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Research Article

Gevrey Regularity for the Noncutoff Nonlinear Homogeneous Boltzmann Equation with Strong Singularity

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The Cauchy problem of the nonlinear spatially homogeneous Boltzmann equation without angular cutoff is studied. By using analytic techniques, one proves the Gevrey regularity of the C^∞ solutions in non-Maxwellian and strong singularity cases.

1. Introduction

The standard form of the initial value problem for the spatially homogeneous nonlinear noncutoff Boltzmann equation is expressed as follows:

$$\frac{\partial f}{\partial t} = Q(f,f), \quad v \in \mathbb{R}^n,\ t \in (0,T]; \qquad f|_{t=0} = f_0(v), \tag{1}$$

where T is a fixed positive number and $f(t,v)$ denotes the density distribution function for velocity v at time t. The Boltzmann collision operator is expressed as follows:

$$Q(g,f) = \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} B(v-v_*,\sigma) \times \left\{g(v'_*) f(v') - g(v_*) f(v)\right\} d\sigma\, dv_*, \tag{2}$$

where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$. For $\sigma \in \mathbb{S}^{n-1}$,

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v+v_*|}{2}\sigma. \tag{3}$$

The Boltzmann collision cross section $B \geq 0$ is a function that was assumed to be the following form:

$$B(|v-v_*|,\sigma) = \Phi(|v-v_*|)\, b(\cos\theta), \qquad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma, \quad \theta \in \left[0, \frac{\pi}{2}\right], \tag{4}$$

where the kinetic factor $\Phi(|v-v_*|) = |v-v_*|^\gamma$. The angular part b has a singularity that satisfies for constant $K > 0$ and $s \in (0,1)$:

$$b(\cos\theta) \approx \frac{K}{\theta^{2+2s}}, \qquad \theta \longrightarrow 0. \tag{5}$$

Cases $0 < s < 1/2$, $1/2 \leq s < 1$ are considered mild singularity and strong singularity, respectively. The following norms of weighted function spaces are introduced:

$$\|f\|_{L^p_r} = \left\|\langle v\rangle^r f(v)\right\|_{L^p}, \qquad \|f\|_{H^s_r} = \left\|\langle D_v\rangle^s \langle v\rangle^r f(v)\right\|_{L^2}, \tag{6}$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$. $\langle D_v\rangle = (1+|D_v|^2)^{1/2}$ is the corresponding pseudo-differential operator. The definition of the Gevrey space can now be listed; compare [1–5].

Definition 1. For $s \geq 1$, the smooth function $u \in G^s(\mathbb{R}^n)$ which is the Gevrey space with index s if there exists a positive constant C such that, for any $k \in \mathbb{N}$,

$$\left\|D_v^k u\right\|_{L^2(\mathbb{R}^n)} \leq C^{k+1}(k!)^s, \tag{7}$$

or, equivalently,

$$\|u\|_{H^k(\mathbb{R}^n)} = \left\|\langle D_v\rangle^k u\right\|_{L^2(\mathbb{R}^n)} \leq C^{k+1}(k!)^s, \tag{8}$$

where

$$\left\|D_v^k u\right\|^2_{L^2(\mathbb{R}^n)} = \sum_{|\beta|=k}\left\|D_v^\beta u\right\|^2_{L^2(\mathbb{R}^n)}. \tag{9}$$

It is indicated that $u \in G^s(\mathbb{R}^n)$ is also equivalent to the fact that there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0 \langle D_v \rangle^{1/s}} u \in L^2(\mathbb{R}^n)$.

Research on the Gevrey regularity of the Boltzmann equation can be traced back to the work of Ukai [6], who constructed a unique local solution in Gevrey space for both spatially homogeneous and inhomogeneous noncutoff Boltzmann equations. In 2004, Desvillettes and Wennberg [7] gave a conjecture of the Gevrey smoothing effect. Five years later, the propagation of Gevrey regularity for solutions of the nonlinear spatially homogeneous Boltzmann equation with Maxwellian molecules is obtained in [8]. In that same year, Morimoto et al. [4] studied linearized cases and proved the Gevrey regularity of solutions without any extra assumption for the initial datum. They then considered the C^∞ solutions with Maxwellian decay in [9]; that is, a positive number δ_0 exists such that, for any $t_0 \in (0, T)$,

$$e^{\delta_0 \langle v \rangle^2} f \in L^\infty \left([t_0, T] ; H^\infty (\mathbb{R}^n) \right). \tag{10}$$

Under the hypotheses of $0 < s < 1/2$, $\gamma \geq 0$, $\gamma + 2s < 1$, and the modified kinetic factor $\Phi(|v|) = (1 + |v|^2)^{\gamma/2}$, they showed the Gevrey smooth property for this type of solutions to the Cauchy problem of the nonlinear homogeneous Boltzmann equation. By using the original definition of kinetic factor, Zhang and Yin [10] extended the above result in a general framework: $0 < s < 1/2$ and $-1 < \gamma + 2s < 1$.

In this paper, the same issue in the strong singularity case $1/2 < s < 1$ is disussed. To discuss this issue properly, some notations are introduced. For any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_+^n$, $v = (v_1, v_2, \ldots, v_n)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, the following expression is denoted:

$$\begin{gathered} |a|_* = a_1 + a_2 + \cdots + a_n, \qquad \xi^a = \xi_1^{a_1} \xi_2^{a_2} \cdots \xi_n^{a_n}, \\ f^{(a)} = D_v^a f = \partial_{v_1}^{a_1} \partial_{v_2}^{a_2} \cdots \partial_{v_n}^{a_n} f. \end{gathered} \tag{11}$$

