{k,n}B_k) \Delta_n^{k-1} x_n - (I - r_{k,n}B_k) p \right. \\
&\qquad\qquad \left. - (\Delta_n^k x_n - p)\right\|^2 \Big) \\
&\quad \le \frac{1}{2} \Big(\left\|\Delta_n^{k-1} x_n - p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2 \\
&\qquad - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} \left(B_k \Delta_n^{k-1} x_n - B_k p\right)\right\|^2 \Big),
\end{aligned} \tag{68}$$

which implies that

$$\begin{aligned}
\left\|\Delta_n^k x_n - p\right\|^2 &\le \left\|\Delta_n^{k-1} x_n - p\right\|^2 \\
&\quad - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} \left(B_k \Delta_n^{k-1} x_n - B_k p\right)\right\|^2 \\
&= \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad - r_{k,n}^2 \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\|^2 \\
&\quad + 2r_{k,n} \langle \Delta_n^{k-1} x_n - \Delta_n^k x_n, B_k \Delta_n^{k-1} x_n - B_k p \rangle \\
&\le \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\quad \times \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\| \\
&\le \left\|x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\quad \times \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\|.
\end{aligned} \tag{69}$$

Also, by Proposition 2(iii), we obtain that for each $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - p\right\|^2 \\
&\quad = \left\|P_C (I - \lambda_{i,n}A_i) \Lambda_n^{i-1} u_n - P_C (I - \lambda_{i,n}A_i) p\right\|^2 \\
&\quad \le \left\langle (I - \lambda_{i,n}A_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n}A_i) p, \Lambda_n^i u_n - p \right\rangle \\
&\quad = \frac{1}{2} \Big(\left\|(I - \lambda_{i,n}A_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n}A_i) p\right\|^2 \\
&\qquad + \left\|\Lambda_n^i u_n - p\right\|^2 \\
&\qquad - \left\|(I - \lambda_{i,n}A_i) \Lambda_n^{i-1} u_n \right. \\
&\qquad\qquad \left. - (I - \lambda_{i,n}A_i) p - \left(\Lambda_n^i u_n - p\right)\right\|^2 \Big) \\
&\quad \le \frac{1}{2} \Big(\left\|\Lambda_n^{i-1} u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 \\
&\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i \Lambda_n^{i-1} u_n - A_i p)\right\|^2 \Big) \\
&\quad \le \frac{1}{2} \Big(\left\|u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 \\
&\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} \left(A_i \Lambda_n^{i-1} u_n - A_i p\right)\right\|^2 \Big),
\end{aligned} \tag{70}$$

which implies

$$\begin{aligned}
\left\|\Lambda_n^i u_n - p\right\|^2 &\le \left\|u_n - p\right\|^2 \\
&\quad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n}(A_i \Lambda_n^{i-1} u_n - A_i p)\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad - \lambda_{i,n}^2 \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|^2 \\
&\quad + 2\lambda_{i,n} \langle \Lambda_n^{i-1} u_n - \Lambda_n^i u_n, A_i \Lambda_n^{i-1} u_n - A_i p \rangle \\
&\le \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\quad \times \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|.
\end{aligned} \tag{71}$$

Thus, utilizing Lemma 7, from (49), (69), and (71) we have

$$\begin{aligned}
&\left\|x_n - p\right\|^2 \\
&\quad = \left\|s_n (\gamma V x_n - \mu F p) + (I - s_n \mu F) T_n v_n \right. \\
&\qquad \left. - (I - s_n \mu F) T_n p\right\|^2 \\
&\quad \le \left\|(I - s_n \mu F) T_n v_n - (I - s_n \mu F) T_n p\right\|^2 \\
&\qquad + 2s_n \langle \gamma V x_n - \mu F p, x_n - p \rangle \\
&\quad \le (1 - s_n \tau)^2 \left\|v_n - p\right\|^2 \\
&\qquad + 2s_n \gamma l \left\|x_n - p\right\|^2 \\
&\qquad + 2s_n \left\|\gamma V p - \mu F p\right\| \left\|x_n - p\right\|
\end{aligned}$$

$$
\begin{aligned}
&= (1 - s_n\tau)^2 \left\|\Lambda_n^N u_n - p\right\|^2 \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \left\|\Lambda_n^i u_n - p\right\|^2 \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \left[\|u_n - p\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \right.\\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left.\times \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \right] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&= (1 - s_n\tau)^2 \left[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \right.\\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left.\times \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \right] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \left[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \right.\\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left.\times \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \right] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le (1 - s_n\tau)^2 \left[\|x_n - p\|^2 - \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 \right.\\
&\qquad\qquad + 2r_{k,n} \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \\
&\qquad\qquad \times \left\|B_k\Delta_n^{k-1}x_n - B_k p\right\| \\
&\qquad\qquad - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left.\times \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\|\right] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le \left(1 - 2s_n(\tau - \gamma l) + s_n^2\tau^2\right) \|x_n - p\|^2 - (1 - s_n\tau)^2 \\
&\quad \times \left(\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \left\|B_k\Delta_n^{k-1}x_n - B_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\| \\
&\le \|x_n - p\|^2 + s_n^2\tau^2\|x_n - p\|^2 - (1 - s_n\tau)^2 \\
&\quad \times \left(\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \left\|B_k\Delta_n^{k-1}x_n - B_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \\
&\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\|.
\end{aligned} \tag{72}
$$

It immediately follows that

$$
\begin{aligned}
&(1 - s_n\tau)^2 \left(\left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\|^2\right) \\
&\quad \le s_n^2\tau^2\|x_n - p\|^2 \\
&\quad\quad + 2r_{k,n} \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| \left\|B_k\Delta_n^{k-1}x_n - B_k p\right\| \\
&\quad\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| \left\|A_i\Lambda_n^{i-1}u_n - A_i p\right\| \\
&\quad\quad + 2s_n \|\gamma Vp - \mu Fp\| \|x_n - p\|.
\end{aligned} \tag{73}
$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, from (67) and $s_n \to 0$ we deduce that

$$
\begin{aligned}
&\lim_{n\to\infty} \left\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\right\| = 0, \\
&\lim_{n\to\infty} \left\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\right\| = 0,
\end{aligned} \tag{74}
$$

for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Hence we get

$$
\begin{aligned}
\|x_n - u_n\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\
&\le \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| \\
&\quad + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\
&\quad + \cdots + \left\|\Delta_n^{M-1}x_n - \Delta_n^M x_n\right\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
\|u_n - v_n\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\
&\le \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| \\
&\quad + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\
&\quad + \cdots + \left\|\Lambda_n^{N-1}u_n - \Lambda_n^N u_n\right\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{76}
$$

So, taking into account that $\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\|$, we have

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{77}$$

Thus, from (77) and $s_n \to 0$ we have

$$\begin{aligned}\|v_n - T_n v_n\| &\le \|v_n - x_n\| + \|x_n - T_n v_n\| \\ &= \|v_n - x_n\| + s_n \|\gamma V x_n - \mu F T_n v_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{78}$$

Now we show that $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. In fact, from the nonexpansivity of T_n, we have

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - v_n\| + \|v_n - T_n v_n\| \\ &\quad + \|T_n v_n - T_n x_n\| \\ &\le 2\|x_n - v_n\| + \|v_n - T_n v_n\|.\end{aligned} \tag{79}$$

By (77) and (78), we get

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{80}$$

From (78) it is easy to see that

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{81}$$

Observe that

$$\begin{aligned}&\|P_C (I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad = \|s_n v_n + (1 - s_n) T_n v_n - v_n\| \\ &\quad = (1 - s_n) \|T_n v_n - v_n\| \\ &\quad \le \|T_n v_n - v_n\|,\end{aligned} \tag{82}$$

where $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. Hence we have

$$\begin{aligned}&\left\|P_C \left(I - \frac{2}{L}\nabla f\right) v_n - v_n\right\| \\ &\quad \le \left\|P_C \left(I - \frac{2}{L}\nabla f\right) v_n - P_C (I - \lambda_n \nabla f) v_n\right\| \\ &\qquad + \|P_C (I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad \le \left\|\left(I - \frac{2}{L}\nabla f\right) v_n - (I - \lambda_n \nabla f) v_n\right\| \\ &\qquad + \|P_C (I - \lambda_n \nabla f) v_n - v_n\| \\ &\quad \le \left(\frac{2}{L} - \lambda_n\right) \|\nabla f (v_n)\| + \|T_n v_n - v_n\|.\end{aligned} \tag{83}$$

