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*Research Article*

# The Existence of Positive Solutions for a Fourth-Order Difference Equation with Sum Form Boundary Conditions

**Yanping Guo,[1] Xuefei Lv,[2] Yude Ji,[2] and Yongchun Liang[1]**

[1] *School of Electrical Engineering, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, China*
[2] *College of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, China*

Correspondence should be addressed to Yongchun Liang; lycocean@163.com

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We consider the fourth-order difference equation: $\Delta(z(k+1)\Delta^3 u(k-1)) = w(k)f(k,u(k))$, $k \in \{1,2,\ldots,n-1\}$ subject to the boundary conditions: $u(0) = u(n+2) = \sum_{i=1}^{n+1} g(i)u(i)$, $a\Delta^2 u(0) - bz(2)\Delta^3 u(0) = \sum_{i=3}^{n+1} h(i)\Delta^2 u(i-2)$, $a\Delta^2 u(n) - bz(n+1)\Delta^3 u(n-1) = \sum_{i=3}^{n+1} h(i)\Delta^2 u(i-2)$, where $a, b > 0$ and $\Delta u(k) = u(k+1) - u(k)$ for $k \in \{0,1,\ldots,n-1\}$, $f : \{0,1,\ldots,n\} \times [0,+\infty) \to [0,+\infty)$ is continuous. $h(i)$ is nonnegative $i \in \{2,3,\ldots,n+2\}$; $g(i)$ is nonnegative for $i \in \{0,1,\ldots,n\}$. Using fixed point theorem of cone expansion and compression of norm type and Hölder's inequality, various existence, multiplicity, and nonexistence results of positive solutions for above problem are derived, which extends and improves some known recent results.

## 1. Introduction

Boundary value problems (BVPs) for ordinary differential equations arise in different areas of applied mathematics and so on. The existence of solutions for second order and higher order nonlocal boundary value problems has been studied by several authors; for example, see [1–11] and the references therein. Many authors have also discussed the existence of positive solutions for higher order difference equation BVPs [12, 13], by using fixed point theorem of cone expansion and compression of norm type, sufficient conditions for the existence of positive solutions for fourth-order and third-order difference equation BVPs are established, respectively. Recently, there has been much attention on the existence of positive solutions for the fourth-order differential equations with integral boundary conditions [14–17]. In [17], Zhang and Ge considered the differential equation BVP:

$$\begin{gathered}
\left(u(t)x'''(t)\right)' = w(t)f(t,x(t)), \quad 0 < t < 1, \\
x(0) = x(1) = \int_0^1 g(s)x(s)\,ds, \\
ax''(0) - b\lim_{t\to 0^+} u(t)x'''(0) = \int_0^1 h(s)x''(s)\,ds, \\
ax''(1) + b\lim_{t\to 1^-} u(t)x'''(1) = \int_0^1 h(s)x''(s)\,ds,
\end{gathered} \tag{1}$$

where $a, b > 0$, $u \in C^1([0,1] \to [0,+\infty))$ is symmetric on $[0,1]$, $w \in L^p[0,1]$ for some $1 \le p \le +\infty$, and it is symmetric on the interval $[0,1]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, and $f(1-t,x) = f(t,x)$ for all $(t,x) \in [0,1] \times [0,+\infty)$, and $g, h \in L^1[0,1]$ are nonnegative, symmetric on $[0,1]$. The authors made use of fixed point theorem of cone expansion and compression of norm type and Hölder inequality to prove the existence of positive solutions for the above problem.

Motivated by the above works, we intend to study the existence and nonexistence of positive solutions of the following fourth-order difference BVP with sum form boundary conditions:

$$\Delta\left(z(k+1)\Delta^3 u(k-1)\right) = w(k)f(k,u(k)), \quad k \in \{1,2,\ldots,n-1\},$$

$$u(0) = u(n+2) = \sum_{i=1}^{n+1} g(i) u(i),$$

$$a\Delta^2 u(0) - bz(2)\Delta^3 u(0) = \sum_{i=3}^{n+1} h(i)\Delta^2 u(i-2),$$

$$a\Delta^2 u(n) + bz(n+1)\Delta^3 u(n-1) = \sum_{i=3}^{n+1} h(i)\Delta^2 u(i-2). \tag{2}$$

Throughout this paper, we make the following assumptions:

$(A_0)$ $z$ is symmetric on $\{2,3,\dots,n+1\}$ and $0 < \sum_{k=2}^{n+1} z(k) < +\infty$;

$(A_1)$ there is a $m > 0$ such that $w(k) \geq m/(n-1)$ for $1 \leq p \leq +\infty$;

$(A_2)$ $f: \{0,1,\dots,n\} \times [0,+\infty) \to [0,+\infty)$ is continuous;

$(A_3)$ $h(i)$ is nonnegative on $\{2,3,\dots,n+2\}$ and $0 \leq \nu < a$, $g(i)$ is nonnegative on $\{0,1,\dots,n+2\}$ and $0 \leq \mu < 1$, where $\nu = \sum_{i=3}^{n+1} h(i)$, $\mu = \sum_{i=1}^{n+1} g(i)$;

$(A_4)$ $a, b > 0$, $Q = 2ab + a^2 \sum_{j=2}^{n+1}(1/z(j)) > 0$.

In order to establish the existence of positive solutions of the problem (2), we need the following definitions, theorem, and lemma.

*Definition 1.* A function $x(t)$ is said to be a solution of problem (2) if $x(t)$ satisfying BVP (2).

*Definition 2* (see [18]). Let $E$ be a real Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$ and

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by $x \leq y$ if and only if $y - x \in P$.