For any $r \in \mathbb{R}$, let

$$\begin{gathered} (a - r)! = (a_1 - r)! \cdot (a_2 - r)! \cdots (a_n - r)!, \\ (ra)! = (ra_1)! \cdot (ra_2)! \cdots (ra_n)! \end{gathered} \tag{12}$$

with a convention that $K! = 1$ if $0 \geq K \in \mathbb{Z}$. For any $a' = (a_1', a_2', \ldots, a_n') \in \mathbb{Z}_+^n$, write $a' \leq a$ if $a_i' \leq a_i$, $i = 1, 2, \ldots, n$. Moreover,

$$C_a^{a'} = \frac{a!}{a'! \, (a - a')!}. \tag{13}$$

Instead of the assumption of Maxwellian decay, the smooth solutions $f(t, \cdot) \in \mathcal{S}(\mathbb{R}^n)$ are considered to satisfy the following inequality (this type of solutions had been studied in some literature. E.g., cf. [11]):

$$\|f\|_{H^{2s}} \leq C_1. \tag{14}$$

For any $P \in \mathbb{R}^+$,

$$\|f\|_{L_P^1} \leq C_1 \cdot 2^P, \tag{15}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the standard Schwartz space and C_1 is a fixed constant. For any $a \in \mathbb{Z}_+^n$,

$$v^a f^{(a)} \leq 0. \tag{16}$$

A preliminary analysis in Section 2 is conducted and Theorem 2 is proved in Section 3.

Theorem 2. *For $v > 1$, $1/2 < s < 1$, and $0 < \gamma < 1$, assume that $f(t, v) \in \mathcal{S}(\mathbb{R}^n)$ is a smooth solution of the Cauchy problem* (1) *that satisfies* (14), (15), *and* (16). *Then for any $0 < t \leq T$, the initial value $f(0, \cdot) \in G^\gamma(\mathbb{R}^n)$ implies that $f(t, \cdot) \in G^\gamma(\mathbb{R}^n)$.*

The proof procedure of Theorem 3 is proved in Section 4.

Theorem 3. *For $1/2 < s < 1$ and $0 < \gamma < 1$, assume that $f(t, v) \in \mathcal{S}(\mathbb{R}^n)$ is a smooth solution of the Cauchy problem* (1) *that satisfies* (14), (15), *and* (16). *A positive number exists $T_0 < T$ such that for any $0 < t \leq T_0$, $f(t, \cdot) \in G^{1/s}(\mathbb{R}^n)$.*

Evidently, the main conclusion of this paper, directly from Theorems 2 and 3 can be obtained.

Theorem 4. *For $1/2 < s < 1$ and $0 < \gamma < 1$, assume that $f(t, v) \in \mathcal{S}(\mathbb{R}^n)$ is a smooth solution of the Cauchy problem* (1) *that satisfies* (14), (15), *and* (16). *Then, for any $0 < t \leq T$, $f(t, \cdot) \in G^{1/s}(\mathbb{R}^n)$.*

2. Preliminary Analysis

In this section, the lemmas are stated and their proof process is provided.

Lemma 5. *Let $l > 0$, $m > 0$ be two given numbers. Assume that f is a function that satisfies* (15). *Then, for any fixed number $\epsilon > 0$, a constant $C = C(\epsilon)$ exists such that $\|f\|_{H_l^m}^2 \leq C \|f\|_{H^{m+\epsilon}}^2$.*

Proof. By Lemma 2.4 in [11],

$$\begin{aligned} \|f\|_{H_l^m}^2 &\leq C \cdot \|f\|_{H_{2l}^{m-\epsilon}} \cdot \|f\|_{H^{m+\epsilon}} \\ &\leq C^{1+(1/2)} \|f\|_{H_{4l}^{m-2\epsilon}}^{1/2} \cdot \|f\|_{H^m}^{1/2} \cdot \|f\|_{H^{m+\epsilon}} \\ &\leq C^{2-(1/2)} \|f\|_{H_{4l}^{m-2\epsilon}}^{1/2} \cdot \|f\|_{H^{m+\epsilon}}^{2-(1/2)} \\ &\vdots \\ &\leq C^{2-(1/2^{k-1})} \|f\|_{H_{2^k l}^{m-k\epsilon}}^{1/2^{k-1}} \cdot \|f\|_{H^{m+\epsilon}}^{2-(1/2^{k-1})} \\ &\leq C^2 \|f\|_{H_{2^k l}^{m-k\epsilon}}^{1/2^{k-1}} \cdot \|f\|_{H^{m+\epsilon}}^2. \end{aligned} \tag{17}$$

A positive integer k is chosen such that $k\epsilon - m > n/2$. For any $q > n/2$, $L^1(\mathbb{R}^n) \subseteq H^{-q}(\mathbb{R}^n)$. By combining this Lemma with (15), the following is obtained:

$$\begin{aligned}\|f\|^{1/2^{k-1}}_{H^{m-k\epsilon}_{2^k l}} &\leq C' \cdot \|f\|^{1/2^{k-1}}_{L^1_{2^k l}} \\ &\leq C' \cdot C_1 \cdot \left(2^{2^k l}\right)^{1/2^{k-1}} \\ &\leq C' \cdot C_1 \cdot 4^l.\end{aligned} \tag{18}$$

Therefore,

$$\|f\|^2_{H^m_l} \leq C\|f\|^2_{H^{m+\epsilon}}. \tag{19}$$

□

Lemma 6. *If $\nu \geq 1$, then, for any $2 \leq r \in \mathbb{N}$, there exists a constant B depending only on r such that, for any $k \in \mathbb{N}$,*

$$\sum_{k' \leq k} C_k^{k'} \cdot \frac{\left\{\left(2k' - r\right)!\right\}^\nu}{\{(2k-r)!\}^\nu} \leq B. \tag{20}$$

Moreover, if $\nu \geq 1$ and $r > 1 + (2\nu/(\nu-1))$, then there exists a constant B' depending on ν and r such that, for any $k \in \mathbb{N}$,

$$\begin{aligned}&\sum_{a' \in \mathbb{Z}^n_+, 0<|a'|_*<2k} C_{2k}^{|a'|_*} \\ &\quad \cdot \frac{\left\{\left(\left|a'\right|_* + 1 - r\right)!\right\}^\nu \cdot \left\{\left(2k - \left|a'\right|_* + 1 - r\right)!\right\}^\nu}{\{(2k-r)!\}^\nu} \\ &\leq B'.\end{aligned} \tag{21}$$