From the boundedness of $\{v_n\}$, $s_n \to 0$ ($\Leftrightarrow \lambda_n \to 2/L$) and $\|T_n v_n - v_n\| \to 0$ (due to (78)), it follows that

$$\lim_{n\to\infty} \left\|v_n - P_C \left(I - \frac{2}{L}\nabla f\right) v_n\right\| = 0. \tag{84}$$

Further, we show that $\omega_w(x_n) \subset \Omega$. Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some w. Note that $\lim_{n\to\infty}\|x_n - u_n\| = 0$ (due to (75)). Hence $x_{n_i} \rightharpoonup w$. Since C is closed and convex, C is weakly closed. So, we have $w \in C$. From (74)-(75), we have that $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, and $v_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \dots, M\}$ and $m \in \{1, 2, \dots, N\}$. First, we prove that $w \in \bigcap_{m=1}^N \mathrm{VI}(C, A_m)$. Let

$$T_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{85}$$

where $m \in \{1, 2, \dots, N\}$. Let $(v, u) \in G(T_m)$. Since $u - A_m v \in N_C v$ and $\Lambda^m_n u_n \in C$, we have

$$\langle v - \Lambda^m_n u_n, u - A_m v\rangle \ge 0. \tag{86}$$

On the other hand, from $\Lambda^m_n u_n = P_C(I - \lambda_{m,n} A_m)\Lambda^{m-1}_n u_n$ and $v \in C$, we have

$$\left\langle v - \Lambda^m_n u_n, \Lambda^m_n u_n - \left(\Lambda^{m-1}_n u_n - \lambda_{m,n} A_m \Lambda^{m-1}_n u_n\right)\right\rangle \ge 0, \tag{87}$$

and hence

$$\left\langle v - \Lambda^m_n u_n, \frac{\Lambda^m_n u_n - \Lambda^{m-1}_n u_n}{\lambda_{m,n}} + A_m \Lambda^{m-1}_n u_n \right\rangle \ge 0. \tag{88}$$

Therefore we have

$$\begin{aligned}&\left\langle v - \Lambda^m_{n_i} u_{n_i}, u\right\rangle \\ &\quad \ge \langle v - \Lambda^m_{n_i} u_{n_i}, A_m v\rangle \\ &\quad \ge \left\langle v - \Lambda^m_{n_i} u_{n_i}, A_m v\right\rangle \\ &\qquad - \left\langle v - \Lambda^m_{n_i} u_{n_i}, \frac{\Lambda^m_{n_i} u_{n_i} - \Lambda^{m-1}_{n_i} u_{n_i}}{\lambda_{m,n_i}} + A_m \Lambda^{m-1}_{n_i} u_{n_i} \right\rangle \\ &\quad = \left\langle v - \Lambda^m_{n_i} u_{n_i}, A_m v - A_m \Lambda^m_{n_i} u_{n_i}\right\rangle \\ &\qquad + \langle v - \Lambda^m_{n_i} u_{n_i}, A_m \Lambda^m_{n_i} u_{n_i} - A_m \Lambda^{m-1}_{n_i} u_{n_i}\rangle \\ &\qquad - \left\langle v - \Lambda^m_{n_i} u_{n_i}, \frac{\Lambda^m_{n_i} u_{n_i} - \Lambda^{m-1}_{n_i} u_{n_i}}{\lambda_{m,n_i}} \right\rangle \\ &\quad \ge \left\langle v - \Lambda^m_{n_i} u_{n_i}, A_m \Lambda^m_{n_i} u_{n_i} - A_m \Lambda^{m-1}_{n_i} u_{n_i}\right\rangle \\ &\qquad - \left\langle v - \Lambda^m_{n_i} u_{n_i}, \frac{\Lambda^m_{n_i} u_{n_i} - \Lambda^{m-1}_{n_i} u_{n_i}}{\lambda_{m,n_i}} \right\rangle.\end{aligned} \tag{89}$$

From (74) and since A_m is Lipschitz continuous, we obtain that $\lim_{n\to\infty}\|A_m \Lambda^m_n u_n - A_m \Lambda^{m-1}_n u_n\| = 0$. From $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \dots, N\}$ and (74), we have

$$\langle v - w, u\rangle \ge 0. \tag{90}$$

Since T_m is maximal monotone, we have $w \in T_m^{-1}0$ and hence $w \in \mathrm{VI}(C, A_m)$, $m = 1, 2, \ldots, N$, which implies $w \in \bigcap_{m=1}^{N} \mathrm{VI}(C, A_m)$. Next we prove that $w \in \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}B_k)\Delta_n^{k-1}x_n$, $n \geq 1$, $k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned}\Theta_k\left(\Delta_n^k x_n, y\right) &+ \varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right) \\ &+ \left\langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n\right\rangle \\ &+ \frac{1}{r_{k,n}}\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n\rangle \geq 0.\end{aligned} \tag{91}$$

By (A2), we have

$$\begin{aligned}\varphi_k(y) &- \varphi_k\left(\Delta_n^k x_n\right) + \left\langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n\right\rangle \\ &+ \frac{1}{r_{k,n}}\left\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n\right\rangle \\ &\geq \Theta_k\left(y, \Delta_n^k x_n\right).\end{aligned} \tag{92}$$

Let $z_t = ty + (1-t)w$, for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned}&\left\langle z_t - \Delta_n^k x_n, B_k z_t\right\rangle \\ &\quad\geq \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) \\ &\qquad+ \langle z_t - \Delta_n^k x_n, B_k z_t\rangle \\ &\qquad- \langle z_t - \Delta_n^k x_n, B_k \Delta_n^{k-1} x_n\rangle \\ &\qquad- \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}}\right\rangle \\ &\qquad+ \Theta_k\left(z_t, \Delta_n^k x_n\right) \\ &\quad= \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) \\ &\qquad+ \langle z_t - \Delta_n^k x_n, B_k z_t - B_k \Delta_n^k x_n\rangle \\ &\qquad+ \langle z_t - \Delta_n^k x_n, B_k \Delta_n^k x_n - B_k \Delta_n^{k-1} x_n\rangle \\ &\qquad- \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}}\right\rangle \\ &\qquad+ \Theta_k\left(z_t, \Delta_n^k x_n\right).\end{aligned} \tag{93}$$

By (74), we have $\|B_k \Delta_n^k x_n - B_k \Delta_n^{k-1} x_n\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of B_k, we obtain $\langle z_t - \Delta_n^k x_n, B_k z_t - B_k \Delta_n^k x_n\rangle \geq 0$. Then, by (A4) we obtain

$$\langle z_t - w, B_k z_t\rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{94}$$

Utilizing (A1), (A4), and (94), we obtain

$$\begin{aligned}0 &= \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\ &\leq t\Theta_k(z_t, y) \\ &\quad+ (1-t)\Theta_k(z_t, w) + t\varphi_k(y) \\ &\quad+ (1-t)\varphi_k(w) - \varphi_k(z_t) \\ &\leq t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] \\ &\quad+ (1-t)\langle z_t - w, B_k z_t\rangle \\ &= t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] \\ &\quad+ (1-t)t\langle y - w, B_k z_t\rangle,\end{aligned} \tag{95}$$

and hence

$$0 \leq \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\langle y - w, B_k z_t\rangle. \tag{96}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \langle y - w, B_k w\rangle. \tag{97}$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$ and hence $w \in \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Further, let us show that $w \in \Gamma$. As a matter of fact, from (84), $v_{n_i} \rightharpoonup w$, and Lemma 9, we conclude that

$$w = P_C\left(I - \frac{2}{L}\nabla f\right)w. \tag{98}$$

So, $w \in \mathrm{VI}(C, \nabla f) = \Gamma$. Therefore, $w \in \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \Gamma =: \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Finally, let us show that $x_n \to q$ as $n \to \infty$, where q is the unique solution of the VIP (50). Indeed, we note that, for $w \in \Omega$ with $x_{n_i} \rightharpoonup w$,

$$\begin{aligned}x_n - w = s_n&(\gamma V x_n - \mu F w) \\ &+ (I - s_n \mu F) T_n v_n - (I - s_n \mu F) w.\end{aligned} \tag{99}$$

