Research Article

Applications of Schauder's Fixed Point Theorem to Semipositone Singular Differential Equations

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We study the existence of positive periodic solutions of second-order singular differential equations. The proof relies on Schauder's fixed point theorem. Our results generalized and extended those results contained in the studies by Chu and Torres (2007) and Torres (2007). In some suitable weak singularities, the existence of periodic solutions may help.

1. Introduction

The theory of second-order periodic differential equation has been widely studied [1–20] because of great practical importance. In this paper, we apply Schauder's fixed point theorem to study the existence of positive solutions of the second-order periodic differential equation

$$x'' + a(t)x = f(t, x) + e(t), \qquad (1)$$

where a, e are continuous and 1-periodic functions and the nonlinearity f is continuous in (t, x), 1-periodic in t, and singular at $x = 0$. In addition, e may take some negative values.

Beginning with the paper of Lazer and Solimini [13], the semilinear singular differential equation

$$x'' + a(t)x = \frac{p(t)}{x^\lambda} + e(t), \qquad (2)$$

where $a, p, e \in C[0, 1]$ and $\lambda > 0$, has received the attention of many researchers during the last two decades [5, 7, 10, 21]. In [2], the author investigated the existence of positive solutions of (2) for three cases when $\widetilde{e}_* > 0$, $\widetilde{e}_* = 0$, and $\widetilde{e}^* \le 0$ through a basic application of Schauder's fixed point theorem. Here, let p^* and p_* denote the essential supremum and infimum

of a function $p \in L^1[0, 1]$, if they exist. For any continuous function p in $[0, 1]$, we set

$$\widetilde{p}(t) = \int_0^1 G(t, s) p(s) \, ds. \qquad (3)$$

In [1], the authors generalized the results of [2] and considered the second-order periodic semilinear singular equation

$$x'' + a(t)x = \frac{p(t)}{x^\alpha} + q(t)x^\beta + e(t), \qquad (4)$$

with $a, p, q, e \in C[0, 1]$ and $\alpha, \beta > 0$. By Schauder's fixed point theorem, they established the existence of positive periodic solutions to (4), if $\widetilde{e}^* > 0$ and $\widetilde{e}_* = 0$.

Although [1, Theorem 3.1] can deal with the cases of the existence of positive periodic solutions to (4), the authors there did not consider these two cases as applications of the existence of positive periodic solutions of (4) if $\widetilde{e}^* \le 0$ or $\widetilde{e}_* < 0 < \widetilde{e}^*$. We also generalize some of the results of [1, 2] in certain ways.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results are given. In Sections 3 and 4, we will state and prove the main results of the paper, as well as some applications to (2) and (4). In Section 5, we apply the results established in this paper to two specific equations.

2. Preliminaries

Throughout this paper, we assume that the equation, known as Hill's equation,

$$x'' + a(t) x = 0 \tag{5}$$

with periodic boundary conditions

$$x(0) = x(1), \qquad x'(0) = x'(1) \tag{6}$$

satisfies the following standing hypothesis:

(A) associated Green's function $G(t, s)$ is nonnegative for all $(t, s) \in [0, 1] \times [0, 1]$.

In other words, the antimaximum principle holds. Under this assumption, the function $\tilde{e}(t)$ is just the unique 1-periodic solution of the linear equation

$$x'' + a(t) x = e(t). \tag{7}$$

Now we make condition (A) clear. When $a(t) = k^2$, condition (A) is equivalent to $0 < k^2 \leq \mu_1 = \pi^2$. Note that μ_1 is the first eigenvalue of the linear problem with Dirichlet conditions $x(0) = 0 = x(1)$. For a nonconstant function $a(t)$, there is a L^p-criterion proved in [16]. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C \|u\|_q^2 \leq \|u'\|_2^2, \quad \forall u \in H_0^1(0, 1). \tag{8}$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \dfrac{2\pi}{q} \left(\dfrac{2}{2+q} \right)^{1-2/q} \left(\dfrac{\gamma(1/q)}{\gamma((1/2) + (1/q))} \right)^2, \\ \qquad\qquad\qquad\qquad \text{if } 1 \leq q < \infty, \\ 4, \qquad\qquad\qquad\quad \text{if } q = \infty, \end{cases} \tag{9}$$

where γ is the gamma function.