**Theorem 3** (see [18]). *Let $P$ be a cone in a real Banach space $E$. Assume that $\Omega_1, \Omega_2$ are bounded open sets in $E$ with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow P \tag{3}$$

*is completely continuous such that either*

(i) $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

*or*

(ii) $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

*then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

**Lemma 4** (Hölder). *Suppose that $u = \{u_1, u_2, \dots, u_n\}$ is a real-valued column; let*

$$\|u\|_p = \begin{cases} \left(\sum_{k=1}^{n} |u_k|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_{k\in\{1,2,\dots,n\}} |u_k|, & p = \infty, \end{cases} \tag{4}$$

*$p$, $q$ satisfy the condition $1/p + 1/q = 1$ which are called conjugate exponent, and $q = \infty$ for $p = 1$. If $1 \leq p \leq \infty$, then*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q, \tag{5}$$

*which can be recorded as*

$$\sum_{k=1}^{n} |u_k v_k| \leq \begin{cases} \left(\sum_{k=1}^{n} |u_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |v_k|^q\right)^{1/q}, & 1 < p < \infty, \\ \left(\sum_{k=1}^{n} |u_k|\right)\left(\sup_{k\in\{1,2,\dots,n\}} |v_k|\right), & p = 1, \\ \left(\sup_{k\in\{1,2,\dots,n\}} |u_k|\right)\left(\sum_{k=1}^{n} |v_k|\right), & p = \infty. \end{cases} \tag{6}$$

## 2. Preliminaries

Let $J = \{0,1,\dots,n+2\}$; $E = \{u(k) : \{0,1,\dots,n+2\} \to \mathbb{R}\}$ is a real Banach space with the norm $\|\cdot\|$ defined by

$$\|u\| = \max_{k\in J} |u(k)|. \tag{7}$$

Let $K$ be a cone of $E$, and

$$K_r = \{u \in K : \|u\| \leq r\},$$

$$\partial K_r = \{u \in K : \|u\| = r\}, \tag{8}$$

$$\overline{K}_{r,R} = \{u \in K : r \leq \|u\| \leq R\},$$

where $0 < r < R$.

In our main results, we will use the following lemmas and properties.

**Lemma 5.** *Suppose that $(A_0)$ and $(A_1)$ hold and $\nu \neq 1$; then, for all $y \in E$, the BVP*

$$-\Delta(z(k+1)\Delta u(k+1)) = y(k+2), \quad k \in \{1,2,\dots,n+1\},$$

$$au(2) - bz(2)\Delta u(2) = \sum_{i=3}^{n+1} h(i) u(i),$$

$$au(n+2) + bz(n+1)\Delta u(n+1) = \sum_{i=3}^{n+1} h(i) u(i) \tag{9}$$

*has unique solution $u$ given by*

$$u(k) = \sum_{i=3}^{n+1} H(k,i) y(i), \tag{10}$$

*where*

$$H(k,i)=G(k,i)+\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}G(i,\tau)h(\tau),$$

$$G(k,i)=\frac{1}{Q}\begin{cases}\left(b+a\sum_{j=k}^{n+1}\frac{1}{z(j)}\right)\left(b+a\sum_{j=2}^{i-1}\frac{1}{z(j)}\right), & 2\le i<k,\\ \left(b+a\sum_{j=i}^{n+1}\frac{1}{z(j)}\right)\left(b+a\sum_{j=2}^{k-1}\frac{1}{z(j)}\right), & k\le i\le n+2,\end{cases}\tag{11}$$

*where* $Q=2ab+a^2\sum_{j=2}^{n+1}(1/z(j))$, $\nu=\sum_{i=3}^{n+1}h(i)$.

**Proposition 6.** *Assume that* $0\le\nu<a$; *then*

$$H(k,i)>0,\quad G(k,i)>0,\quad k,i\in\{2,3,\dots,n+2\}.\tag{12}$$

**Proposition 7.** *Assume that* $k,i\in\{2,3,\dots,n+2\}$; *then*

$$\begin{gathered}\frac{1}{Q}b^2\le G(k,i)\le G(i,i)\le\frac{1}{Q}D,\\ G(n+4-k,n+4-i)=G(k,i),\end{gathered}\tag{13}$$

*where*

$$D=\left(b+a\sum_{j=2}^{n+1}\frac{1}{z(j)}\right)^2,\quad k,i\in\{2,3,\dots,n+2\}.\tag{14}$$

**Proposition 8.** *Suppose that* $0\le\nu<a$; *then*

$$\begin{gathered}\frac{1}{Q}\nu b^2\gamma<H(k,i)\le H(i,i)\le\frac{1}{Q}\gamma Da,\\ k,i\in\{2,3,\dots,n+2\},\end{gathered}\tag{15}$$

*where*

$$\begin{gathered}D=\left(b+a\sum_{j=2}^{n+1}\frac{1}{z(j)}\right)^2,\qquad\gamma=\frac{1}{a-\nu},\\ k,i\in\{2,3,\dots,n+2\}.\end{gathered}\tag{16}$$

*Proof.* From Lemma 5 and Proposition 7, we have

$$\begin{aligned}H(k,i)&=G(k,i)+\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}G(i,\tau)h(\tau)\\&>\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}G(i,\tau)h(\tau)\\&\ge\frac{1}{a-\nu}\frac{1}{Q}b^2\sum_{\tau=3}^{n+1}h(\tau)\\&=\frac{1}{Q}\gamma b^2\nu.\end{aligned}\tag{17}$$

On the other hand, using $G(k,i)\le G(i,i)\le(1/Q)D$, we get

$$\begin{aligned}H(k,i)&=G(k,i)+\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}G(i,\tau)h(\tau)\\&\le G(i,i)+\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}G(i,\tau)h(\tau)\\&\le\frac{1}{Q}D+\frac{1}{Q}D\frac{1}{a-\nu}\sum_{\tau=3}^{n+1}h(\tau)\\&\le\frac{1}{Q}D\left(1+\frac{\nu}{a-\nu}\right)\\&=\frac{1}{Q}Da\gamma.\end{aligned}\tag{18}$$