By (61) and Lemma 13, we obtain that

$$\begin{aligned}&\|x_n - w\|^2 \\ &\quad= s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle \\ &\qquad+ \langle(I - s_n\mu F) T_n v_n - (I - s_n \mu F) w, x_n - w\rangle \\ &\quad= s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle \\ &\qquad+ \|(I - s_n\mu F) T_n v_n - (I - s_n \mu F) w\| \|x_n - w\| \\ &\quad\leq s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle \\ &\qquad+ (1 - s_n\tau)\|v_n - w\|\|x_n - w\| \\ &\quad\leq s_n\langle\gamma V x_n - \mu F w, x_n - w\rangle \\ &\qquad+ (1 - s_n\tau)\|x_n - w\|^2.\end{aligned} \tag{100}$$

Hence it follows that

$$\begin{aligned}\|x_n - w\|^2 &\le \frac{1}{\tau}\langle \gamma V x_n - \mu F w, x_n - w\rangle \\ &= \frac{1}{\tau}\left(\gamma \langle V x_n - V w, x_n - w\rangle \right. \\ &\quad \left. + \langle \gamma V w - \mu F w, x_n - w\rangle\right) \\ &\le \frac{1}{\tau}\left(\gamma l \|x_n - w\|^2 + \langle \gamma V w - \mu F w, x_n - w\rangle\right),\end{aligned} \tag{101}$$

which hence leads to

$$\|x_n - w\|^2 \le \frac{\langle \gamma V w - \mu F w, x_n - w\rangle}{\tau - \gamma l}. \tag{102}$$

In particular, we have

$$\|x_{n_i} - w\|^2 \le \frac{\langle \gamma V w - \mu F w, x_{n_i} - w\rangle}{\tau - \gamma l}. \tag{103}$$

Since $x_{n_i} \rightharpoonup w$, it follows from (103) that $x_{n_i} \to w$ as $i \to \infty$.

Now we show that w solves the VIP (50). Since $x_n = s_n \gamma V x_n + (I - s_n \mu F) T_n v_n$, we have

$$\begin{aligned}&(\mu F - \gamma V) x_n \\ &\quad = -\frac{1}{s_n}\left((I - s_n\mu F) x_n - (I - s_n \mu F) T_n v_n\right).\end{aligned} \tag{104}$$

It follows that, for each $p \in \Omega$,

$$\begin{aligned}&\langle (\mu F - \gamma V) x_n, x_n - p\rangle \\ &\quad = -\frac{1}{s_n}\langle (I - s_n\mu F) x_n - (I - s_n\mu F) T_n v_n, x_n - p\rangle \\ &\quad = -\frac{1}{s_n}\langle (I - s_n\mu F) x_n - (I - s_n\mu F) T_n v_n, x_n - p\rangle \\ &\quad = -\frac{1}{s_n}\left\langle (I - s_n\mu F) x_n - (I - s_n\mu F) T_n \Lambda_n^N \Delta_n^M x_n, x_n - p\right\rangle \\ &\quad = -\frac{1}{s_n}\left\langle \left(I - T_n\Lambda_n^N\Delta_n^M\right) x_n - \left(I - T_n\Lambda_n^N\Delta_n^M\right) p, x_n - p\right\rangle \\ &\qquad + \left\langle \mu F x_n - \mu F T_n \Lambda_n^N \Delta_n^M x_n, x_n - p\right\rangle \\ &\quad \le \left\langle \mu F x_n - \mu F T_n \Lambda_n^N \Delta_n^M x_n, x_n - p\right\rangle\end{aligned} \tag{105}$$

since $I - T_n\Lambda_n^N\Delta_n^M$ is monotone (i.e., $\langle (I - T_n\Lambda_n^N\Delta_n^M)x - (I - T_n\Lambda_n^N\Delta_n^M)y, x - y\rangle \ge 0$, for all $x, y \in H$. This is due to the nonexpansivity of $T_n\Lambda_n^N\Delta_n^M$). Since $\|x_n - T_n v_n\| = \|(I - T_n\Lambda_n^N\Delta_n^M)x_n\| \to 0$ as $n \to \infty$, by replacing n in (105) with n_i and letting $i \to \infty$, we get

$$\begin{aligned}\langle (\mu F - \gamma V) w, w - p\rangle &= \lim_{i\to\infty}\left\langle (\mu F - \gamma V) x_{n_i}, x_{n_i} - p\right\rangle \\ &\le \lim_{i\to\infty}\left\langle \mu F x_{n_i} - \mu F T_{n_i} v_{n_i}, x_{n_i} - p\right\rangle \\ &= 0.\end{aligned} \tag{106}$$

That is, $w \in \Omega$ is a solution of VIP (50).

Finally we show that the sequence $\{x_n\}$ converges strongly to q. To this end, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widehat{w}$. By the same arguments as above, we have $\widehat{w} \in \Omega$. Moreover, it follows from (106) that

$$\langle (\mu F - \gamma V) w, w - \widehat{w}\rangle \le 0. \tag{107}$$

Interchanging w and $\widehat{w}$, we obtain

$$\langle (\mu F - \gamma V) \widehat{w}, \widehat{w} - w\rangle \le 0. \tag{108}$$

Utilizing Lemma 12 and adding the two inequalities (107) and (108), we have

$$\begin{aligned}&(\mu\eta - \gamma l)\|w - \widehat{w}\|^2 \\ &\quad \le \langle (\mu F - \gamma V) w - (\mu F - \gamma V)\widehat{w}, w - \widehat{w}\rangle \le 0.\end{aligned} \tag{109}$$

Hence $w = \widehat{w}$. Therefore we conclude that $x_n \to w$ as $n \to \infty$. Taking into account the uniqueness of solutions of VIP (50), we have $w = q$. The VIP (50) can be rewritten as

$$\langle (I - \mu F + \gamma V) q - q, q - p\rangle \ge 0, \quad \forall p \in \Omega. \tag{110}$$

By Proposition 2(i), this is equivalent to the fixed point equation

$$P_\Omega (I - \mu F + \gamma V) q = q. \tag{111}$$

This completes the proof. □

Corollary 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B : H \to H$ and $A_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $i = 1, 2$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, B) \cap VI(C, A_1) \cap VI(C, A_2) \cap \Gamma \ne \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle B x_n, y - u_n\rangle \\ &\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ &v_n = P_C(I - \lambda_{2,n}A_2) P_C(I - \lambda_{1,n}A_1) u_n, \\ &x_n = s_n\gamma V x_n + (I - s_n\mu F) T_n v_n, \quad \forall n \ge 1,\end{aligned} \tag{112}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow$ $\lim_{n\to\infty}\lambda_n = 2/L$);*

(ii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for* $i = 1, 2$;

(iii) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$, *which is a unique solution of the VIP:*

$$\langle (\mu F - \gamma V) q, p - q \rangle \geq 0, \quad \forall p \in \Omega. \tag{113}$$

Corollary 17. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B : H \to H$ and $A : C \to H$ be ζ-inverse strongly monotone and ξ-inverse strongly monotone, respectively. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, B) \cap VI(C, A) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\
&\quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
&v_n = P_C(I - \rho_n A) u_n, \\
&x_n = s_n \gamma V x_n + (I - s_n \mu F) T_n v_n, \quad \forall n \geq 1,
\end{aligned} \tag{114}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) $s_n \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty} \lambda_n = 2/L$);

(ii) $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$;

(iii) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$, *which is a unique solution of the VIP:*

$$\langle (\mu F - \gamma V) q, p - q \rangle \geq 0, \quad \forall p \in \Omega. \tag{115}$$

4. Explicit Iterative Algorithm and Its Convergence Criteria

We next state and prove the second main result of this paper.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M and N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \bigcap_{k=1}^{M} GMEP(\Theta_k, \varphi_k, B_k) \cap \bigcap_{i=1}^{N} VI(C, A_i) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
u_n &= T_{r_{M,n}}^{(\Theta_M, \varphi_M)} (I - r_{M,n} B_M) T_{r_{M-1,n}}^{(\Theta_{M-1}, \varphi_{M-1})} (I - r_{M-1,n} B_{M-1}) \\
&\quad \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} (I - r_{1,n} B_1) x_n, \\
v_n &= P_C (I - \lambda_{N,n} A_N) P_C (I - \lambda_{N-1,n} A_{N-1}) \\
&\quad \cdots P_C (I - \lambda_{2,n} A_2) P_C (I - \lambda_{1,n} A_1) u_n, \\
x_{n+1} &= s_n \gamma V x_n + \beta_n x_n \\
&\quad + ((1 - \beta_n) I - s_n \mu F) T_n v_n, \quad \forall n \geq 1,
\end{aligned} \tag{116}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) $s_n \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$, *and* $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty} \lambda_n = 2/L$);

(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *and* $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$, *for all* $i \in \{1, 2, \ldots, N\}$;

(iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$, *for all* $k \in \{1, 2, \ldots, M\}$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$, *which is a unique solution of VIP* (50).