We write $p \succ 0$ if $p \geq 0$ for a.e. $t \in [0, 1]$ and p is positive in a set of positive measures.

Lemma 1 (see [16]). *Assume that $a(t) \succ 0$ and $a \in L^p[0, 1]$ for some $1 \leq p \leq \infty$. If*

$$\|a\|_p \leq K(2p^*), \tag{10}$$

then the standing hypothesis (A) holds.

Remark 2. In [3, 12, 16], the existence results are based on the positiveness of $G(t, s) > 0$, which plays a very important role in employing some fixed point theorems in cones for completely continuous operators. Our assumption (A) only needs that $G(t, s)$ is nonnegative, and therefore our results cover the critical case, which was not covered in the above three papers.

3. Main Results

In this section, we establish the main result about the existence of positive periodic solutions for (1).

Theorem 3. *Suppose that the following conditions hold.*

(H_1) *There exist nonnegative continuous functions b, c, d on $[0, 1]$ and g, h on $(0, +\infty)$ such that $b \geq c > 0$, $g > 0$ is strictly decreasing, and h/g is increasing. Moreover, there holds for any $t \in [0, 1]$ and $x \in (0, +\infty)$*

$$c(t) g(x) \leq f(t, x) \leq b(t) g(x) + d(t) h(x). \tag{11}$$

(H_2) *There exists a positive constant R such that*

$$R > g(R) \left(\tilde{b}^* + \frac{h(R)}{g(R)} \tilde{d}^* \right) + \tilde{e}^*, \tag{12}$$

$$\tilde{e}_* \geq g^{-1} \left(\frac{R - \tilde{e}^*}{\tilde{b}^* + (h(R)/g(R)) \tilde{d}^*} \right) - g(R) \tilde{c}_*. \tag{13}$$

Then (1) has at least one positive periodic solution.

Proof. Let C_1 denote the set of all continuous 1-periodic functions. We define a completely continuous map $T : C_1 \to C_1$ by

$$(Tx)(t) := \int_0^1 G(t, s) \left[f(s, x(s)) + e(s) \right] ds \tag{14}$$
$$= \int_0^1 G(t, s) f(s, x(s)) ds + \tilde{e}(t).$$

It is clear that a periodic solution of (1) is just a fixed point of T. It suffices to prove the existence of the fixed point of map T.

Let R be the positive constant in (H_2) and set

$$r := g^{-1} \left(\frac{R - \tilde{e}^*}{\tilde{b}^* + (h(R)/g(R)) \tilde{d}^*} \right). \tag{15}$$

Combining (12) with the fact that g is positive and strictly decreasing, we have $R > r > 0$. We introduce a set by

$$\Omega := \left\{ x \in C_1 : r \leq x(t) \leq R, \forall t \in [0, 1] \right\}. \tag{16}$$

Obviously, Ω is a closed convex set. For any $x \in \Omega$ and $t \in [0, 1]$, by assumption (11), we get

$$(Tx)(t) \geq \int_0^1 G(t, s) c(s) g(x(s)) ds + \tilde{e}(t). \tag{17}$$

Since $x(s) \leq R$ and g is strictly decreasing, there holds

$$(Tx)(t) \geq g(R) \tilde{c}(t) + \tilde{e}(t) \geq g(R) \tilde{c}_* + \tilde{e}_*. \tag{18}$$

Together with assumption (13), the above inequality leads to

$$(Tx)(t) \geq g^{-1} \left(\frac{R - \tilde{e}^*}{\tilde{b}^* + (h(R)/g(R)) \tilde{d}^*} \right) = r. \tag{19}$$