Proof. By using Proposition 3.1 in [9], the following is obtained:

$$\begin{aligned}&\sum_{k' \leq k} C_k^{k'} \cdot \frac{\left\{\left(2k' - r\right)!\right\}^\nu}{\{(2k-r)!\}^\nu} \\ &= \sum_{k' \leq k} C_k^{k'} \cdot \frac{\left\{\left(k' - r\right)!\right\}^\nu}{\{(k-r)!\}^\nu} \\ &\quad \cdot \left\{\frac{\left(2k' - r\right) \cdot \left(2k' - r - 1\right) \cdots \left(k' - r + 1\right)}{(2k-r) \cdot (2k-r-1) \cdots (k-r+1)}\right\}^\nu \\ &\leq \sum_{k' \leq k} C_k^{k'} \cdot \frac{\left\{\left(k' - r\right)!\right\}^\nu}{\{(k-r)!\}^\nu} \leq B.\end{aligned} \tag{22}$$

This completes the proof of the first inequality. Thereafter, the same analysis technique is applied as the proof of Proposition 3.1 in [9] to discuss the second inequality. Notice that

$$\frac{2k\,(2k-1)\cdots(2k-r+1)}{\left(2k-k'\right)\left(2k-k'-1\right)\cdots\left(2k-k'-r+1\right)} \leq 3^r \tag{23}$$

if $2k - k' \geq 2k/2$ and $2k \geq 4r$. Therefore, for $\nu > 1$ and $r > 1 + (2\nu/(\nu-1))$, one has

$$\begin{aligned}&\sum_{0<|a'|_*<2k} C_{2k}^{|a'|_*} \cdot \frac{\left\{\left(\left|a'\right|_* + 1 - r\right)!\right\}^\nu \cdot \left\{\left(2k - \left|a'\right|_* + 1 - r\right)!\right\}^\nu}{\{(2k-r)!\}^\nu} \\ &= \sum_{0<k'<2k} C_{2k}^{k'} \cdot \frac{\left\{\left(k' + 1 - r\right)!\right\}^\nu \cdot \left\{\left(2k - k' + 1 - r\right)!\right\}^\nu}{\{(2k-r)!\}^\nu} \\ &\leq \sum_{0<k'<2k} \Big((2k\,(2k-1)\cdots(2k-r+1)) \\ &\quad \times \left(k'\left(k'-1\right)\cdots\left(k'-r+1\right)\left(2k-k'\right)\right. \\ &\quad \left.\times \left(2k-k'-1\right)\cdots\left(2k-k'-r+1\right)\right)^{-1}\Big) \\ &\quad \cdot \frac{\left(k'+1-r\right)^\nu \cdot \left(2k-k'+1-r\right)^\nu}{\{(2k-r)\,(2k-r-1)\cdots(2k-2r+1)\}^{\nu-1}} \\ &\leq 3^r \sum_{0<k'<2k} \frac{\left(k'+1-r\right)^\nu \cdot \left(2k-k'+1-r\right)^\nu}{\{(2k-r)\,(2k-r-1)\cdots(2k-2r+1)\}^{\nu-1}} \\ &\leq B'.\end{aligned} \tag{24}$$

This completes the proof of the second inequality. □

3. Proof of Theorem 2

Suppose that $\nu > 1$, $1/2 < s < 1$, and $0 < \gamma < 1$. Let $f(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ be a smooth solution of the Cauchy problem (1) that satisfies (14), (15), and (16). Write

$$M_k f = (1-\Delta)^k f = \sum_{k' \leq k} \sum_{|a|_* = k'} (-1)^{k'} \cdot C_k^{k'} \cdot \frac{k'!}{a!} f^{(2a)}. \tag{25}$$

Then the Fourier transform $\widehat{M_k f}(\xi) = (1+|\xi|^2)^k \widehat{f}(\xi)$. For any a', $a \in \mathbb{Z}^n_+$, $|a|_* = k$, and $a' \leq a$, it follows from (16) that

$$\begin{aligned}\nu^{a'} f^{(a)} f^{(a-a')} &= \nu^a f^{(a)} \cdot \nu^{a'-a} f^{(a-a')} \\ &\geq 0,\end{aligned} \tag{26}$$

which implies that, for any integer $l \geq 0$,

$$\langle \nu \rangle^{2l} f^{(a)} \left(\langle \nu \rangle^{2l}\right)^{(a')} f^{(a-a')} \geq 0. \tag{27}$$

Therefore,

$$\begin{aligned}\|f\|_{H^k_{2l}}^2 &= \left(M_k\langle v\rangle^{2l} f, \langle v\rangle^{2l} f\right)_{L^2}\\ &= \left(\sum_{k'\le k}\sum_{|a|_*=k'} (-1)^{k'}\cdot C_k^{k'}\cdot\frac{k'!}{a!}\left(\langle v\rangle^{2l} f\right)^{(2a)}, \langle v\rangle^{2l} f\right)_{L^2}\\ &= \sum_{k'\le k}\sum_{|a|_*=k'} C_k^{k'}\cdot\frac{k'!}{a!}\left(\left(\langle v\rangle^{2l} f\right)^{(a)}, \left(\langle v\rangle^{2l} f\right)^{(a)}\right)_{L^2}\\ &\ge \sup_{|a|_*=k}\frac{k!}{a!}\left(\langle v\rangle^{2l} f^{(a)} + \sum_{0<a'\le a}\left(\langle v\rangle^{2l}\right)^{(a')} f^{(a-a')},\right.\\ &\qquad\left.\langle v\rangle^{2l} f^{(a)} + \sum_{0<a'\le a}\left(\langle v\rangle^{2l}\right)^{(a')} f^{(a-a')}\right)_{L^2}\\ &\ge \sup_{|a|_*=k}\frac{k!}{a!}\left(\langle v\rangle^{2l} f^{(a)}, \langle v\rangle^{2l} f^{(a)}\right)_{L^2}\\ &= \sup_{|a|_*=k}\frac{k!}{a!}\left\|f^{(a)}\right\|_{L^2_{2l}}^2.\end{aligned} \tag{28}$$