Proof. First of all, repeating the same arguments as in Theorem 15, we can write

$$\begin{aligned}
P_C (I - \lambda_n \nabla f) &= \frac{2 - \lambda_n L}{4} I + \frac{2 + \lambda_n L}{4} T_n \\
&= s_n I + (1 - s_n) T_n,
\end{aligned} \tag{117}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{118}$$

Put

$$\begin{aligned}
\Delta_n^k &= T_{r_{k,n}}^{(\Theta_k, \varphi_k)} (I - r_{k,n} B_k) T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})} (I - r_{k-1,n} B_{k-1}) \\
&\quad \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} (I - r_{1,n} B_1) x_n,
\end{aligned} \tag{119}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned}
\Lambda_n^i &= P_C (I - \lambda_{i,n} A_i) P_C (I - \lambda_{i-1,n} A_{i-1}) \\
&\quad \cdots P_C (I - \lambda_{2,n} A_2) P_C (I - \lambda_{1,n} A_1),
\end{aligned} \tag{120}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$. In addition, taking into consideration conditions (i) and (ii), we may assume, without loss of generality, that $s_n \leq 1 - \beta_n$, for all $n \geq 1$.

We divide the remainder of the proof into several steps.

Step 1. Let us show that $\|x_n - p\| \leq \max\{\|x_1 - p\|, \|\gamma Vp - \mu Fp\|/(\tau - \gamma l)\}$, for all $n \geq 1$ and $p \in \Omega$. Indeed, take $p \in \Omega$ arbitrarily. Repeating the same arguments as those of (59)–(61) in the proof of Theorem 15, we obtain

$$\begin{aligned} \|u_n - p\| &\leq \|x_n - p\|, \\ \|v_n - p\| &\leq \|u_n - p\|, \\ \|v_n - p\| &\leq \|x_n - p\|. \end{aligned} \tag{121}$$

Taking into account conditions (i) and (ii), we may assume, without loss of generality, that $s_n \leq 1 - \beta_n$, for all $n \geq 1$. Then from (121), $T_n p = p$, and Lemma 13, we have

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|s_n(\gamma Vx_n - \mu Fp) + \beta_n(x_n - p) \\ &\quad + ((1-\beta_n)I - s_n\mu F)T_n v_n \\ &\quad - ((1-\beta_n)I - s_n\mu F)T_n p\| \\ &\leq s_n\|\gamma Vx_n - \mu Fp\| \\ &\quad + \beta_n\|x_n - p\| + (1-\beta_n) \\ &\quad \times \left\|\left(I - \frac{s_n}{1-\beta_n}\mu F\right)T_n v_n - \left(I - \frac{s_n}{1-\beta_n}\mu F\right)T_n p\right\| \\ &\leq s_n(\|\gamma Vx_n - \gamma Vp\| + \|\gamma Vp - \mu Fp\|) \\ &\quad + \beta_n\|x_n - p\| \\ &\quad + (1-\beta_n)\left(1 - \frac{s_n\tau}{1-\beta_n}\right)\|v_n - p\| \\ &\leq s_n(\|\gamma Vx_n - \gamma Vp\| + \|\gamma Vp - \mu Fp\|) \\ &\quad + \beta_n\|x_n - p\| \\ &\quad + (1 - \beta_n - s_n\tau)\|x_n - p\| \\ &\leq s_n\gamma l\|x_n - p\| + s_n\|\gamma Vp - \mu Fp\| \\ &\quad + (1 - s_n\tau)\|x_n - p\| \\ &= (1 - s_n(\tau - \gamma l))\|x_n - p\| \\ &\quad + s_n(\tau - \gamma l)\frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}. \end{aligned} \tag{122}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}, \quad \forall n \geq 1. \tag{123}$$

Hence $\{x_n\}$ is bounded. According to (121), $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{Vx_n\}$, and $\{FT_n v_n\}$ are also bounded.

Step 2. Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. To this end, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \quad \forall n \geq 1. \tag{124}$$

Observe that, from the definition of z_n,

$$\begin{aligned} &z_{n+1} - z_n \\ &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{s_{n+1}\gamma Vx_{n+1} + ((1-\beta_{n+1})I - s_{n+1}\mu F)T_{n+1}v_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{s_n\gamma Vx_n + ((1-\beta_n)I - s_n\mu F)T_n v_n}{1-\beta_n} \\ &= \frac{s_{n+1}}{1-\beta_{n+1}}\gamma Vx_{n+1} - \frac{s_n}{1-\beta_n}\gamma Vx_n \\ &\quad + T_{n+1}v_{n+1} - T_n v_n \\ &\quad + \frac{s_n}{1-\beta_n}\mu FT_n v_n - \frac{s_{n+1}}{1-\beta_{n+1}}\mu FT_{n+1}v_{n+1} \\ &= \frac{s_{n+1}}{1-\beta_{n+1}}(\gamma Vx_{n+1} - \mu FT_{n+1}v_{n+1}) \\ &\quad + \frac{s_n}{1-\beta_n}(\mu FT_n v_n - \gamma Vx_n) \\ &\quad + T_{n+1}v_{n+1} - T_n v_n. \end{aligned} \tag{125}$$

Thus, it follows that

$$\begin{aligned} &\|z_{n+1} - z_n\| \\ &\leq \frac{s_{n+1}}{1-\beta_{n+1}}(\gamma\|Vx_{n+1}\| + \mu\|FT_{n+1}v_{n+1}\|) \\ &\quad + \frac{s_n}{1-\beta_n}(\mu\|FT_n v_n\| + \gamma\|Vx_n\|) \\ &\quad + \|T_{n+1}v_{n+1} - T_n v_n\|. \end{aligned} \tag{126}$$

On the other hand, since ∇f is $(1/L)$-ism, $P_C(I-\lambda_n\nabla f)$ is nonexpansive for $\lambda_n \in (0,2/L)$. So, it follows that, for any given $p\in\Omega$,

$$\begin{aligned}
&\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right\| \\
&\quad\le \left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n-p\right\|+\|p\| \\
&\quad= \left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n-P_C\left(I-\lambda_{n+1}\nabla f\right)p\right\|+\|p\| \\
&\quad\le \left\|v_n-p\right\|+\|p\| \\
&\quad\le \left\|v_n\right\|+2\|p\|.
\end{aligned}\tag{127}$$