Similar discussion shows that there holds, for any $x \in \Omega$ and $t \in [0,1]$,

$$
\begin{aligned}
(Tx)(t) \\
&\leq \int_0^1 G(t,s) g(x(s)) \left(b(s) + \frac{h(x(s))}{g(x(s))} d(s) \right) ds + \tilde{e}(t) \\
&\leq g(r) \left(\tilde{b}(t) + \frac{h(R)}{g(R)} \tilde{d}(t) \right) + \tilde{e}(t) \\
&\leq g(r) \left(\tilde{b}_* + \frac{h(R)}{g(R)} \tilde{d}_* \right) + \tilde{e}_* = R.
\end{aligned}
\tag{20}
$$

Consequently, we get the conclusion that there holds $T(\Omega) \subset \Omega$. By Schauder's fixed point theorem, we obtain that there exists at least one fixed point of T, which completes the proof. $\square$

Applying Theorem 3 to a special case, we have the following result.

Corollary 4. *Suppose that $\tilde{e}^* \leq 0$ and there hold condition (H_1) and the following one.*

(H_2^*) *There exists a positive constant R such that*

$$
R > g(R) \left(\tilde{b}^* + \frac{h(R)}{g(R)} \tilde{d}^* \right),
\tag{21}
$$

$$
\tilde{e}_* \geq g^{-1} \left(\frac{R}{\tilde{b}^* + (h(R)/g(R)) \tilde{d}^*} \right) - g(R) \tilde{c}_*.
$$

Then (1) has at least one positive periodic solution.

4. Applications

4.1. The Case $\tilde{e}^* \leq 0$. In this subsection, we consider (4) in the cases when $\tilde{e}^* \leq 0$. To meet the condition in Corollary 4, we let

$$
g(x) = x^{-\alpha}, \quad h(x) = x^{\beta}, \quad x \in (0, +\infty).
\tag{22}
$$

We note that to verify the existence of the positive periodic solution of (4) by Corollary 4 an important step is to find $R > 0$ such that

$$
\tilde{e}_* \geq \left(\frac{\tilde{b}^* + \tilde{d}^* R^{\alpha+\beta}}{R} \right)^{1/\alpha} - \frac{\tilde{c}_*}{R^\alpha},
\tag{23}
$$

where b, c, d are the functions in condition (H_1). Let $m = 1/R$. Then finding $R > 0$ with (23) is equivalent to finding $m > 0$ satisfying

$$
\tilde{e}_* \geq \left[m \left(\tilde{b}^* + \tilde{d}^* \left(\frac{1}{m} \right)^{\alpha+\beta} \right) \right]^{1/\alpha} - \tilde{c}_* m^\alpha.
\tag{24}
$$

To this end, we consider the following function:

$$
\begin{aligned}
F(m) &:= \left[m \left(\tilde{b}^* + \tilde{d}^* \left(\frac{1}{m} \right)^{\alpha+\beta} \right) \right]^{1/\alpha} - \tilde{c}_* m^\alpha \\
&= m^\alpha \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} - \tilde{c}_* \right], \\
&\qquad\qquad m \in (0, +\infty).
\end{aligned}
\tag{25}
$$

By direct calculation, we have

$$
\begin{aligned}
&F'(m) \\
&= \alpha m^{\alpha-1} \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} - \tilde{c}_* \right] \\
&\quad + m^\alpha \left[\frac{1}{\alpha} \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right. \\
&\qquad\qquad \times \left(\left(1 - \alpha^2 \right) \tilde{b}^* m^{-\alpha^2} \right. \\
&\qquad\qquad\quad \left.\left. + \left(1 - \alpha - \beta - \alpha^2 \right) \tilde{d}^* m^{-\alpha-\beta-\alpha^2} \right) \right] \\
&= \frac{1}{\alpha} m^{\alpha-1} \\
&\quad \times \left\{ \alpha^2 \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} - \tilde{c}_* \right] \right. \\
&\qquad + m \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right. \\
&\qquad\qquad \times \left(\left(1 - \alpha^2 \right) \tilde{b}^* m^{-\alpha^2} \right. \\
&\qquad\qquad\quad + \left(1 - \alpha - \beta - \alpha^2 \right) \\
&\qquad\qquad\quad \left.\left.\left. \times \tilde{d}^* m^{-\alpha-\beta-\alpha^2} \right) \right] \right\} \\
&= \frac{1}{\alpha} m^{\alpha-1} \\
&\quad \times \left\{ \alpha^2 \left[\left(\tilde{b}^* m^{1-\alpha^2} + \overline{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} - \tilde{c}_* \right] \right. \\
&\qquad + \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right. \\
&\qquad\qquad \times \left(\left(1 - \alpha^2 \right) \tilde{b}^* m^{1-\alpha^2} \right. \\
&\qquad\qquad\quad \left.\left.\left. + \left(1 - \alpha - \beta - \alpha^2 \right) \overline{d}^* m^{1-\alpha-\beta-\alpha^2} \right) \right] \right\}
\end{aligned}
$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left[\alpha^2 \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} \right. \right.$$