Now Theorem 2 is proved. By multiplying both sides of (1) by $M_{2k}f$, one gets

$$\begin{aligned}\frac{1}{2}\frac{d\|M_k f\|_{L^2}^2}{dt} &= \left(\frac{\partial f}{\partial t}, M_{2k}f\right)_{L^2}\\ &= \left(Q(f,f), M_{2k}f\right)_{L^2}.\end{aligned} \tag{29}$$

Consequently,

$$\begin{aligned}\left\|M_k f(t)\right\|_{L^2}^2 &= \left\|M_k f(0)\right\|_{L^2}^2 + 2\int_0^t \left(Q(f,f), M_{2k}f\right)_{L^2} d\tau\\ &= \left\|M_k f(0)\right\|_{L^2}^2 + 2\int_0^t \left(I_1 + I_2\right) d\tau,\end{aligned} \tag{30}$$

where

$$\begin{aligned}I_1 &= \left(Q(f, M_k f), M_k f\right)_{L^2};\\ I_2 &= \left(Q(f,f), M_{2k}f\right)_{L^2} - \left(Q(f, M_k f), M_k f\right)_{L^2}.\end{aligned} \tag{31}$$

The following lemma is cited to estimate I_1.

Lemma 7 (part of Theorem 3.1 in [12]). *Let $0 < \gamma < 1$ and $1/2 < s < 1$. Suppose that $\Phi(|z|) = |z|^\gamma$. Then*

$$\begin{aligned}-\left(Q(g,F), F\right)_{L^2} &\ge C_g\|F\|_{H^s_{\gamma/2}}^2 - \|g\|_{L^1_\gamma}\cdot\|F\|_{H^\eta_{\gamma/2}}^2\\ &\quad - C\left(\|g\|_{L^1_{2-\gamma}}^{2s} + \|g\|_{L^1_{2-\gamma}}\right)\|F\|_{L^2_{\gamma/2}}^2,\end{aligned} \tag{32}$$

where $C_g > 0$ is a constant that depends on g and $0 < \eta < s$ depends on γ, s.

By using this lemma with $g = f$ and $F = M_k f$, the following is obtained:

$$\begin{aligned}I_1 + C_0\|M_k f\|_{H^s_{\gamma/2}}^2 &\le C\left(\|f\|_{L^1_{2-\gamma}}^{2s} + \|f\|_{L^1_{2-\gamma}}\right)\|M_k f\|_{L^2_{\gamma/2}}^2\\ &\quad + \|f\|_{L^1_\gamma}\|M_k f\|_{H^\eta_{\gamma/2}}^2,\end{aligned} \tag{33}$$

where $0 < \eta < s$ and C_0 is a constant that depends only on f. Given that

$$\|g\|_{H^\eta}^2 \le \|g\|_{H^s}^{\eta/s}\cdot\|g\|_{L^2}^{(s-\eta)/s} \le \varepsilon\|g\|_{H^s}^2 + \varepsilon^{-\eta/(s-\eta)}\|g\|_{L^2}^2, \tag{34}$$

one obtains

$$\|f\|_{L^1_\gamma}\|M_k f\|_{H^\eta_{\gamma/2}}^2 \le \varepsilon\|M_k f\|_{H^s_{\gamma/2}}^2 + C_\varepsilon\|f\|_{L^1_\gamma}^{s/(s-\eta)}\|M_k f\|_{L^2_{\gamma/2}}^2. \tag{35}$$

One chooses $\varepsilon = C_0/2$ and applies (15) to deduce that

$$\begin{aligned}I_1 + \frac{1}{2}C_0\|M_k f\|_{H^s_{\gamma/2}}^2 &\le C\cdot\left(\|f\|_{L^1_{2-\gamma}}^{2s} + \|f\|_{L^1_{2-\gamma}} + \|f\|_{L^1_\gamma}^{s/(s-\eta)}\right)\\ &\quad\cdot\|M_k f\|_{L^2_{\gamma/2}}^2\\ &\le C\cdot\|M_k f\|_{L^2_{\gamma/2}}^2.\end{aligned} \tag{36}$$

By combining the above inequality and (30), one yields the following:

$$\begin{aligned}&\left\|M_k f(t)\right\|_{L^2}^2 + C_0\int_0^t \|M_k f\|_{H^s_{\gamma/2}}^2 d\tau\\ &\quad\le \left\|M_k f(0)\right\|_{L^2}^2 + C\int_0^t \|M_k f\|_{L^2_{\gamma/2}}^2 d\tau + 2\int_0^t I_2 d\tau.\end{aligned} \tag{37}$$

Next it is planned to give an estimation of I_2. By using the conclusion in page 146 of [9] (see also page 1177 of [10]), one has

$$\begin{aligned}\left(Q(f,f)\right)^{(a)} &= Q\left(f, f^{(a)}\right) + Q\left(f^{(a)}, f\right)\\ &\quad + \sum_{0<a'<a} C_a^{a'}\cdot Q\left(f^{(a')}, f^{(a-a')}\right).\end{aligned} \tag{38}$$

Thus,

$$\begin{aligned}I_2 &= \left(Q(f,f), M_{2k}f\right)_{L^2} - \left(Q(f, M_k f), M_k f\right)_{L^2}\\ &= \left(M_k Q(f,f), M_k f\right)_{L^2} - \left(Q(f, M_k f), M_k f\right)_{L^2}\end{aligned}$$

$$= \left(\sum_{k' \le k} \sum_{|a|_* = k'} (-1)^{k'} C_k^{k'} \cdot \frac{k'!}{a!} (Q(f,f))^{(2a)}, M_k f \right)_{L^2} - (Q(f, M_k f), M_k f)_{L^2}$$
$$= \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} (-1)^{k'} \cdot C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \left(Q\left(f^{(a')}, f^{(2a-a')}\right), M_k f \right)_{L^2} + (Q(M_k f, f), M_k f)_{L^2}$$
$$= I_{21} + I_{22}. \tag{39}$$

One refers to the estimation from Proposition 3.6 in [13].