This together with the boundedness of $\{v_n\}$ implies that $\{P_C(I-\lambda_{n+1}\nabla f)v_n\}$ is bounded. Also, observe that

$$\begin{aligned}
&\left\|T_{n+1}v_n-T_nv_n\right\| \\
&\quad=\left\|\frac{4P_C\left(I-\lambda_{n+1}\nabla f\right)-\left(2-\lambda_{n+1}L\right)I}{2+\lambda_{n+1}L}v_n\right. \\
&\qquad\left.-\frac{4P_C\left(I-\lambda_{n}\nabla f\right)-\left(2-\lambda_{n}L\right)I}{2+\lambda_{n}L}v_n\right\| \\
&\quad\le\left\|\frac{4P_C\left(I-\lambda_{n+1}\nabla f\right)}{2+\lambda_{n+1}L}v_n-\frac{4P_C\left(I-\lambda_{n}\nabla f\right)}{2+\lambda_{n}L}v_n\right\| \\
&\qquad+\left\|\frac{2-\lambda_nL}{2+\lambda_nL}v_n-\frac{2-\lambda_{n+1}L}{2+\lambda_{n+1}L}v_n\right\| \\
&\quad=\left\|\left(4\left(2+\lambda_nL\right)P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right.\right. \\
&\qquad\left.-4\left(2+\lambda_{n+1}L\right)P_C\left(I-\lambda_n\nabla f\right)v_n\right) \\
&\qquad\left.\times\left(\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)\right)^{-1}\right\| \\
&\qquad+\frac{4L\left|\lambda_{n+1}-\lambda_n\right|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\| \\
&\quad=\left\|\left(4L\left(\lambda_n-\lambda_{n+1}\right)P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right.\right. \\
&\qquad+4\left(2+\lambda_{n+1}L\right) \\
&\qquad\left.\times\left(P_C\left(I-\lambda_{n+1}\nabla f\right)v_n-P_C\left(I-\lambda_n\nabla f\right)v_n\right)\right) \\
&\qquad\left.\times\left(\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)\right)^{-1}\right\| \\
&\qquad+\frac{4L\left|\lambda_{n+1}-\lambda_n\right|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\| \\
&\quad\le\frac{4L\left|\lambda_n-\lambda_{n+1}\right|\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right\|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)} \\
&\qquad+\Big(\Big(4\left(2+\lambda_{n+1}L\right) \\
&\qquad\quad\times\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n-P_C\left(I-\lambda_n\nabla f\right)v_n\right\|\Big) \\
&\qquad\quad\times\left(\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)\right)^{-1}\Big) \\
&\qquad+\frac{4L\left|\lambda_{n+1}-\lambda_n\right|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\| \\
&\quad\le\left|\lambda_{n+1}-\lambda_n\right| \\
&\qquad\times\left[L\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right\|+4\left\|\nabla f\left(v_n\right)\right\|+L\left\|v_n\right\|\right] \\
&\quad\le\widetilde{M}\left|\lambda_{n+1}-\lambda_n\right|,
\end{aligned}\tag{128}$$

where $\sup_{n\ge1}\{L\|P_C(I-\lambda_{n+1}\nabla f)v_n\|+4\|\nabla f(v_n)\|+L\|v_n\|\}\le\widetilde{M}$ for some $\widetilde{M}>0$. So, by (128), we have that

$$\begin{aligned}
\left\|T_{n+1}v_{n+1}-T_nv_n\right\|&\le\left\|T_{n+1}v_{n+1}-T_{n+1}v_n\right\| \\
&\quad+\left\|T_{n+1}v_n-T_nv_n\right\| \\
&\le\left\|v_{n+1}-v_n\right\|+\widetilde{M}\left|\lambda_{n+1}-\lambda_n\right| \\
&\le\left\|v_{n+1}-v_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right).
\end{aligned}\tag{129}$$

Note that

$$\begin{aligned}
\left\|v_{n+1}-v_n\right\|&=\left\|\Lambda_{n+1}^Nu_{n+1}-\Lambda_n^Nu_n\right\| \\
&=\left\|P_C\left(I-\lambda_{N,n+1}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\
&\qquad\left.-P_C\left(I-\lambda_{N,n}A_N\right)\Lambda_{n}^{N-1}u_{n}\right\| \\
&\le\left\|P_C\left(I-\lambda_{N,n+1}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\
&\qquad\left.-P_C\left(I-\lambda_{N,n}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\
&\quad+\left\|P_C\left(I-\lambda_{N,n}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\
&\qquad\left.-P_C\left(I-\lambda_{N,n}A_N\right)\Lambda_{n}^{N-1}u_{n}\right\| \\
&\le\left\|\left(I-\lambda_{N,n+1}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\
&\qquad\left.-\left(I-\lambda_{N,n}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\
&\quad+\left\|\left(I-\lambda_{N,n}A_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\right. \\
&\qquad\left.-\left(I-\lambda_{N,n}A_N\right)\Lambda_{n}^{N-1}u_{n}\right\| \\
&\le\left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\
&\quad+\left\|\Lambda_{n+1}^{N-1}u_{n+1}-\Lambda_{n}^{N-1}u_{n}\right\| \\
&\le\left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\
&\quad+\left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left\|A_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\Lambda_{n+1}^{N-2}u_{n+1}-\Lambda_n^{N-2}u_n\right\| \\
&\vdots \\
&\leq \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|A_N\Lambda_{n+1}^{N-1}u_{n+1}\right\| \\
&\quad+\left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left\|A_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\| \\
&\quad+\cdots+\left|\lambda_{1,n+1}-\lambda_{1,n}\right|\left\|A_1\Lambda_{n+1}^{0}u_{n+1}\right\| \\
&\quad+\left\|\Lambda_{n+1}^{0}u_{n+1}-\Lambda_n^{0}u_n\right\| \\
&\leq \widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\|,
\end{aligned}
\tag{130}
$$

where $\sup_{n\geq 1}\{\sum_{i=1}^{N}\|A_i\Lambda_{n+1}^{i-1}u_{n+1}\|\}\leq \widetilde{M}_0$ for some $\widetilde{M}_0>0$. Also, utilizing Proposition 1(ii), (v) we deduce that

$$
\begin{aligned}
&\left\|u_{n+1}-u_n\right\| \\
&=\left\|\Delta_{n+1}^{M}x_{n+1}-\Delta_n^{M}x_n\right\| \\
&=\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}B_M\right)\Delta_{n}^{M-1}x_{n}\Big\| \\
&\leq\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\| \\
&\quad+\Big\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}B_M\right)\Delta_{n}^{M-1}x_{n}\Big\| \\
&\leq\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\| \\
&\quad+\Big\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\| \\
&\quad+\left\|\left(I-r_{M,n}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}-\left(I-r_{M,n}B_M\right)\Delta_{n}^{M-1}x_{n}\right\| \\
&\leq\frac{\left|r_{M,n+1}-r_{M,n}\right|}{r_{M,n+1}} \\
&\quad\times\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad -\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\| \\
&\quad+\left|r_{M,n+1}-r_{M,n}\right|\left\|B_M\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\quad+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\| \\
&=\left|r_{M,n+1}-r_{M,n}\right| \\
&\quad\times\Big[\left\|B_M\Delta_{n+1}^{M-1}x_{n+1}\right\| \\
&\qquad+\frac{1}{r_{M,n+1}}\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad\qquad -\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\|\Big] \\
&\quad+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\| \\
&\vdots \\
&\leq\left|r_{M,n+1}-r_{M,n}\right| \\
&\quad\times\Big[\left\|B_M\Delta_{n+1}^{M-1}x_{n+1}\right\|+\frac{1}{r_{M,n+1}} \\
&\qquad\times\Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1} \\
&\qquad\qquad -\left(I-r_{M,n+1}B_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\|\Big] \\
&\quad+\cdots+\left|r_{1,n+1}-r_{1,n}\right| \\
&\quad\times\Big[\left\|B_1\Delta_{n+1}^{0}x_{n+1}\right\|+\frac{1}{r_{1,n+1}} \\
&\qquad\times\Big\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n+1}B_1\right)\Delta_{n+1}^{0}x_{n+1} \\
&\qquad\qquad -\left(I-r_{1,n+1}B_1\right)\Delta_{n+1}^{0}x_{n+1}\Big\|\Big] \\
&\quad+\left\|\Delta_{n+1}^{0}x_{n+1}-\Delta_{n}^{0}x_{n}\right\| \\
&\leq\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|+\left\|x_{n+1}-x_n\right\|,
\end{aligned}
\tag{131}
$$

where $\widetilde{M}_1>0$ is a constant such that, for each $n\geq 1$,

$$
\begin{aligned}
&\sum_{k=1}^{M}\Big[\left\|B_k\Delta_{n+1}^{k-1}x_{n+1}\right\| \\
&\quad+\frac{1}{r_{k,n+1}}\Big\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n+1}B_k\right)\Delta_{n+1}^{k-1}x_{n+1} \\
&\qquad -\left(I-r_{k,n+1}B_k\right)\Delta_{n+1}^{k-1}x_{n+1}\Big\|\Big]\leq\widetilde{M}_1.
\end{aligned}
\tag{132}
$$