□

**Lemma 9.** *Suppose that* $\mu\ne1$, *for all* $y\in E$; *the BVP*

$$\begin{gathered}-\Delta^2u(k-1)=y(k),\quad k\in\{1,2,\dots,n+1\},\\ u(0)=u(n+2)=\sum_{i=1}^{n+1}g(i)u(i)\end{gathered}\tag{19}$$

*has unique solution* $u$ *given by*

$$u(k)=\sum_{i=1}^{n+1}H_1(k,i)y(i),\tag{20}$$

*where*

$$\begin{gathered}H_1(k,i)=G_1(k,i)+\frac{1}{1-\mu}\sum_{\tau=1}^{n+1}G_1(\tau,i)g(\tau),\\ G_1(k,i)=\begin{cases}\frac{i}{n+2}(n+2-k), & 0\le i<k,\\ \frac{k}{n+2}(n+2-i), & k\le i<n+2.\end{cases}\end{gathered}\tag{21}$$

*Proof.* From the properties of difference operator, we can get

$$-\Delta u(k)+\Delta u(k-1)=y(k);\tag{22}$$

then we have

$$\begin{gathered}-\Delta u(1)+\Delta u(0)=y(1),\\ -\Delta u(2)+\Delta u(1)=y(2),\\ -\Delta u(3)+\Delta u(2)=y(3),\\ \vdots\\ -\Delta u(k-1)+\Delta u(k-2)=y(k-1).\end{gathered}\tag{23}$$

It can imply that

$$-\Delta u(k-1)+\Delta u(0)=\sum_{i=1}^{k-1}y(i).\tag{24}$$

Let $\Delta u(0) = A$; we have

$$-\Delta u(k-1) = A - \sum_{i=1}^{k-1} y(i). \tag{25}$$

That is

$$u(k) - u(k-1) = A - \sum_{i=1}^{k-1} y(i); \tag{26}$$

thus,

$$\begin{aligned} u(1) - u(0) &= A, \\ u(2) - u(1) &= A - \sum_{i=1}^{1} y(i), \\ u(3) - u(2) &= A - \sum_{i=1}^{2} y(i), \\ &\vdots \\ u(k) - u(k-1) &= A - \sum_{i=1}^{k-1} y(i). \end{aligned} \tag{27}$$

We can get

$$\begin{aligned} u(k) &= u(0) + kA - \sum_{j=1}^{k-1}\sum_{i=1}^{j} y(i), \\ u(n+2) &= u(0) + (n+2)A - \sum_{j=1}^{n+1}\sum_{i=1}^{j} y(i). \end{aligned} \tag{28}$$

From the boundary conditions, we have

$$A = \frac{1}{n+2}\sum_{j=1}^{n+1}\sum_{i=1}^{j} y(i); \tag{29}$$

thus,

$$\begin{aligned} u(k) = {} & \sum_{i=1}^{n+1} g(i)u(i) + \frac{k}{n+2}\sum_{j=1}^{n+1}\sum_{i=1}^{j} y(i) \\ & - \sum_{j=1}^{k-1}\sum_{i=1}^{j} y(i). \end{aligned} \tag{30}$$

Because

$$\begin{aligned} & \frac{k}{n+2}\sum_{j=1}^{n+1}\sum_{i=1}^{j} y(i) \\ & = \frac{k}{n+2} \\ & \quad \times [y(1) + (y(1) + y(2)) \\ & \quad + (y(1) + y(2) + y(3)) \\ & \quad + \cdots + (y(1) + y(2) + \cdots + y(n+1))] \\ & = \frac{k}{n+2}[(n+1)y(1) + ny(2) \\ & \quad + (n-1)y(3) + \cdots + y(n+1)] \\ & = \frac{k}{n+2}(n+1)y(1) + \frac{k}{n+2}ny(2) \\ & \quad + \cdots + \frac{k}{n+2}y(n+1), \\ & \sum_{j=1}^{k-1}\sum_{i=1}^{j} y(i) = y(1) + (y(1) + y(2)) \\ & \quad + \cdots + (y(1) + y(2) + \cdots + y(k-1)) \\ & = (k-1)y(1) + (k-2)y(2) + \cdots + y(k-1), \end{aligned} \tag{31}$$

we have

$$\begin{aligned} & \frac{k}{n+2}\sum_{j=1}^{n+1}\sum_{i=1}^{j} y(i) - \sum_{j=1}^{k-1}\sum_{i=1}^{j} y(i) \\ & = \left[\frac{k(n+1)}{n+2} - (k-1)\right] y(1) \\ & \quad + \left[\frac{kn}{n+2} - (k-2)\right] y(2) \\ & \quad + \cdots + \left[\frac{k}{n+2}(n-k+3) - 1\right] y(k-1) \\ & \quad + \left[\frac{k}{n+2}(n-k+2)\right] y(k) + \cdots + \frac{k}{n+2}y(n+1). \end{aligned} \tag{32}$$

Thus, we get

$$u(k) = \sum_{i=1}^{n+1} g(i)u(i) + \sum_{i=1}^{n+1} G_1(k,i)y(i), \tag{33}$$

where

$$G_1(k,i) = \begin{cases} \dfrac{i}{n+2}(n+2-k), & 0 \le i < k, \\ \dfrac{k}{n+2}(n+2-i), & k \le i < n+2. \end{cases} \tag{34}$$

Multiplying the above equation with $g(k)$, summing them from 1 to $n+1$, we have

$$\begin{aligned} \sum_{k=1}^{n+1} g(k)u(k) = {} & \sum_{k=1}^{n+1} g(k)\sum_{i=1}^{n+1} g(i)u(i) \\ & + \sum_{k=1}^{n+1} g(k)\sum_{i=1}^{n+1} G_1(k,i)y(i), \end{aligned} \tag{35}$$

$$\sum_{i=1}^{n+1} g(i)u(i) = \frac{1}{1-\sum_{k=1}^{n+1} g(k)}\sum_{k=1}^{n+1} g(k)\sum_{i=1}^{n+1} G_1(k,i)y(i).$$