$$+ \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1}$$

$$\times \left(\left(1 - \alpha^2 \right) \widetilde{b}^* m^{1-\alpha^2} + \left(1 - \alpha - \beta - \alpha^2 \right) \right.$$

$$\left. \left. \times \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right) \right] - \alpha^2 \widetilde{c}_* \right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right.$$

$$\times \left[\alpha^2 \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right) \right.$$

$$+ \left(\left(1 - \alpha^2 \right) \widetilde{b}^* m^{1-\alpha^2} + \left(1 - \alpha - \beta - \alpha^2 \right) \right.$$

$$\left. \left. \times \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right) \right] - \alpha^2 \widetilde{c}_* \right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right.$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2} + \left(\alpha^2 + \left(1 - \alpha - \beta - \alpha^2 \right) \right) \right.$$

$$\left. \left. \times \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right] - \alpha^2 \widetilde{c}_* \right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1} \right.$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2} + \left(1 - \alpha - \beta \right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right]$$

$$\left. - \alpha^2 \widetilde{c}_* \right\}. \tag{26}$$

Suppose that $1 - \alpha^2 > 0$ and $1 - \alpha - \beta - \alpha^2 > 0$. Set

$$\Phi(m) =: \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1}$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2} + \left(1 - \alpha - \beta \right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right], \tag{27}$$

$$m \in [0, +\infty).$$

It is clear that Φ is continuous and increasing on $[0, +\infty)$. Since there hold $\Phi(0) = 0$ and $\lim_{m \to +\infty} \Phi(m) = +\infty$, there exists a unique $m_0 \in (0, +\infty)$ such that $\Phi(m_0) = \alpha^2 \widetilde{c}_*$. It

follows that $F'(m_0) = 0$. In addition, we note that $F'(m) < 0$ for $0 < m < m_0$ and $F'(m) > 0$ for $m > m_0$. Hence, we get the conclusion that F possesses a minimum at $m = m_0$. Based upon the above discussion, we have that m_0 is the best choice for satisfying inequality (24).

Theorem 5. *Suppose that $0 < \alpha, \beta < 1$, $1 - \alpha - \alpha^2 - \beta > 0$, and there exist nonnegative continuous functions b, c, d on $[0, 1]$ such that $b \geq c > 0$ and*

$$c(t) x^{-\alpha} \leq p(t) x^{-\alpha} + q(t) x^\beta \leq b(t) x^{-\alpha} + d(t) x^\beta,$$
$$t \in [0, 1], \quad x \in (0, +\infty). \tag{28}$$

If there holds

$$0 \geq \widetilde{e}_* \geq m_0^\alpha \left[\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} - \widetilde{c}_* \right], \tag{29}$$

where m_0 is the unique positive solution of the equation

$$\left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right)^{(1/\alpha)-1}$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2} + \left(1 - \alpha - \beta \right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2} \right] = \alpha^2 \widetilde{c}_*, \tag{30}$$

then (4) has at least one positive periodic solution.

Proof. Let $g(x) = x^{-\alpha}$, $h(x) = x^\beta$, and $x \in (0, +\infty)$. It is obvious that $g > 0$ is strictly decreasing and h/g is nondecreasing. According to the assumption about b, c, d, we have that condition (H_1) of Theorem 3 holds.