Combining (126)–(131), we get

$$
\begin{aligned}
&\left\|z_{n+1}-z_n\right\|-\left\|x_{n+1}-x_n\right\| \\
&\quad\leq\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)
\end{aligned}
$$

$$\begin{aligned}
&+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&+\left\|T_{n+1}v_{n+1}-T_nv_n\right\|-\left\|x_{n+1}-x_n\right\|\\
\le{}&\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&+\left\|v_{n+1}-v_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)\\
&-\left\|x_{n+1}-x_n\right\|\\
\le{}&\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\|\\
&+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
\le{}&\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|\\
&+\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|\\
&+\left\|x_{n+1}-x_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)\\
&-\left\|x_{n+1}-x_n\right\|\\
={}&\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|\\
&+\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|\\
&+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right).
\end{aligned}\tag{133}$$

Thus, it follows from (133) and conditions (i)–(iv) that

$$\limsup_{n\to\infty}\left(\left\|z_{n+1}-z_n\right\|-\left\|x_{n+1}-x_n\right\|\right)\le 0.\tag{134}$$

Hence by Lemma 11 we have

$$\lim_{n\to\infty}\left\|z_n-x_n\right\|=0.\tag{135}$$

Consequently,

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=\lim_{n\to\infty}\left(1-\beta_n\right)\left\|z_n-x_n\right\|=0,\tag{136}$$

and by (129)–(131),

$$\begin{gathered}
\lim_{n\to\infty}\left\|u_{n+1}-u_n\right\|=0,\\
\lim_{n\to\infty}\left\|v_{n+1}-v_n\right\|=0,\\
\lim_{n\to\infty}\left\|T_{n+1}v_{n+1}-T_nv_n\right\|=0.
\end{gathered}\tag{137}$$

Step 3. Let us show that $\|B_k\Delta_n^{k-1}x_n-B_kp\|\to 0$ and $\|A_i\Lambda_n^{i-1}u_n-A_ip\|\to 0$, for all $k\in\{1,2,\ldots,M\}$ and $i\in\{1,2,\ldots,N\}$.

Indeed, since

$$x_{n+1}=s_n\gamma Vx_n+\beta_nx_n+\left(\left(1-\beta_n\right)I-s_n\mu F\right)T_nv_n,\tag{138}$$

we have

$$\begin{aligned}
&\left\|x_n-T_nv_n\right\|\\
&\quad\le\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-T_nv_n\right\|\\
&\quad\le\left\|x_n-x_{n+1}\right\|+s_n\left\|\gamma Vx_n-\mu FT_nv_n\right\|\\
&\qquad+\beta_n\left\|x_n-T_nv_n\right\|;
\end{aligned}\tag{139}$$

that is,

$$\begin{aligned}
\left\|x_n-T_nv_n\right\|\le{}&\frac{1}{1-\beta_n}\left\|x_n-x_{n+1}\right\|\\
&+\frac{s_n}{1-\beta_n}\left(\gamma\left\|Vx_n\right\|+\mu\left\|FT_nv_n\right\|\right).
\end{aligned}\tag{140}$$

So, from $s_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, and condition (ii), it follows that

$$\lim_{n\to\infty}\left\|x_n-T_nv_n\right\|=0.\tag{141}$$

Also, from (29) it follows that, for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$,

$$\begin{aligned}
&\|v_n - p\|^2 \\
&\quad = \|\Lambda_n^N u_n - p\|^2 \\
&\quad \le \|\Lambda_n^i u_n - p\|^2 \\
&\quad = \|P_C(I - \lambda_{i,n} A_i)\Lambda_n^{i-1} u_n - P_C(I - \lambda_{i,n} A_i) p\|^2 \\
&\quad \le \|(I - \lambda_{i,n} A_i)\Lambda_n^{i-1} u_n - (I - \lambda_{i,n} A_i) p\|^2 \\
&\quad \le \|\Lambda_n^{i-1} u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2 \\
&\quad \le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2 \\
&\quad \le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2, \\
&\|u_n - p\|^2 \\
&\quad = \|\Delta_n^M x_n - p\|^2 \\
&\quad \le \|\Delta_n^k x_n - p\|^2 \\
&\quad = \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n} B_k)\Delta_n^{k-1} x_n \\
&\qquad - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n} B_k) p\|^2 \\
&\quad \le \|(I - r_{k,n} B_k)\Delta_n^{k-1} x_n - (I - r_{k,n} B_k) p\|^2 \\
&\quad \le \|\Delta_n^{k-1} x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|B_k \Delta_n^{k-1} x_n - B_k p\|^2 \\
&\quad \le \|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|B_k \Delta_n^{k-1} x_n - B_k p\|^2.
\end{aligned} \tag{142}$$

Furthermore, utilizing Lemma 7, we deduce from (116) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|s_n(\gamma V x_n - \mu F p) + \beta_n(x_n - T_n v_n) \\
&\qquad + (I - s_n \mu F) T_n v_n - (I - s_n \mu F) p\|^2 \\
&\quad \le \|\beta_n(x_n - T_n v_n) \\
&\qquad + (I - s_n \mu F) T_n v_n - (I - s_n \mu F) p\|^2 \\
&\qquad + 2 s_n \langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle \\
&\quad \le [\beta_n \|x_n - T_n v_n\| \\
&\qquad + \|(I - s_n \mu F) T_n v_n - (I - s_n \mu F) T_n p\|]^2 \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad \le [\beta_n \|x_n - T_n v_n\| + (1 - s_n \tau)\|v_n - p\|]^2 \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad = (1 - s_n \tau)^2 \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|.
\end{aligned} \tag{143}$$

From (142)-(143), it follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad \le \|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2 \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad \le \|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|B_k \Delta_n^{k-1} x_n - B_k p\|^2 \\
&\qquad + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2 \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|,
\end{aligned} \tag{144}$$

and so

$$\begin{aligned}
&r_{k,n}(2\mu_k - r_{k,n})\|B_k \Delta_n^{k-1} x_n - B_k p\|^2 \\
&\qquad + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|A_i \Lambda_n^{i-1} u_n - A_i p\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad \le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n \tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2 s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|.
\end{aligned} \tag{145}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, by (136), (141), and (145) we conclude immediately that

$$\lim_{n\to\infty} \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\| = 0,$$
$$\lim_{n\to\infty} \left\| B_k \Delta_n^{k-1} x_n - B_k p \right\| = 0, \tag{146}$$

for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$.

Step 4. Let us show that $\|x_n - u_n\| \to 0$, $\|u_n - v_n\| \to 0$, and $\|v_n - T_n v_n\| \to 0$ as $n \to \infty$.

Indeed, by Proposition 1(ii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
&\left\| \Delta_n^k x_n - p \right\|^2 \\
&\quad = \left\| T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n} B_k\right) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n} B_k\right) p \right\|^2 \\
&\quad \le \left\langle \left(I - r_{k,n} B_k\right) \Delta_n^{k-1} x_n - \left(I - r_{k,n} B_k\right) p, \Delta_n^k x_n - p \right\rangle \\
&\quad = \frac{1}{2} \left(\left\| \left(I - r_{k,n} B_k\right) \Delta_n^{k-1} x_n - \left(I - r_{k,n} B_k\right) p \right\|^2 + \left\| \Delta_n^k x_n - p \right\|^2 \right. \\
&\qquad \left. - \left\| \left(I - r_{k,n} B_k\right) \Delta_n^{k-1} x_n - \left(I - r_{k,n} B_k\right) p - \left(\Delta_n^k x_n - p\right) \right\|^2 \right) \\
&\quad \le \frac{1}{2} \left(\left\| \Delta_n^{k-1} x_n - p \right\|^2 + \left\| \Delta_n^k x_n - p \right\|^2 - \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} (B_k \Delta_n^{k-1} x_n - B_k p) \right\|^2 \right),
\end{aligned} \tag{147}$$

which implies that

$$\begin{aligned}
&\left\| \Delta_n^k x_n - p \right\|^2 \\
&\quad \le \left\| \Delta_n^{k-1} x_n - p \right\|^2 - \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} (B_k \Delta_n^{k-1} x_n - B_k p) \right\|^2 \\
&\quad = \left\| \Delta_n^{k-1} x_n - p \right\|^2 - \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n \right\|^2 - r_{k,n}^2 \left\| B_k \Delta_n^{k-1} x_n - B_k p \right\|^2 \\
&\qquad + 2 r_{k,n} \left\langle \Delta_n^{k-1} x_n - \Delta_n^k x_n, B_k \Delta_n^{k-1} x_n - B_k p \right\rangle \\
&\quad \le \left\| \Delta_n^{k-1} x_n - p \right\|^2 - \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n \right\|^2 + 2 r_{k,n} \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n \right\| \left\| B_k \Delta_n^{k-1} x_n - B_k p \right\| \\
&\quad \le \left\| x_n - p \right\|^2 - \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n \right\|^2 + 2 r_{k,n} \left\| \Delta_n^{k-1} x_n - \Delta_n^k x_n \right\| \left\| B_k \Delta_n^{k-1} x_n - B_k p \right\|.
\end{aligned} \tag{148}$$