So, we can get

$$u(k) = \sum_{i=1}^{n+1} G_1(k,i)\, y(i) + \frac{1}{1-\sum_{k=1}^{n+1} g(k)} \sum_{k=1}^{n+1} g(k) \sum_{i=1}^{n+1} G_1(k,i)\, y(i) = \sum_{i=1}^{n+1} H_1(k,i)\, y(i). \tag{36}$$

□

**Proposition 10.** *Assume that $0 \le \mu < 1$; then*

$$\begin{gathered} H_1(k,i) > 0, \qquad G_1(k,i) > 0, \\ k, i \in \{1,2,\dots,n+1\}, \\ H_1(k,i) \ge 0, \qquad G_1(k,i) \ge 0, \\ k, i \in J. \end{gathered} \tag{37}$$

**Proposition 11.** *For $k, i \in J$, one has*

$$\begin{gathered} \frac{1}{n+2} e(i)\, e(k) \le G_1(k,i) \le G_1(i,i) \\ = i\left(1 - \frac{i}{n+2}\right) = e(i) \le \frac{n+2}{4}, \\ G_1(n+2-k, n+2-i) = G_1(k,i), \end{gathered} \tag{38}$$

*where $e(i) = (i/(n+2))(n+2-i)$.*

**Proposition 12.** *If $0 \le \mu < 1$, then, for all $k, i \in J$, one has*

$$\rho e(i) \le H_1(k,i) \le H_1(i,i) \le \gamma^* \frac{i}{n+2}(n+2-i) \le \frac{n+2}{4}\gamma^*, \tag{39}$$

*where $\gamma^* = 1/(1-\mu)$, $\rho = \sum_{\tau=1}^{n+1} e(\tau) g(\tau)/(n+2)(1-\mu)$.*

We construct a cone on $E$ by

$$K = \left\{ u \in E : u \ge 0, \Delta^2 u(k) \le 0 \text{ on } \{0,1,\dots,n\}, \text{ and } \min_{k\in J} u(k) \ge \delta_* \|u\| \right\}, \tag{40}$$

where

$$\delta_* = \frac{2(n+3)\, v b^2 \rho}{3(n+2)\gamma^* a \left(b + a\sum_{j=2}^{n+1} (1/z(j))\right)^2}. \tag{41}$$

Obviously, $K$ is a closed convex cone of $E$.

Define an operator $T : E \to E$ as

$$(Tu)(k) = \sum_{\tau=1}^{n+1} H_1(k,\tau) \sum_{i=3}^{n+1} H(\tau+1, i)\, w(i-2)\, f(i-2, u(i-2)). \tag{42}$$

Let

$$\Psi(k,i) = \sum_{\tau=1}^{n+1} H_1(k,\tau)\, H(\tau+1, i). \tag{43}$$

Then we can obtain the following properties.

**Proposition 13.** *If $(A_3)$ and $(A_4)$ hold, then*

$$\begin{gathered} 0 < \Psi(k,i) \le \frac{(n+2)(n+1)}{4(1-\mu)} H(i,i), \\ k \in J, \quad i \in \{2,3,\dots,n+2\}. \end{gathered} \tag{44}$$

**Proposition 14.** *If $(A_3)$ and $(A_4)$ hold, then, for all $k \in J$, $i \in \{2,3,\dots,n+2\}$, one has*

$$\frac{(n+1)(n+3)}{6Q} v b^2 \gamma \rho < \Psi(k,i) \le \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a. \tag{45}$$

**Lemma 15.** *Suppose that $(A_0)$–$(A_4)$ hold; if $u \in E$ is a solution of the equation*

$$u(k) = (Tu)(k) = \sum_{i=3}^{n+1} \Psi(k,i)\, w(i-2)\, f(i-2, u(i-2)), \tag{46}$$

*then $u$ is a solution of the BVP (2).*

**Lemma 16.** *Assume that $(A_0)$–$(A_4)$ hold; then $T(K) \subset K$ and $T : K \to K$ is completely continuous.*

*Proof.* From above works, for all $u \in K$, we have

$$\Delta^2 (Tu)(k-1) = -\sum_{i=3}^{n+1} H(k+1, i)\, w(i-2)\, f(i-2, u(i-1)) \le 0, \quad k \in \{1,2,\dots,n+1\}. \tag{47}$$

Because

$$(Tu)(0) = (Tu)(n+2) = \sum_{\tau=1}^{n+1} H_1(0,\tau) \times \sum_{i=3}^{n+1} H(\tau+1, i)\, w(i-2)\, f(i-2, u(i-2))$$

$$= \sum_{\tau=1}^{n+1} H_1 (n+2, \tau)$$
$$\times \sum_{i=3}^{n+1} H(\tau+1, i) w(i-2) f(i-2, u(i-2))$$
$$= \sum_{\tau=1}^{n+1} \frac{1}{1-\mu} \sum_{j=1}^{n+1} G_1 (j, \tau) G(j)$$
$$\times \sum_{i=3}^{n+1} H(\tau+1, i) w(i-2) f(i-2, u(i-2))$$
$$\geq 0. \tag{48}$$

That is to say $(Tu)(k) \geq 0$ for $k \in J$.