Lemma 8. *Suppose that* $\gamma + 2s > 0$ *and* $0 < s < 1$. *Then, for any* $\sigma \in [2s-1, 2s]$ *and* $p \in [0, \gamma + 2s]$,

$$\left| (Q(f,g), h)_{L^2} \right| \le C \|f\|_{L^1_{\gamma+2s}} \|g\|_{H^\sigma_{\gamma+2s-p}} \|h\|_{H^{2s-\sigma}_p}. \tag{40}$$

By using this lemma with $\sigma = 2s$ and $p = \gamma + 2s$, one gets

$$I_{22} = (Q(M_k f, f), M_k f)_{L^2} \le C \|M_k f\|_{L^1_{\gamma+2s}} \cdot \|f\|_{H^{2s}} \cdot \|M_k f\|_{L^2_{\gamma+2s}} \le C \|M_k f\|^2_{L^2_{2l}}, \tag{41}$$

where $2l > \gamma + 2s + n/2$. The final inequality is used in hypothesis (14) and the fact that $L^2_m(\mathbb{R}^n) \subseteq L^1_{\gamma+2s}(\mathbb{R}^n)$ if $m > \gamma + 2s + n/2$. Write

$$|||f(t)|||_k = \sup_{|a|_* \le k} \frac{\|f(t)\|_{H^{|a|_*}} \cdot \rho^{|a|_*}}{\{(|a|_* - r)!\}^\nu}, \tag{42}$$

where $0 < \rho < 1$, $\nu > 1$, $r > 1 + (2\nu/(\nu-1))$. Combining (28) and Lemma 8 with $p = \gamma + 2s$, the following is obtained:

$$I_{21} = \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} (-1)^{k'} \cdot C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \left(Q\left(f^{(a')}, f^{(2a-a')}\right), M_k f \right)_{L^2}$$
$$\le C \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \left\| f^{(a')} \right\|_{L^1_{\gamma+2s}} \left\| f^{(2a-a')} \right\|_{H^\sigma} \times \|M_k f\|_{H^{2s-\sigma}_{\gamma+2s}}$$
$$\le C \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \left\| f^{(a')} \right\|_{L^2_{2l}} \left\| f^{(2a-a')} \right\|_{H^\sigma} \times \|M_k f\|_{H^{2s-\sigma}_{\gamma+2s}}$$
$$\le C \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \cdot \frac{a'!}{|a'|_*!} \cdot \frac{(2a-a')!}{|2a-a'|_*!} \cdot \|f\|_{H^{|a'|_*}_{2l}} \cdot \|f\|_{H^{|2a-a'|_*+\sigma}} \cdot \|f\|_{H^{2k+2s-\sigma}_{\gamma+2s}}, \tag{43}$$

where $s < \sigma < 1$, $2l > \gamma + 2s + n/2$. By choosing $s' \in (2s - \sigma, s)$ and applying Lemmas 5 and 6 and the fact that

$$\sup_{|a|_* = k'} \frac{C_{2a}^a}{C_{2k'}^{k'}} \le 1, \qquad \sup_{|a|_* = k'} \frac{a!}{k'!} \le C, \tag{44}$$

one has

$$I_{21} \le C \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} \left(C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \cdot \frac{a'!}{|a'|_*!} \cdot \frac{(2a-a')!}{|2a-a'|_*!} \cdot \|f\|_{H^{|a'|_*+1}} \cdot \|f\|_{H^{|2a-a'|_*+1}} \cdot \|f\|_{H^{s'+2k}} \right)$$
$$\le C\{(2k-r)!\}^\nu \times \sum_{k' \le k} \sum_{|a|_* = k'} \sum_{0<a'<2a} \left(C_k^{k'} \cdot \frac{k'!}{a!} \cdot C_{2a}^{a'} \cdot \frac{a'!}{|a'|_*!} \cdot \frac{(2a-a')!}{|2a-a'|_*!} \cdot \left(\left(\{(|a'|_* + 1 - r)!\}^\nu \cdot \{(|2a-a'|_* + 1 - r)!\}^\nu \right) \right.\right.$$