Also, by Proposition 2(iii), we obtain that for each $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
&\left\| \Lambda_n^i u_n - p \right\|^2 \\
&\quad = \left\| P_C (I - \lambda_{i,n} A_i) \Lambda_n^{i-1} u_n - P_C (I - \lambda_{i,n} A_i) p \right\|^2 \\
&\quad \le \left\langle \left(I - \lambda_{i,n} A_i\right) \Lambda_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i\right) p, \Lambda_n^i u_n - p \right\rangle \\
&\quad = \frac{1}{2} \left(\left\| \left(I - \lambda_{i,n} A_i\right) \Lambda_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i\right) p \right\|^2 + \left\| \Lambda_n^i u_n - p \right\|^2 \right. \\
&\qquad \left. - \left\| \left(I - \lambda_{i,n} A_i\right) \Lambda_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i\right) p - \left(\Lambda_n^i u_n - p\right) \right\|^2 \right) \\
&\quad \le \frac{1}{2} \left(\left\| \Lambda_n^{i-1} u_n - p \right\|^2 + \left\| \Lambda_n^i u_n - p \right\|^2 - \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (A_i \Lambda_n^{i-1} u_n - A_i p) \right\|^2 \right) \\
&\quad \le \frac{1}{2} \left(\left\| u_n - p \right\|^2 + \left\| \Lambda_n^i u_n - p \right\|^2 - \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (A_i \Lambda_n^{i-1} u_n - A_i p) \right\|^2 \right),
\end{aligned} \tag{149}$$

which implies

$$\begin{aligned}
&\left\| \Lambda_n^i u_n - p \right\|^2 \\
&\quad \le \left\| u_n - p \right\|^2 - \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (A_i \Lambda_n^{i-1} u_n - A_i p) \right\|^2 \\
&\quad = \left\| u_n - p \right\|^2 - \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n \right\|^2 - \lambda_{i,n}^2 \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\|^2 \\
&\qquad + 2 \lambda_{i,n} \left\langle \Lambda_n^{i-1} u_n - \Lambda_n^i u_n, A_i \Lambda_n^{i-1} u_n - A_i p \right\rangle \\
&\quad \le \left\| u_n - p \right\|^2 - \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n \right\|^2 + 2 \lambda_{i,n} \left\| \Lambda_n^{i-1} u_n - \Lambda_n^i u_n \right\| \left\| A_i \Lambda_n^{i-1} u_n - A_i p \right\|.
\end{aligned} \tag{150}$$

Thus, from (143), (148), and (150), we have

$$\begin{aligned}
&\left\| x_{n+1} - p \right\|^2 \\
&\quad \le \left(1 - s_n \tau\right)^2 \left\| v_n - p \right\|^2 + \beta_n^2 \left\| x_n - T_n v_n \right\|^2 \\
&\qquad + 2 \left(1 - s_n \tau\right) \beta_n \left\| v_n - p \right\| \left\| x_n - T_n v_n \right\| \\
&\qquad + 2 s_n \left\| \gamma V x_n - \mu F p \right\| \left\| x_{n+1} - p \right\|
\end{aligned}$$

$$
\begin{aligned}
&= (1 - s_n\tau)^2 \left\|\Lambda_n^N u_n - p\right\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\le (1 - s_n\tau)^2 \left\|\Lambda_n^i u_n - p\right\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\le (1 - s_n\tau)^2 \\
&\quad \times \Big[\|u_n - p\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|\Big] \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&= (1 - s_n\tau)^2 \\
&\quad \times \Big[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|\Big] \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\le (1 - s_n\tau)^2 \\
&\quad \times \Big[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|\Big] \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\le (1 - s_n\tau)^2 \\
&\quad \times \Big[\|x_n - p\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\| \\
&\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\|\Big] \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 - (1 - s_n\tau)^2 \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\| \\
&\quad - (1 - s_n\tau)^2 \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\| \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|;
\end{aligned} \tag{151}
$$

that is,

$$
\begin{aligned}
&(1 - s_n\tau)^2 \left[\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2\right] \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\| \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\| \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\quad \le (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|B_k \Delta_n^{k-1} x_n - B_k p\right\| \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|A_i \Lambda_n^{i-1} u_n - A_i p\right\| \\
&\qquad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\qquad + 2(1 - s_n\tau)\beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\qquad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\|.
\end{aligned} \tag{152}
$$

So, from $s_n \to 0$, (136), (141), and (146) we immediately get

$$
\begin{gathered}
\lim_{n\to\infty} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|, \\
\lim_{n\to\infty} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| = 0,
\end{gathered} \tag{153}
$$

for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Note that

$$\begin{aligned} \|x_n - u_n\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\ &\le \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\ &\quad + \cdots + \left\|\Delta_n^{M-1} x_n - \Delta_n^M x_n\right\|, \\ \|u_n - v_n\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\ &\le \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\ &\quad + \cdots + \left\|\Lambda_n^{N-1} u_n - \Lambda_n^N u_n\right\|. \end{aligned} \tag{154}$$

Thus, from (153) we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u_n\| &= 0, \\ \lim_{n\to\infty} \|u_n - v_n\| &= 0. \end{aligned} \tag{155}$$

It is easy to see that as $n \to \infty$

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \longrightarrow 0. \tag{156}$$

Also, observe that

$$\|T_n v_n - v_n\| \le \|T_n v_n - x_n\| + \|x_n - v_n\|. \tag{157}$$

Hence we have from (141)

$$\lim_{n\to\infty} \|T_n v_n - v_n\| = 0. \tag{158}$$

Step 5. Let us show that $\limsup_{n\to\infty}\langle(\mu F - \gamma V)q, q - x_n\rangle \le 0$, where $q \in \Omega$ is the same as in Theorem 15; that is, $q \in \Omega$ is a unique solution of VIP (50). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \langle (\mu F - \gamma V) q, q - x_n\rangle \\ &\quad = \lim_{i\to\infty} \left\langle (\mu F - \gamma V) q, q - x_{n_i}\right\rangle. \end{aligned} \tag{159}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From Step 4, we have that $\Delta_{n_i}^k x_{n_i} \rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, and $v_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$ and $m \in \{1, 2, \ldots, N\}$. Since $v_n - T_n v_n \to 0$ by Step 4, by the same arguments as in the proof of Theorem 15, we get $w \in \Omega$. Since $q = P_\Omega(I - \mu F + \gamma V)q$, it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \langle (\mu F - \gamma V) q, q - x_n\rangle \\ &\quad = \lim_{i\to\infty} \left\langle (\mu F - \gamma V) q, q - x_{n_i}\right\rangle \\ &\quad = \langle (\mu F - \gamma V) q, q - w\rangle \le 0. \end{aligned} \tag{160}$$

Step 6. Let us show that $\lim_{n\to\infty}\|x_n - q\| = 0$, where $q \in \Omega$ is the same as in Theorem 15; that is, $q \in \Omega$ is a unique solution of VIP (50). From (116), we know that

$$\begin{aligned} x_{n+1} - q &= s_n(\gamma V x_n - \mu F q) \\ &\quad + \beta_n(T_n v_n - q) \\ &\quad + ((1-\beta_n) I - s_n \mu F) T_n v_n \\ &\quad - ((1-\beta_n) I - s_n \mu F) q. \end{aligned} \tag{161}$$