On the other hand, for $k \in J$, we have

$$(Tu)(k) = \sum_{i=3}^{n+1} \Psi(k, i) w(i-2) f(i-2, u(i-2))$$
$$\leq \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left( b + a \sum_{j=2}^{n+1} \frac{1}{z(j)} \right)^2 \tag{49}$$
$$\times a \sum_{i=3}^{n+1} w(i-2) f(i-2, u(i-2)).$$

Similarly, we can get

$$(Tu)(k) = \sum_{i=3}^{n+1} \Psi(k, i) w(i-2) f(i-2, u(i-2))$$
$$\geq \frac{(n+2)(n+1)}{6Q} v b^2 \gamma \rho \sum_{i=3}^{n+1} w(i-2) f(i-2, u(i-2))$$
$$= \delta_* \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left( b + a \sum_{j=2}^{n+1} \frac{1}{z(j)} \right)^2$$
$$\times a \sum_{i=3}^{n+1} w(i-2) f(i-2, u(i-2))$$
$$\geq \delta_* \|Tu\|, \quad k \in J. \tag{50}$$

So, $Tu \in K$ and $T(K) \subset K$. It is easy to see that $T : K \to K$ is completely continuous. □

## 3. The Existence of One Positive Solution

In this part, we apply Theorem 3 and Lemma 4 to prove the existence of one positive solution for BVP (2). We need consider the following cases for: $p > 1$, $p = 1$, and $p = \infty$.

Let

$$f^\beta = \lim_{n \to \beta} \sup \max_{k \in J} \frac{f(k, u)}{u}, \qquad f_\beta = \lim_{u \to \beta} \inf \min_{k \in J} \frac{f(k, u)}{u}, \tag{51}$$

where $\beta$ denotes 0 or $\infty$, and

$$B = \max \left\{ \frac{(n+2)(n+1)}{4(1-\mu)} \left( \sum_{i=3}^{n+1} |H|^p \right)^{1/p} \left( \sum_{i=3}^{n+1} |w|^q \right)^{1/q}, \right.$$
$$\frac{(n+2)(n+1)}{4Q} \gamma \gamma^* a$$
$$\left. \times \left( b + a \sum_{j=2}^{n+1} \frac{1}{z(j)} \right) \left( \sum_{i=3}^{n+1} |w| \right) \right\},$$
$$\eta = \left[ (n+2)(n+1) \gamma \gamma^* \left( b + a \sum_{j=2}^{n+1} \frac{1}{z(j)} \right)^2 a \right]^{-1}. \tag{52}$$

*Remark 17.* If we only consider the case $p = \infty$, then we can take

$$B = \frac{(n+2)(n+1)}{4Q} \gamma \left( b + a \sum_{j=2}^{n+1} \frac{1}{z(j)} \right)^2 \left( \sum_{i=3}^{n+1} |w| \right). \tag{53}$$

Firstly, the following theorem deals with the case $p > 1$.

**Theorem 18.** *Suppose that $(A_0)$–$(A_4)$ hold. If there exist two constants r, R with $0 < r < \delta_* R$ such that*

- $(C_1)$ *$f(k, u) \leq B^{-1} r$ for all $(k, u) \in J \times [\delta_* r, r]$ and $f(k, u) \geq (4\eta / m\delta_*) QR$ for all $(k, u) \in J \times [\delta_* R, R]$;*

  *or*
- $(C_2)$ *$f(k, u) \geq (4\eta / m\delta_*) QR$ for all $(k, u) \in J \times [\delta_* r, r]$ and $f(k, u) \leq B^{-1} R$ for all $(k, u) \in J \times [\delta_* R, R]$, then BVP (2) has at least one positive solution.*

*Proof.* We only consider condition $(C_1)$. For $u \in K$, from the definition of $K$ we obtain that

$$\min_{k \in J} u(k) \geq \delta_* \|u\|. \tag{54}$$

So, for $u \in \partial K_r$, we have $u(k) \in [\delta_* r, r]$, which implies that $f(k, u(k)) \leq B^{-1} \gamma$. Thus, for $k \in J$, from $(C_1)$ we get

$$(Tu)(k) = \sum_{i=3}^{n+1} \Psi(k, i) w(i-2) f(i-2, u(i-2))$$
$$\leq \frac{(n+2)(n+1)}{4(1-\mu)} B^{-1} \gamma \sum_{i=3}^{n+1} H(i, i) w(i-2)$$
$$\leq \frac{(n+2)(n+1)}{4(1-\mu)} \left( \sum_{i=3}^{n+1} |H|^p \right)^{1/p} \left( \sum_{i=3}^{n+1} |w|^q \right)^{1/q} B^{-1} \gamma$$
$$= r = \|u\|; \tag{55}$$

that is, $u \in \partial K_r$ implies that

$$\|Tu\| \leq \|u\|. \tag{56}$$

On the other hand, for $u \in \partial K_R$, we have $u(k) \in [\delta_* R, R]$, which implies that $f(k, u(k)) \geq (4\eta/m\delta_*)QR$; therefore for $k \in J$, from $(C_1)$ we have

$$\begin{aligned}(Tu)(k) &= \sum_{i=3}^{n+1} \Psi(k,i)\, w(i-2)\, f(i-2, u(i-2)) \\ &\geq \frac{(n+1)(n+3)}{6Q} v b^2 \gamma\rho \frac{4\eta}{m\delta_*} QR \sum_{i=3}^{n+1} w(i-2) \\ &\geq \delta_* \frac{(n+1)(n+2)}{4Q} \gamma^* \gamma \\ &\quad \times \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a \frac{4\eta}{m\delta_*} QRm \\ &= R = \|u\|;\end{aligned} \tag{57}$$

that is, $u \in \partial K_R$ implies that

$$\|Tu\| \geq \|u\|. \tag{58}$$

From the above works, we apply (i) of Theorem 3 to yield that $T$ has a fixed point $u^* \in \overline{K}_{r,R}$, $r \leq \|u^*\| \leq R$ and $u^*(k) \geq \delta_* \|u^*\| > 0$, $k \in J$. Thus, it follows that BVP (2) has at least one positive solution $u^*$. □

The following theorem deals with the case $p = 1$.