Let $R = 1/m_0$. It follows from assumption (29) that there holds

$$\widetilde{e}_* \geq g^{-1} \left(\frac{R}{\widetilde{b}^* + (h(R)/g(R)) \widetilde{d}^*} \right) - g(R) \widetilde{c}_*. \tag{31}$$

It suffices to verify that R satisfies

$$R > g(R) \left(\widetilde{b}^* + \frac{h(R)}{g(R)} \widetilde{d}^* \right), \tag{32}$$

which is equivalent to the inequality

$$m_0^{\alpha+1} \left(\overline{b}^* m_0^{1-\alpha^2} + \overline{d}^* m_0^{1-\alpha-\beta-\alpha^2} \right)^{1/\alpha} < 1. \tag{33}$$

Due to (30), we get

$$\alpha^2 \widetilde{c}_* \geq \left(\widetilde{b}^* m_0^{1-\alpha^2} \right)^{(1/\alpha)-1} \left(\widetilde{b}^* m_0^{1-\alpha^2} \right) = \left(\widetilde{b}^* m_0^{1-\alpha^2} \right)^{1/\alpha}, \tag{34}$$

which implies

$$m_0 \leq \left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}} \right)^{\alpha/(1-\alpha^2)}. \tag{35}$$

We also obtain by (30)

$$\alpha^2 \widetilde{c}_* \geq \left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right)^{(1/\alpha)-1}$$
$$\times \left[(1-\alpha-\beta)\widetilde{b}^* m_0^{1-\alpha^2} + (1-\alpha-\beta)\widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right]$$
$$= (1-\alpha-\beta)\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right)^{1/\alpha},$$
(36)

which leads to

$$\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right)^{1/\alpha} \leq \frac{\alpha^2 \widetilde{c}_*}{(1-\alpha-\beta)}. \quad (37)$$

Combining (35) with (37), we have

$$m_0^{\alpha+1}\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right)^{1/\alpha}$$
$$< \left(\left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}}\right)^{\alpha/(1-\alpha^2)}\right)^{\alpha+1} \frac{\alpha^2 \widetilde{c}_*}{(1-\alpha-\beta)}$$
$$= \left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}}\right)^{\alpha/(1-\alpha)} \frac{\alpha^2 \widetilde{c}_*}{(1-\alpha-\beta)}$$
$$= \alpha^{2\alpha/(1-\alpha)}\left(\frac{\widetilde{c}_*}{\widetilde{b}^*}\right)^{\alpha/(1-\alpha)} \frac{\alpha^2}{(1-\alpha-\beta)} < 1,$$
(38)

which completes the proof. □

4.2. The Case $\widetilde{e}_* < 0 < \widetilde{e}^*$. In this subsection, we consider (2) and (4) in the case when $\widetilde{e}_* < 0 < \widetilde{e}^*$. In order to study (2) by Theorem 3, we have to set $g(x) = x^{-\lambda}$, $h(x) = 0$, and $x \in (0, +\infty)$. Let b, c be the functions in condition (H_1). For the purpose of looking for $R > 0$ satisfying

$$\widetilde{e}_* \geq \left(\frac{\widetilde{b}^*}{R-\widetilde{e}^*}\right)^{1/\lambda} - \frac{\widetilde{c}_*}{R^\lambda}, \quad (39)$$

we introduce a function on $(0, +\infty)$ by

$$F(m) := \left(\widetilde{b}^* m\right)^{1/\lambda} - \frac{\widetilde{c}_* m^\lambda}{(1+\widetilde{e}^* m)^\lambda}$$
$$= m^\lambda \left[\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda} - \widetilde{c}_*(1+\widetilde{e}^* m)^{-\lambda}\right]. \quad (40)$$