Applying Lemmas 7 and 13 and noticing that $T_n q = q$ and $\|v_n - q\| \le \|x_n - q\|$, for all $n \ge 1$, we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad \le \|\beta_n(T_n v_n - q) + ((1-\beta_n) I - s_n\mu F) T_n v_n \\ &\qquad - ((1-\beta_n) I - s_n \mu F) q\|^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \mu F q, x_{n+1} - q\rangle \\ &\quad \le [\beta_n \|T_n v_n - q\| \\ &\qquad + \|((1-\beta_n) I - s_n\mu F) T_n v_n - ((1-\beta_n) I - s_n\mu F) q\|]^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad = \Bigg[\beta_n \|T_n v_n - q\| + (1-\beta_n) \\ &\qquad \times \left\|\left(I - \frac{s_n}{1-\beta_n}\mu F\right) T_n v_n - \left(I - \frac{s_n}{1-\beta_n}\mu F\right) T_n q\right\|\Bigg]^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad \le \left[\beta_n \|v_n - q\| + (1-\beta_n)\left(1 - \frac{s_n\tau}{1-\beta_n}\right)\|v_n - q\|\right]^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad = [\beta_n \|v_n - q\| + (1 - \beta_n - s_n\tau)\|v_n - q\|]^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad \le [\beta_n \|x_n - q\| + (1 - \beta_n - s_n\tau)\|x_n - q\|]^2 \\ &\qquad + 2 s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q\rangle \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad \le (1 - s_n\tau)^2 \|x_n - q\|^2 \\ &\qquad + 2 s_n \gamma l \|x_n - q\| \|x_{n+1} - q\| \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle \\ &\quad \le (1 - \tau s_n)^2 \|x_n - q\|^2 \\ &\qquad + s_n \gamma l \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\ &\qquad + 2 s_n \langle \gamma V q - \mu F q, x_{n+1} - q\rangle. \end{aligned} \tag{162}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\quad\le \frac{1-2\tau s_n+\tau^2 s_n^2+s_n\gamma l}{1-s_n\gamma l}\|x_n-q\|^2\\
&\qquad+\frac{2s_n}{1-s_n\gamma l}\langle\gamma Vq-\mu Fq,x_{n+1}-q\rangle\\
&\quad=\left(1-\frac{2(\tau-\gamma l)s_n}{1-s_n\gamma l}\right)\|x_n-q\|^2\\
&\qquad+\frac{\tau^2 s_n^2}{1-s_n\gamma l}\|x_n-q\|^2\\
&\qquad+\frac{2s_n}{1-s_n\gamma l}\langle\gamma Vq-\mu Fq,x_{n+1}-q\rangle\\
&\quad\le\left(1-\frac{2(\tau-\gamma l)}{1-s_n\gamma l}s_n\right)\|x_n-q\|^2+\frac{2(\tau-\gamma l)s_n}{1-s_n\gamma l}\\
&\qquad\times\left(\frac{\tau^2 s_n}{2(\tau-\gamma l)}\widetilde{M}_2+\frac{1}{\tau-\gamma l}\langle\mu Fq-\gamma Vq,q-x_{n+1}\rangle\right)\\
&\quad=(1-\sigma_n)\|x_n-q\|^2+\sigma_n\delta_n,
\end{aligned}\tag{163}$$

where $\widetilde{M}_2=\sup_{n\ge1}\|x_n-q\|^2$, $\sigma_n=(2(\tau-\gamma l)/(1-s_n\gamma l))s_n$, and

$$\delta_n=\frac{\tau^2 s_n}{2(\tau-\gamma l)}\widetilde{M}_2+\frac{1}{\tau-\gamma l}\langle\mu Fq-\gamma Vq,q-x_{n+1}\rangle.\tag{164}$$

From condition (i) and Step 5, it is easy to see that $\sigma_n\to 0$, $\sum_{n=0}^{\infty}\sigma_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. Hence, by Lemma 10, we conclude that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f: C\to\mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi: C\to\mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B: H\to H$ and $A_i: C\to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $i=1,2$. Let $F: H\to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa,\eta>0$. Let $V: H\to H$ be an l-Lipschitzian mapping with constant $l\ge 0$. Let $0<\mu<2\eta/\kappa^2$ and $0\le\gamma l<\tau$, where $\tau=1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Assume that $\Omega:=GMEP(\Theta,\varphi,B)\cap VI(C,A_1)\cap VI(C,A_2)\cap\Gamma\neq\emptyset$ and that either (B1) or (B2) holds. For arbitrarily given $x_1\in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n,y-u_n\rangle\\
&\qquad+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad\forall y\in C,\\
&v_n=P_C(I-\lambda_{2,n}A_2)P_C(I-\lambda_{1,n}A_1)u_n,\\
&x_{n+1}=s_n\gamma Vx_n+\beta_n x_n\\
&\qquad+((1-\beta_n)I-s_n\mu F)T_n v_n,\quad\forall n\ge 1,
\end{aligned}\tag{165}$$

where $P_C(I-\lambda_n\nabla f)=s_nI+(1-s_n)T_n$ (here T_n is nonexpansive and $s_n=(2-\lambda_nL)/4\in(0,1/2)$ for each $\lambda_n\in(0,2/L)$). Assume that the following conditions hold:

(i) *$s_n\in(0,1/2)$ for each $\lambda_n\in(0,2/L)$, and $\lim_{n\to\infty}s_n=0$ ($\Leftrightarrow\lim_{n\to\infty}\lambda_n=2/L$);*
(ii) *$\{\beta_n\}\subset(0,1)$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;*
(iii) *$\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i)$ and $\lim_{n\to\infty}|\lambda_{i,n+1}-\lambda_{i,n}|=0$ for $i=1,2$;*
(iv) *$\{r_n\}\subset[e,f]\subset(0,2\zeta)$ and $\lim_{n\to\infty}|r_{n+1}-r_n|=0$.*

Then $\{x_n\}$ converges strongly as $\lambda_n\to 2/L$ ($\Leftrightarrow s_n\to 0$) to a point $q\in\Omega$, which is a unique solution of VIP (113).

Corollary 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f: C\to\mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi: C\to\mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $B: H\to H$ and $A: C\to H$ be ζ-inverse strongly monotone and ξ-inverse strongly monotone, respectively. Let $F: H\to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa,\eta>0$. Let $V: H\to H$ be an l-Lipschitzian mapping with constant $l\ge 0$. Let $0<\mu<2\eta/\kappa^2$ and $0\le\gamma l<\tau$, where $\tau=1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Assume that $\Omega:=\mathrm{GMEP}(\Theta,\varphi,B)\cap\mathrm{VI}(C,A)\cap\Gamma\neq\emptyset$ and that either (B1) or (B2) holds. For arbitrarily given $x_1\in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n,y-u_n\rangle\\
&\qquad+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad\forall y\in C,\\
&v_n=P_C(I-\rho_nA)u_n,\\
&x_{n+1}=s_n\gamma Vx_n+\beta_n x_n\\
&\qquad+((1-\beta_n)I-s_n\mu F)T_n v_n,\quad\forall n\ge 1,
\end{aligned}\tag{166}$$

where $P_C(I-\lambda_n\nabla f)=s_nI+(1-s_n)T_n$ (here T_n is nonexpansive and $s_n=(2-\lambda_nL)/4\in(0,1/2)$ for each $\lambda_n\in(0,2/L)$). Assume that the following conditions hold:

(i) *$s_n\in(0,1/2)$ for each $\lambda_n\in(0,2/L)$, and $\lim_{n\to\infty}s_n=0$ ($\Leftrightarrow\lim_{n\to\infty}\lambda_n=2/L$);*
(ii) *$\{\beta_n\}\subset(0,1)$ and $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;*
(iii) *$\{\rho_n\}\subset[a,b]\subset(0,2\xi)$ and $\lim_{n\to\infty}|\rho_{n+1}-\rho_n|=0$;*
(iv) *$\{r_n\}\subset[e,f]\subset(0,2\zeta)$ and $\lim_{n\to\infty}|r_{n+1}-r_n|=0$.*

Then $\{x_n\}$ converges strongly as $\lambda_n\to 2/L$ ($\Leftrightarrow s_n\to 0$) to a point $q\in\Omega$, which is a unique solution of VIP (115).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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