**Theorem 19.** *Suppose that $(A_0)$–$(A_4)$ hold and $(C_1)$ or $(C_2)$ holds. Then BVP (2) has at least one positive solution.*

*Proof.* Let $(\sum_{i=3}^{n+1} |H(i,i)|)(\sup_{i\in\{3,4,\ldots,n+1\}} |w(i-2)|)$ replace $(\sum_{i=3}^{n+1} |H(i,i)|^p)^{1/p} (\sum_{i=3}^{n+1} |w(i-2)|^q)^{1/q}$ and repeat the argument of Theorem 18. □

Finally we consider the case $p = \infty$.

**Theorem 20.** *Suppose that $(A_0)$–$(A_4)$ hold and $(C_1)$ or $(C_2)$ holds. Then BVP (2) has at least one positive solution.*

*Proof.* Similar to the proof of Theorem 18, for $x \in \partial K_r$, we have

$$\begin{aligned}(Tu)(k) & \\ &= \sum_{i=3}^{n+1} \Psi(k,i)\, w(i-2)\, f(i-2, u(i-2)) \\ &\leq \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a B^{-1} r \sum_{i=3}^{n+1} w(i-2) \\ &\leq \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a B^{-1} r \sum_{i=3}^{n+1} |w| \\ &\leq r = \|u\|.\end{aligned} \tag{59}$$

So, for $x \in \partial K_r$, we have $\|Tu\| \leq \|u\|$. And from the proof of Theorem 18, $\|Tu\| \geq \|u\|$, $k \in J$ for $u \in \partial K_R$. Thus Theorem 20 is proved. □

**Theorem 21.** *Assume that $(A_0)$–$(A_4)$ hold. If one of the following conditions is satisfied*

$(C_3)$ *$f_0 > 4\eta Q/m\delta_*$ and $f^\infty < 1/B$ (particularly, $f_0 = \infty$ and $f^\infty = 0$);*

$(C_4)$ *$f^0 < 1/B$ and $f_\infty > 4\eta Q/m\delta_*^2$ (particularly, $f^0 = 0$ and $f_\infty = \infty$),*

*then BVP (2) has at least one positive solution.*

*Proof.* We only consider the case $(C_3)$. The proof of case $(C_4)$ is similar to case $(C_3)$. Considering $f_0 > 4\eta Q/m\delta_*^2$, there exists $r_1 > 0$ such that $f(k,u) \geq (f_0 - \varepsilon_1)u$ for $k \in J$, $u \in [0, r_1]$, where $\varepsilon_1 > 0$ satisfies $f_0 - \varepsilon_1 \geq 4\eta Q/m\delta_*^2$. Then, for $k \in J$, $u \in \partial K_{r_1}$, from $(C_3)$ we have

$$\begin{aligned}(Tu)(k) & \\ &= \sum_{i=3}^{n+1} \Psi(k,i)\, w(i-2)\, f(i-2, u(i-2)) \\ &\geq \frac{(n+1)(n+3)}{6Q} v b^2 \gamma\rho \sum_{i=3}^{n+1} w(i-2)\, f(i-2, u(i-2)) \\ &\geq \delta_*^2 \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a(f_0 - \varepsilon_1) \\ &\quad \times \sum_{i=3}^{n+1} w(i-2)\, u(i-2) \\ &\geq \delta_*^2 \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a(f_0 - \varepsilon_1) m \|u\| \\ &\geq \delta_*^2 \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \left(b + a\sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a \frac{4\eta Q}{m\delta_*^2} m \|u\| \\ &\geq \|u\|;\end{aligned} \tag{60}$$

that is, $x \in \partial K_{r_1}$ implies that $\|Tu\| \geq \|u\|$.

Next, considering $f^\infty < 1/B$, then there exists $\overline{R}_1 > 0$ such that

$$f(k,u) \leq (f^\infty + \varepsilon_2)u, \quad k \in J,\ u \in (\overline{R}_1, \infty), \tag{61}$$

where $\varepsilon_2 > 0$ satisfies $B(f^\infty + \varepsilon_2) < 1$; assume that

$$M = \max_{0 \leq u \leq \overline{R}_1, k \in J} f(k,u); \tag{62}$$

then

$$f(k,u) \leq M + (f^\infty + \varepsilon_2) u\varepsilon. \tag{63}$$

Choosing $R_1 > \max\{r_1, \overline{R}_1, MB(1 - B(f^\infty + \varepsilon_2))^{-1}\}$.

Thus, for $u \in K_{R_1}$, we obtain

$$\begin{aligned}
&\|(Tu)(k)\| \\
&= \sum_{i=3}^{n+1} \Psi(k,i) w(i-2) f(i-2,u(i-2)) \\
&\leq \frac{(n+2)(n+1)}{4(1-\mu)} \sum_{i=3}^{n+1} H(i,i) w(i-2) f(i-2,u(i-2)) \\
&\leq \frac{(n+2)(n+1)}{4(1-\mu)} \sum_{i=3}^{n+1} H(i,i) w(i-2) \left(M + \left(f^{\infty} + \varepsilon_2\right)\|u\|\right) \\
&\leq \frac{(n+2)(n+1)}{4(1-\mu)} \left(\sum_{i=3}^{n+1} |H|^p\right)^{1/p} \left(\sum_{i=3}^{n+1} |w|^q\right)^{1/q} \\
&\quad \times \left(M + \left(f^{\infty} + \varepsilon_2\right)\|u\|\right) \\
&\leq B\left(M + \left(f^{\infty} + \varepsilon_2\right)\|u\|\right) \\
&\leq BM + \left(f^{\infty} + \varepsilon_2\right)\|u\| B \\
&< R_1 - R_1\left(f^{\infty} + \varepsilon_2\right) B + \left(f^{\infty} + \varepsilon_2\right)\|u\| B \\
&\leq R_1 - \left(f^{\infty} + \varepsilon_2\right)\|u\| B + \left(f^{\infty} + \varepsilon_2\right)\|u\| B \\
&= R_1;
\end{aligned} \tag{64}$$

that is, $u \in \partial K_{R_1}$, we have $\|Tu\| < \|u\|$.