It suffices to find $m_0 \in (0, +\infty)$ such that $F(m_0) \leq \widetilde{e}_*$. To this end, we compute the derivative of F. There holds

$$F'(m)$$
$$= \lambda m^{\lambda-1}\left[\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda} - \widetilde{c}_*(1+\widetilde{e}^* m)^{-\lambda}\right]$$
$$+ m^\lambda \left[\frac{1-\lambda^2}{\lambda}\widetilde{b}^{*(1/\lambda)} m^{((1-\lambda^2)/\lambda)-1}\right.$$
$$\left. +\lambda \widetilde{c}_* \widetilde{e}^*(1+\widetilde{e}^* m)^{-\lambda-1}\right]$$
$$= \frac{1}{\lambda} m^{\lambda-1}\left\{\lambda^2\left[\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda} - \widetilde{c}_*(1+\widetilde{e}^* m)^{-\lambda}\right]\right.$$
$$+ \lambda m \left[\frac{1-\lambda^2}{\lambda}\widetilde{b}^{*(1/\lambda)} m^{((1-\lambda^2)/\lambda)-1}\right.$$
$$\left.\left. + \lambda \widetilde{c}_* \widetilde{e}^*(1+\widetilde{e}^* m)^{-\lambda-1}\right]\right\}$$
$$= \frac{1}{\lambda} m^{\lambda-1}\left[\lambda^2 \widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda} - \lambda^2 \widetilde{c}_*(1+\widetilde{e}^* m)^{-\lambda}\right.$$
$$+ \left(1-\lambda^2\right)\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda}$$
$$\left. + \lambda^2 \widetilde{c}_* \frac{\widetilde{e}^* m}{1+\widetilde{e}^* m}(1+\widetilde{e}^* m)^{-\lambda}\right]$$
$$= \frac{1}{\lambda} m^{\lambda-1}\left[\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda} - \lambda^2 \widetilde{c}_* \frac{1}{1+\widetilde{e}^* m}(1+\widetilde{e}^* m)^{-\lambda}\right]$$
$$= \frac{m^{\lambda-1}}{\lambda(1+\widetilde{e}^* m)^{\lambda+1}}\left[\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda}(1+\widetilde{e}^* m)^{\lambda+1} - \lambda^2 \widetilde{c}_*\right]. \quad (41)$$

Assume $0 < \lambda < 1$. By letting

$$\Phi(m) =: \widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda}(1+\widetilde{e}^* m)^{\lambda+1}, \quad m \in [0, +\infty), \quad (42)$$

we note that Φ is increasing, with $\Phi(0) = 0$, and $\lim_{m \to +\infty}\Phi(m) = +\infty$. Then the equation $\Phi(m) = \lambda^2 \widetilde{c}_*$ has a unique solution $m_0 \in (0, +\infty)$, which is also the unique solution of $F(m) = 0$. Since $F'(m) < 0$ for $0 < m < m_0$ and $F'(m) > 0$ for $m > m_0$, F attains the minimum at $m = m_0$, which shows that we get the best choice of positive constant satisfying $F(m) \leq \widetilde{e}_*$.

Theorem 6. *Suppose that $0 < \lambda < 1$ and there exist continuous and nonnegative functions b, c such that $b \geq p \geq c > 0$. If $\widetilde{e}_* < 0 < \widetilde{e}^*$ and*

$$\widetilde{e}_* \geq m_0^\lambda \left[\widetilde{b}^{*(1/\lambda)} m_0^{(1-\lambda^2)/\lambda} - \widetilde{c}_*(1+\widetilde{e}^* m_0)^{-\lambda}\right], \quad (43)$$

where m_0 is the unique positive solution of the equation

$$\widetilde{b}^{*(1/\lambda)} m^{(1-\lambda^2)/\lambda}(1+\widetilde{e}^* m)^{\lambda+1} = \lambda^2 \widetilde{c}_*, \quad (44)$$

then (2) has at least one positive periodic solution.