$$
\begin{aligned}
&\times\left(\left\{\left(2k'-r\right)!\right\}^{\nu}\right)^{-1}\Big) \\
&\cdot\frac{\left\{\left(2k'-r\right)!\right\}^{\nu}}{\{(2k-r)!\}^{\nu}}\cdot |||f|||_{2k}^{2} \\
&\cdot\|f\|_{H^{s'+2k}}\cdot\rho^{-2k'-2}\Big) \\
&\leq C'\cdot\rho^{-2k}\cdot\{(2k-r)!\}^{\nu}\cdot|||f|||_{2k}^{2}\cdot\|f\|_{H^{s'+2k}} \\
&\cdot\sum_{k'\leq k}\sum_{|a|_*=k'}\sum_{0<a'<2a}\left(C_k^{k'}\cdot C_{2k'}^{|a'|_*}\cdot\frac{a!}{k'!}\cdot\frac{C_{2a}^{a}}{C_{2k'}^{k'}}\cdot\frac{\left\{\left(2k'-r\right)!\right\}^{\nu}}{\{(2k-r)!\}^{\nu}}\right. \\
&\cdot\Big(\Big(\left\{\left(\left|a'\right|_*+1-r\right)!\right\}^{\nu} \\
&\cdot\left\{\left(2k'-\left|a'\right|_*+1-r\right)!\right\}^{\nu}\Big) \\
&\times\left.\left(\left\{\left(2k'-r\right)!\right\}^{\nu}\right)^{-1}\Big)\right) \\
&\leq C'\cdot\rho^{-2k}\cdot\{(2k-r)!\}^{\nu}\cdot|||f|||_{2k}^{2}\cdot\|f\|_{H^{s'+2k}} \\
&\cdot\sum_{k'\leq k}\left(C_k^{k'}\frac{\left\{\left(2k'-r\right)!\right\}^{\nu}}{\{(2k-r)!\}^{\nu}}\sum_{|a|_*=k'}\frac{a!}{k'!}\sum_{0<|a'|_*<2k'}C_{2k'}^{|a'|_*}\right. \\
&\left.\cdot\frac{\left\{\left(\left|a'\right|_*+1-r\right)!\right\}^{\nu}\cdot\left\{\left(2k'-\left|a'\right|_*+1-r\right)!\right\}^{\nu}}{\left\{\left(2k'-r\right)!\right\}^{\nu}}\right) \\
&\leq C''\rho^{-2k}\cdot\{(2k-r)!\}^{\nu}\cdot|||f|||_{2k}^{2}\cdot\|f\|_{H^{s'+2k}} \\
&\leq C\rho^{-4k}\cdot\{(2k-r)!\}^{2\nu}\cdot|||f|||_{2k}^{4}+\|f\|_{H^{s'+2k}}^{2}.
\end{aligned}
\tag{45}
$$

Combining the above expression with (41) yields

$$
\begin{aligned}
I_2 = I_{21}+I_{22} &\leq C\rho^{-4k}\cdot\{(2k-r)!\}^{2\nu}\cdot|||f|||_{2k}^{4} \\
&+\|f\|_{H^{s'+2k}}^{2}+C\left\|M_k f\right\|_{L_{2l}^{2}}^{2}.
\end{aligned}
\tag{46}
$$

Plugging this estimation into (37) and applying Lemma 5 again, one obtains

$$
\begin{aligned}
&\|f(t)\|_{H^{2k}}^{2}+C_0\int_0^t\|f\|_{H^{s+2k}}^{2}d\tau \\
&\leq\left\|M_k f(t)\right\|_{L^2}^{2}+C_0\int_0^t\left\|M_k f\right\|_{H_{\gamma/2}^{s}}^{2}d\tau \\
&\leq\left\|M_k f(0)\right\|_{L^2}^{2}+C\int_0^t\left\|M_k f\right\|_{L_{\gamma/2}^{2}}^{2}d\tau \\
&+C\int_0^t\left\|M_k f\right\|_{L_{2l}^{2}}^{2}d\tau+2\int_0^t\|f\|_{H^{s'+2k}}^{2}d\tau \\
&+C\rho^{-4k}\cdot\{(2k-r)!\}^{2\nu}\cdot\int_0^t|||f|||_{2k}^{4}d\tau \\
&\leq\|f(0)\|_{H^{2k}}^{2}+C\rho^{-4k}\cdot\{(2k-r)!\}^{2\nu} \\
&\cdot\int_0^t|||f|||_{2k}^{4}d\tau+C\int_0^t\|f\|_{H^{s'+2k}}^{2}d\tau.
\end{aligned}
\tag{47}
$$

By using Remark 2 in [11],

$$
C\int_0^t\|f\|_{H^{s'+2k}}^{2}d\tau\leq C_0\int_0^t\|f\|_{H^{s+2k}}^{2}d\tau+C'\int_0^t\|f\|_{H^{2k}}^{2}d\tau.
\tag{48}
$$

Therefore,

$$
\begin{aligned}
\|f(t)\|_{H^{2k}}^{2}\leq&\|f(0)\|_{H^{2k}}^{2}+C\rho^{-4k}\cdot\{(2k-r)!\}^{2\nu} \\
&\cdot\int_0^t|||f|||_{2k}^{4}d\tau+C\int_0^t\|f\|_{H^{2k}}^{2}d\tau.
\end{aligned}
\tag{49}
$$

By multiplying both sides of (1) by $M_{2k+1}f$ and using the same analysis techniques, one can also get

$$
\begin{aligned}
\|f(t)\|_{H^{2k+1}}^{2}\leq&\|f(0)\|_{H^{2k+1}}^{2}+C\rho^{-2(2k+1)} \\
&\cdot\{(2k+1-r)!\}^{2\nu}\cdot\int_0^t|||f|||_{2k+1}^{4}d\tau \\
&+C\int_0^t\|f\|_{H^{2k+1}}^{2}d\tau.
\end{aligned}
\tag{50}
$$

Thus, for any $k\in\mathbb{N}$,

$$
\begin{aligned}
\|f(t)\|_{H^{k}}^{2}\leq&\|f(0)\|_{H^{k}}^{2}+C\rho^{-2k}\cdot\{(k-r)!\}^{2\nu} \\
&\cdot\int_0^t|||f|||_{k}^{4}d\tau+C\int_0^t\|f\|_{H^{k}}^{2}d\tau.
\end{aligned}
\tag{51}
$$

That is,

$$
\begin{aligned}
\left[\frac{\|f(t)\|_{H^k}}{\{(k-r)!\}^{\nu}}\cdot\rho^{k}\right]^{2}\leq&|||f(0)|||_{k}^{2} \\
&+C\int_0^t\left(|||f|||_{k}^{2}+|||f|||_{k}^{4}\right)d\tau,
\end{aligned}
\tag{52}
$$

which yields

$$
|||f(t)|||_{k}^{2}\leq|||f(0)|||_{k}^{2}+C\int_0^t\left(|||f|||_{k}^{2}+|||f|||_{k}^{4}\right)d\tau.
\tag{53}
$$