From above works and (ii) of Theorem 3 we know that $T$ has a fixed point $u^* \in \overline{K}_{r_1,R_1}$, $r_1 \leq u^* < R_1$, and $u^*(k) \geq \delta_* \|u^*\| > 0$, $k \in J$. Thus, it implies that BVP (2) has a positive solution $u^*$. □

**Theorem 22.** *Suppose that $(A_0)$–$(A_4)$ hold. If there exist two constants $r_2, R_2$ with $0 < r_2 < R_2$ such that*

$(C_5)$ *$f(k,\cdot)$ is nondecreasing on $[0,R_2]$ for all $k \in J$;*

$(C_6)$ *$f(k,\delta_* r_2) \geq (4\eta Q/m\delta_*) r_2$ and $f(k,R_2) \leq B^{-1} R_2$ for all $k \in J$,*

*then the BVP (2) has at least one positive solution.*

*Proof.* For $u \in K$, from the definition of $K$ we have $\min_{k\in J} u(t) \geq \delta_* \|u\|$. So, for $u \in \partial K_{r_2}$, we have $u(k) \geq \delta_* \|u\| = \delta_* r_2, k \in J$, from conditions $(C_5)$ and $(C_6)$, we obtain

$$\begin{aligned}
&(Tu)(k) \\
&\quad = \sum_{i=3}^{n+1} \Psi(k,i) w(i-2) f(i-2,u(i-2)) \\
&\quad \geq \sum_{i=3}^{n+1} \Psi(k,i) w(i-2) f\left(i-2,\delta_* r_2\right) \\
&\quad \geq \frac{(n+1)(n+3)}{6Q} \nu b^2 \gamma \rho \frac{4\eta Q}{m\delta_*} r_2 \sum_{i=3}^{n+1} w(i-2) \\
&\quad = \delta_* \frac{(n+2)(n+1)}{4Q} \gamma^* \gamma \\
&\quad \times \left(b + a \sum_{j=2}^{n+1} \frac{1}{z(j)}\right)^2 a \frac{4\eta Q}{m\delta_*} r_2 \sum_{i=3}^{n+1} w(i-2) \\
&\quad = r_2 = \|u\|,
\end{aligned} \tag{65}$$

that is, for $u \in \partial K_{r_2}$, we can imply that

$$\|Tu\| \geq \|u\|. \tag{66}$$

On the other hand, for $u \in \partial K_{R_2}$, we have $u(k) \leq R_2$, $k \in J$, this together with $(C_5)$ and $(C_6)$, we have

$$\begin{aligned}
&(Tu)(k) \\
&\quad = \sum_{i=3}^{n+1} \Psi(k,i) w(i-2) f(i-2,u(i-2)) \\
&\quad \leq \sum_{i=3}^{n+1} \Psi(k,i) w(i-2) f\left(i-2,R_2\right) \\
&\quad \leq \frac{(n+2)(n+1)}{4(1-\mu)} \sum_{i=3}^{n+1} H(i,i) w(i-2) f\left(i-2,R_2\right) \\
&\quad \leq \frac{(n+2)(n+1)}{4(1-\mu)} \left(\sum_{i=3}^{n+1} |H|^p\right)^{1/p} \left(\sum_{i=3}^{n+1} |w|^q\right)^{1/q} B^{-1} R_2 \\
&\quad = R_2 = \|u\|,
\end{aligned} \tag{67}$$

that is, for $u \in \partial K_{R_2}$, we can imply that

$$\|Tu\| \leq \|u\|. \tag{68}$$

From above works and Theorem 3, we prove that $T$ has a fixed point $u^* \in \overline{K}_{r_2,R_2}$, $r_2 \leq \|u^*\| \leq R_2$, and $u^*(k) \geq \delta_* \|u^*\| > 0$, $k \in J$. So the BVP (2) has at least one positive solution. □

## 4. The Existence of Multiple Positive Solutions

**Theorem 23.** *Assume that $(A_0)$–$(A_4)$ hold, and the following two conditions hold:*

$(C_7)$ *$f_0 > 4\eta Q/m\delta_*^2$ and $f_\infty > 4\eta Q/m\delta_*^2$ (particularly, $f_0 = f_\infty = \infty$);*

$(C_8)$ *there exists $l > 0$ such that $\max_{k\in J, u\in\partial K_l} f(k,u) < B^{-1} l$.*

*Then BVP (2) has at least two positive solutions $u^*(k)$, $u^{**}(k)$, which satisfy*

$$0 < \left\|u^{**}\right\| < l < \left\|u^*\right\|. \tag{69}$$

*Proof.* We can take two constants $r, R$ and suppose that $0 < r < l < R$. If $f_0 > 4\eta Q/m\delta_*^2$; from the proof of Theorem 21 we have

$$\|Tu\| \geq \|u\|, \quad u \in \partial K_r. \tag{70}$$

If $f_\infty > 4\eta Q/m\delta_*^2$, similarly, we have

$$\|Tu\| \geq \|u\|, \quad u \in \partial K_R. \tag{71}$$

On the other hand, from $(C_8)$, for $u \in \partial K_l$, we have

$$\begin{aligned}\|Tu\| &= \sum_{i=3}^{n+1}\Psi(k,i)\,w(i-2)\,f(i-2,u(i-2))\\ &< \frac{(n+2)(n+1)}{4(1-\mu)}\sum_{i=3}^{n+1}H(i,i)\,w(i-2)\,B^{-1}l\\ &\leq \frac{(n+2)(n+1)}{4(1-\mu)}\left(\sum_{i=3}^{n+1}|H|^p\right)^{1/p}\left(\sum_{i=3}^{n+1}|w|^q\right)^{1/q}B^{-1}l = l;\end{aligned} \tag{72}$$

that is,

$$\|Tu\| < l = \|u\|, \quad u \in \partial K_l. \tag{73}$$

Applying Theorem 3, we can prove that $T$ has a fixed point $u^{**} \in \overline{K}_{r,l}$ and a fixed point $u^* \in \overline{K}_{l,R}$. Thus, we prove that BVP (2) has two positive solutions $u^*, u^{**}$. From above formula we obtain $\|u^*\| \neq l$ and $\|u^{**}\| \neq l$. So $0 < \|u^{**}\| < l < \|u^*\|$. □