Proof. Let $g(x) = x^{-\alpha}$, $h(x) = 0$, $x \in (0, +\infty)$. Since $b \geq p \geq c$, we have, for any $t \in [0, 1]$ and any and $x \in (0, +\infty)$, $c(t)g(x) \leq p(t)g(x) \leq b(t)g(x)$. Hence, we conclude that condition (H_1) in Theorem 3 is satisfied. Taking $R = (1/m_0) + \tilde{e}^*$, we obtain by inequality (43)

$$\tilde{e}_* \geq m_0^\lambda \left[\tilde{b}^{*(1/\lambda)} m_0^{(1-\lambda^2)/\lambda} - \tilde{c}_* \left(1 + \tilde{e}^* m_0\right)^{-\lambda} \right]$$

$$= \left(\tilde{b}^* m_0\right)^{1/\lambda} - \frac{\tilde{c}_* m_0^\lambda}{\left(1 + \tilde{e}^* m_0\right)^\lambda} = \left(\frac{\tilde{b}^*}{R - \tilde{e}^*}\right)^{1/\lambda} - \frac{\tilde{c}_*}{R^\lambda}. \tag{45}$$

It suffices to verify

$$R > R^{-\lambda} \tilde{b}^* + \tilde{e}^*, \tag{46}$$

which is equivalent to

$$\tilde{b}^{*(1/\lambda)} m_0^{1+(1/\lambda)} < 1 + \tilde{e}^* m_0. \tag{47}$$

It follows from (44) that $\tilde{b}^{*(1/\lambda)} m_0^{(1-\lambda^2)/\lambda} \leq \lambda^2 \tilde{c}_*$, which implies

$$m_0 \leq \left(\frac{\lambda^2 \tilde{c}_*}{\tilde{b}^{*(1/\lambda)}}\right)^{\lambda/(1-\lambda^2)}. \tag{48}$$

Together with the assumption that $b \geq c$, the above inequality shows

$$\tilde{b}^{*(1/\lambda)} m_0^{1+(1/\lambda)} \leq \tilde{b}^{*(1/\lambda)} \left(\left(\frac{\lambda^2 \tilde{c}_*}{\tilde{b}^{*(1/\lambda)}}\right)^{\lambda/(1-\lambda^2)}\right)^{(1+\lambda)/\lambda}$$

$$= \tilde{b}^{*(1/\lambda)} \left(\frac{\lambda^2 \tilde{c}_*}{\tilde{b}^{*(1/\lambda)}}\right)^{1/(1-\lambda)} \tag{49}$$

$$= \lambda^{2/(1-\lambda)} \left(\frac{\tilde{c}_*}{\tilde{b}^*}\right)^{1/(1-\lambda)} < 1 < 1 + \tilde{e}^* m_0.$$

The proof is finished. □

In the following, we consider (2). We also let $g(x) = x^{-\alpha}$, $h(x) = x^\beta$, and $x \in (0, +\infty)$. We will deal with this case in a similar manner. Let b, c, d be the functions in condition (H_1). For this purpose, we note that looking for $R > 0$ satisfying

$$\tilde{e}_* \geq \left(\frac{\tilde{b}^* + \tilde{d}^* R^{\alpha+\beta}}{R - \tilde{e}^*}\right)^{1/\alpha} - \frac{\tilde{c}_*}{R^\alpha} \tag{50}$$

is equivalent to finding $m = 1/(R - \tilde{e}^*) > 0$ such that

$$\tilde{e}_* \geq \left[m \left(\tilde{b}^* + \tilde{d}^* \left(\frac{1 + \tilde{e}^* m}{m}\right)^{\alpha+\beta}\right) \right]^{1/\alpha} - \frac{\tilde{c}_* m^\alpha}{\left(1 + \tilde{e}^* m\right)^\alpha}. \tag{51}$$

Based upon this observation, we have to define

$$F(m) := \left[m \left(\tilde{b}^* + \tilde{d}^* \left(\frac{1 + \tilde{e}^* m}{m}\right)^{\alpha+\beta}\right) \right]^{1/\alpha} - \frac{\tilde{c}_* m^\alpha}{\left(1 + \tilde{e}^* m\right)^\alpha}$$

$$= m^\alpha \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)^{1/\alpha} \right.$$

$$\left. - \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \right], \quad m \in (0, +\infty). \tag{52}$$