Thereafter, set

$$
F(t)=|||f(0)|||_{k}^{2}+C\int_0^t\left(|||f|||_{k}^{2}+|||f|||_{k}^{4}\right)d\tau.
\tag{54}
$$

A straightforward calculation provides the following:

$$
\begin{aligned}
F'(t)=C\cdot\left(|||f|||_{k}^{2}+|||f|||_{k}^{4}\right)\leq C\cdot\left(F(t)+F^{2}(t)\right), \\
t\in[0,T].
\end{aligned}
\tag{55}
$$

If the initial value $f(0) \in G^{\nu}(\mathbb{R}^n)$, one can use a small ρ to satisfy $|||f(0)|||_k^2 \cdot e^{CT} < 1/2$. By using (53),

$$|||f(t)|||_k^2 \le F(t) \le \frac{|||f(0)|||_k^2 \cdot e^{Ct}}{1 - |||f(0)|||_k^2 \cdot e^{Ct}} \le \frac{|||f(0)|||_k^2 \cdot e^{CT}}{1 - |||f(0)|||_k^2 \cdot e^{CT}} \le 1. \quad (56)$$

Therefore, for any $k \in \mathbb{N}$,

$$\|f(t)\|_{H^k} \le \rho^{-k} \cdot \{(k-r)!\}^{\nu} \le \rho^{-k} \cdot \{k!\}^{\nu}. \quad (57)$$

That is, $f(t) \in G^{\nu}(\mathbb{R}^n)$. This completes the proof of Theorem 2.

4. Proof of Theorem 3

In this section, the proof of Theorem 3 is provided. That is, for $1/2 < s < 1$ and $0 < \gamma < 1$, considering the solution $f(t,\cdot)$ of the Cauchy problem (1) that satisfies the hypotheses in Theorem 3, one shows that there is a positive number T_0 that exists such that $f(t,\cdot) \in G^{1/s}(\mathbb{R}^n)$ if $t \in (0, T_0]$. To do this, one assumes that $\nu = s^{-1}$ and $A = 2k$, $k \in \mathbb{N}$. By multiplying both sides of (29) by $(\rho t^{\nu})^{2A}$, one obtains

$$\begin{aligned} 2(Q(f,f), M_{2k}f)_{L^2} \cdot (\rho t^{\nu})^{2A} &= (\rho t^{\nu})^{2A} \cdot \frac{d\|M_k f\|_{L^2}^2}{dt} \\ &= \frac{d}{dt}\left[(\rho t^{\nu})^{2A} \cdot \|M_k f\|_{L^2}^2\right] - \rho^{1/\nu} \\ &\quad \cdot (2\nu A) \cdot (\rho t^{\nu})^{2A-(1/\nu)} \cdot \|M_k f\|_{L^2}^2. \end{aligned} \quad (58)$$

By integrating the above equation from zero to t, the following expression is obtained:

$$\begin{aligned} (\rho t^{\nu})^{2A} \cdot \|M_k f(t)\|_{L^2}^2 - (\rho t^{\nu})^{2A} \cdot \|M_k f\|_{L^2|t=0}^2 &= 2\int_0^t (Q(f,f), M_{2k}f)_{L^2} \cdot (\rho \tau^{\nu})^{2A} d\tau \\ &\quad + \int_0^t \rho^{1/\nu}(2\nu A) \cdot (\rho\tau^{\nu})^{2A-(1/\nu)} \|M_k f\|_{L^2}^2 d\tau. \end{aligned} \quad (59)$$

By writing

$$[[[f(t)]]]_k = \sup_{|a|_* \le k} \frac{\|f(t)\|_{H^{|a|_*}} \cdot (\rho t^{\nu})^{|a|_*}}{\{(|a|_* - r)!\}^{\nu}}, \quad (60)$$

one can get

$$\begin{aligned} &2\int_0^t (Q(f,f), M_{2k}f)_{L^2} \cdot (\rho\tau^{\nu})^{2A} d\tau \\ &\quad + C_0 \int_0^t \|M_k f\|_{H^s_{\gamma/2}}^2 \cdot (\rho\tau^{\nu})^{2A} d\tau \\ &\le C\left[\int_0^t \|f\|_{H^{s'+2k}}^2 \cdot (\rho\tau^{\nu})^{2A} d\tau \right. \\ &\quad \left. + \int_0^t \{(A-r)!\}^{2\nu} \cdot [[[f]]]_A^4 d\tau\right], \end{aligned} \quad (61)$$

where $0 < 2s-1 < 2s-\sigma < s' < s < 1$. Considering that the analytical method is quite similar to the one in Section 3, the proof of the above inequality is omitted. Therefore,

$$\begin{aligned} &(\rho t^{\nu})^{2A} \|f(t)\|_{H^A}^2 + C_0 \int_0^t \|f\|_{H^{A+s}}^2 \cdot (\rho\tau^{\nu})^{2A} d\tau \\ &\le (\rho t^{\nu})^{2A} \|M_k f(t)\|_{L^2}^2 \\ &\quad + C_0 \int_0^t \|M_k f\|_{H^s_{\gamma/2}}^2 \cdot (\rho\tau^{\nu})^{2A} d\tau \\ &\le (\rho t^{\nu})^{2A} \|M_k f\|_{L^2|t=0}^2 \\ &\quad + C\int_0^t \|f\|_{H^{A+s'}}^2 \cdot (\rho\tau^{\nu})^{2A} d\tau \\ &\quad + C\int_0^t \{(A-r)!\}^{2\nu} \cdot [[[f]]]_A^4 d\tau \\ &\quad + \int_0^t \rho^{1/\nu}(2\nu A) \cdot (\rho\tau^{\nu})^{2A-(1/\nu)} \|M_k f\|_{L^2}^2 d\tau. \end{aligned} \quad (62)$$