*Remark 24.* From the proof of Theorem 23 we obtain that if $(C_8)$ holds and $f_0 > 4\eta Q/m\delta_*^2$ (or $f_\infty > 4\eta Q/m\delta_*^2$), then BVP (2) has at least one positive solution $u$. It satisfies $0 < \|u\| < l$ (or $l < \|u\|$).

Using a similar method we can obtain the following results.

**Theorem 25.** *Suppose that $(A_0)$–$(A_4)$ hold, and the following two conditions hold*

$(C_9)$ *$f^0 < 1/B$ and $f^\infty < 1/B$;*

$(C_{10})$ *there exists $L > 0$ such that $\min_{k\in J, u\in\partial K_2} f(k,u) > (4\eta Q/m\delta_*)L$,*

*then BVP (2) has at least two positive solutions $u^*(k)$, $u^{**}(k)$, which satisfy*

$$0 < \left\|u^{**}\right\| < L < \left\|u^*\right\|. \tag{74}$$

*Remark 26.* If $(C_{10})$ holds and $f^0 < 1/B$ (or $f^\infty < 1/B$), then BVP (2) has at least one positive solution $u$ satisfying $0 < \|u\| < L$ (or $L < \|u\|$).

**Theorem 27.** *Assume that $(A_0)$–$(A_4)$ hold. If there exist $2n$ positive numbers $d_t, D_t$, $t = 1, 2, \ldots, n$, with $d_1 < \delta_* D_1 < D_1 < d_2 < \delta_* D_2 < D_2 < \cdots < d_n < \delta_* D_n < D_n$ such that*

$(C_{11})$ *$f(k,u) \leq B^{-1}d_t$, for $(k,u) \in J \times (\delta_* d_t, d_t)$ and $f(k,u) \geq (4\eta Q/m\delta_*)D_t$ for $(k,u) \in J \times [\delta_* D_t, D_t]$, $k = 1, 2, \ldots, n$; or*

$(C_{12})$ *$f(k,u) \geq (4\eta Q/m\delta_*)d_t$, for $(k,u) \in J \times (\delta_* d_t, d_t)$ and $f(k,u) \leq B^{-1}D_t$ for $(k,u) \in J \times [\delta_* D_t, D_t]$, $t = 1, 2, \ldots, n$,*

*then BVP (2) has at least $n$ positive solutions $u_k$ satisfying $d_t \leq \|u_t\| \leq D_t$, $t = 1, 2, \ldots, n$.*

**Theorem 28.** *Assume that $(A_0)$–$(A_4)$ hold. If there exist $2n$ positive numbers $d_t, D_t$, $t = 1, 2, \ldots, n$, with $d_1 < D_1 < d_2 < D_2 < \cdots < d_n < D_n$ such that*

$(C_{13})$ *$f(k,\cdot)$ is nondecreasing on $[0, D_n]$ for all $k \in J$;*

$(C_{14})$ *$f(k, \delta_* d_k) \geq (4\eta Q/m\delta_*)d_t$, and $f(k, D_t) \leq B^{-1}D_t$, $t = 1, 2, \ldots, n$,*

*then BVP (2) has at least $n$ positive solutions $u_t$ satisfying $d_t \leq \|u_t\| \leq D_t$, $t = 1, 2, \ldots, n$.*

## 5. The Nonexistence of Positive Solution

**Theorem 29.** *Suppose that $(A_0)$–$(A_4)$ hold, and $Bf(k,u) < u$, for all $k \in J$, $u > 0$; then BVP (2) has no positive solution.*

*Proof.* Assume that $u(k)$ is a positive solution of BVP (2). Then $u \in K$, $u(k) > 0$, $k \in \{0, 1, \ldots, n+2\}$ and

$$\begin{aligned}\|Tu\| &= \|u\| = \max_{k\in J}\|u(k)\|\\ &= \max_{k\in J}\sum_{i=3}^{n+1}\Psi(k,i)\,w(i-2)\,f(i-2,u(i-2))\\ &\leq \frac{(n+2)(n+1)}{4(1-\mu)}\sum_{i=3}^{n+1}H(i,i)\,w(i-2)\,f(i-2,u(i-2))\\ &< \frac{(n+2)(n+1)}{4(1-\mu)}\sum_{i=3}^{n+1}H(i,i)\,w(i-2)\,B^{-1}u(i-2)\\ &\leq B^{-1}\|u\|\frac{(n+2)(n+1)}{4(1-\mu)}\sum_{i=3}^{n+1}H(i,i)\,w(i-2)\\ &\leq B^{-1}\|u\|\frac{(n+2)(n+1)}{4(1-\mu)}\left(\sum_{i=3}^{n+1}|H|^p\right)^{1/p}\left(\sum_{i=3}^{n+1}|w|^q\right)^{1/q}\\ &= \|u\|,\end{aligned} \tag{75}$$

which is a contradiction. □

Similarly, we can get the following result.

**Theorem 30.** *Suppose that $(A_0)$–$(A_4)$ hold and $(m\delta_*^2/4\eta Q)f(k,u) > u$, for all $k \in J$, $u > 0$; then BVP (2) has no positive solution.*

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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