By using the conclusion in page 157 of [9],

$$\rho^{1/\nu} \cdot A \cdot \left(\rho^{1/\nu}\tau\right)^{-1} \ll \frac{1}{2}(\rho\tau^{\nu})^{-2} |(A-r)|^{2\nu} \cdot \langle\xi\rangle^{2s-2} + \langle\xi\rangle^{2s}, \quad (63)$$

provided that ρ is sufficiently small. Thus,

$$\begin{aligned} &\rho^{1/\nu} \cdot A \cdot (\rho\tau^{\nu})^{2A-(1/\nu)} \langle\xi\rangle^{2A} \\ &= \rho^{1/\nu} \cdot A \cdot \left(\rho^{1/\nu}\tau\right)^{-1} \cdot \tau^{2A\nu} \cdot \langle\xi\rangle^{2A} \cdot \rho^{2A} \\ &\ll \frac{1}{2}(\rho\tau^{\nu})^{-2} |(A-r)|^{2\nu} \cdot \langle\xi\rangle^{2A+2s-2} \cdot \tau^{2A\nu} \cdot \rho^{2A} \\ &\quad + \langle\xi\rangle^{2A+2s} \cdot \tau^{2A\nu} \cdot \rho^{2A} \\ &= \frac{1}{2}(\rho\tau^{\nu})^{2(A-1)} \cdot |(A-r)|^{2\nu} \cdot \langle\xi\rangle^{2A+2s-2} \\ &\quad + (\rho\tau^{\nu})^{2A} \cdot \langle\xi\rangle^{2A+2s}. \end{aligned} \quad (64)$$

That is,

$$\frac{1}{\{(A-r)!\}^{2\nu}}\int_0^t \rho^{1/\nu}(2\nu A)\cdot(\rho\tau^\nu)^{2A-(1/\nu)}\|M_k f\|_{L^2}^2 d\tau \ll 2\nu\cdot\left(\frac{1}{2}\int_0^t \frac{(\rho\tau^\nu)^{2(A-1)}\cdot\|f\|_{H^{A+s-1}}^2}{\{(A-1-r)!\}^{2\nu}}d\tau + \int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^{A+s}}^2}{\{(A-r)!\}^{2\nu}}d\tau\right). \tag{65}$$

Let

$$[f]_k = \sup_{|\alpha|_*\le k}\int_0^t \frac{(\rho\tau^\nu)^{2|\alpha|_*}\cdot\|f\|_{H^{|\alpha|_*+s}}^2}{\{(|\alpha|_*-r)!\}^{2\nu}}d\tau. \tag{66}$$

By using (65), if ρ is a sufficiently small number,

$$\frac{1}{\{(A-r)!\}^{2\nu}}\int_0^t \rho^{1/\nu}(2\nu A)\cdot(\rho\tau^\nu)^{2A-(1/\nu)}\|M_k f\|_{L^2}^2 d\tau \le \frac{C_0}{2}[f]_A. \tag{67}$$

Thus, the following inequality in this case is obtained:

$$\begin{aligned}\frac{(\rho t^\nu)^{2A}\cdot\|f(t)\|_{H^A}^2}{\{(A-r)!\}^{2\nu}} + C_0\int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^{A+s}}^2}{\{(A-r)!\}^{2\nu}}d\tau &\le \frac{(\rho t^\nu)^{2A}\cdot\|M_k f\|_{L^2|t=0}^2}{\{(A-r)!\}^{2\nu}} + C\int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^{A+s'}}^2}{\{(A-r)!\}^{2\nu}}d\tau \\ &\quad + C\int_0^t [[[f]]]_A^4 d\tau + \frac{C_0}{2}[f]_A \\ &\le \|f(0)\|_{L^2}^2 + C\int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^{A+s'}}^2}{\{(A-r)!\}^{2\nu}}d\tau \\ &\quad + C\int_0^t [[[f]]]_A^4 d\tau + \frac{C_0}{2}[f]_A.\end{aligned} \tag{68}$$

Combing the above inequality with Remark 2 in [11] yields

$$\begin{aligned}\frac{(\rho\tau^\nu)^{2A}\cdot\|f(t)\|_{H^A}^2}{\{(A-r)!\}^{2\nu}} + \frac{C_0}{2}\int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^{A+s}}^2}{\{(A-r)!\}^{2\nu}}d\tau &\le \|f(0)\|_{L^2}^2 + C\int_0^t \frac{(\rho\tau^\nu)^{2A}\cdot\|f\|_{H^A}^2}{\{(A-r)!\}^{2\nu}}d\tau \\ &\quad + C\int_0^t [[[f]]]_A^4 d\tau + \frac{C_0}{2}[f]_A \\ &\le \|f(0)\|_{L^2}^2 + C\int_0^t \left([[[f]]]_A^4 + [[[f]]]_A^4\right)d\tau \\ &\quad + \frac{C_0}{2}[f]_A.\end{aligned} \tag{69}$$

By using the same approach, one can prove that the above inequality in the case $A = 2k+1$, $k \in \mathbb{N}$. By taking the supremum in each term on the left hand side of this inequality, one obtains the following for any $k \in \mathbb{N}$:

$$[[[f(t)]]]_k^2 \le \|f(0)\|_{L^2}^2 + C\int_0^t \left([[[f]]]_k^2 + [[[f]]]_k^4\right)d\tau. \tag{70}$$

Choosing a suitable number $T_0 \in (0,T)$ satisfies the following:

$$\|f(0)\|_{L^2}^2 \cdot e^{CT_0} < \frac{1}{2}. \tag{71}$$

Then, for any $0 < t < T_0$,

$$[[[f(t)]]]_k^2 \le \frac{\|f(0)\|_{L^2}^2\cdot e^{CT_0}}{1-\|f(0)\|_{L^2}^2\cdot e^{CT_0}} \le 1, \tag{72}$$

which provides the Gevrey smoothing effect in $(0,T_0)$. This completes the proof of Theorem 3.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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