The derivative of F can be calculated as follows:

$$F'(m)$$

$$= \alpha m^{\alpha-1} \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right. \right.$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)^{1/\alpha}$$

$$\left. - \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \right]$$

$$+ m^\alpha \left[\frac{1}{\alpha} \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right. \right.$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)^{(1/\alpha)-1}$$

$$\times \left(\left(1 - \alpha^2\right) \tilde{b}^* m^{-\alpha^2}\right.$$

$$+ \left(\alpha + \beta\right) \tilde{d}^* \tilde{e}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta-1}$$

$$+ \left(1 - \alpha - \beta - \alpha^2\right) \tilde{d}^* m^{-\alpha-\beta-\alpha^2}$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)$$

$$+ \alpha \tilde{c}_* \tilde{e}^* \left(1 + \tilde{e}^* m\right)^{-\alpha-1} \Big]$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \alpha^2 \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right. \right. \right.$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)^{1/\alpha}$$

$$\left. - \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \right]$$

$$+ m \left[\left(\tilde{b}^* m^{1-\alpha^2}\right. \right.$$

$$+ \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)^{(1/\alpha)-1}$$

$$\times \left(\left(1-\alpha^2\right) \tilde{b}^* m^{-\alpha^2} \right.$$

$$+ (\alpha+\beta)\,\tilde{d}^*\tilde{e}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta-1}$$

$$+ \left(1-\alpha-\beta-\alpha^2\right)\tilde{d}^* m^{-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)$$

$$\left. +\alpha^2\tilde{c}_*\tilde{e}^* \left(1+\tilde{e}^* m\right)^{-\alpha-1}\right]\right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \alpha^2 \left[\left(\tilde{b}^* m^{1-\alpha^2} \right. \right.\right.$$

$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{1/\alpha}$$

$$\left. -\, \tilde{c}_* \left(1+\tilde{e}^* m\right)^{-\alpha}\right]$$

$$+ \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.\right.$$

$$\left.\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$

$$\times \left(\left(1-\alpha^2\right) \tilde{b}^* m^{1-\alpha^2}\right.$$

$$+ (\alpha+\beta)\,\tilde{d}^*\tilde{e}^* m\, m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta-1}$$

$$+ \left(1-\alpha-\beta-\alpha^2\right)\tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)$$

$$\left.\left. + \alpha^2\tilde{c}_*\tilde{e}^* m \left(1+\tilde{e}^* m\right)^{-\alpha-1}\right]\right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \alpha^2 \left[\left(\tilde{b}^* m^{1-\alpha^2} \right.\right.\right.$$

$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{1/\alpha}$$

$$\left. -\, \tilde{c}_* \left(1+\tilde{e}^* m\right)^{-\alpha}\right]$$

$$+ \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.\right.$$

$$\left.\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$

$$\times \left(\left(1-\alpha^2\right) \tilde{b}^* m^{1-\alpha^2}\right.$$

$$+ (\alpha+\beta)\,\tilde{d}^*\tilde{e}^* \frac{m}{1+\tilde{e}^* m} m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}$$

$$+ \left(1-\alpha-\beta-\alpha^2\right)\tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)$$

$$\left. + \alpha^2\tilde{c}_*\tilde{e}^* \frac{m}{1+\tilde{e}^* m}\left(1+\tilde{e}^* m\right)^{-\alpha}\right]\right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left[\alpha^2 \left(\tilde{b}^* m^{1-\alpha^2} \right.\right.\right.$$

$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{1/\alpha}$$

$$\left. - \alpha^2\tilde{c}_* \left(1+\tilde{e}^* m\right)^{-\alpha}\right]$$

$$+ \left[\left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.\right.$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)^{(1/\alpha)-1}$$

$$\times \left(\left(1-\alpha^2\right) \tilde{b}^* m^{1-\alpha^2}\right.$$

$$+ (\alpha+\beta)\,\tilde{d}^* \frac{\tilde{e}^* m}{1+\tilde{e}^* m} m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}$$

$$+ \left(1-\alpha-\beta-\alpha^2\right)\tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)$$

$$\left. + \alpha^2\tilde{c}_* \frac{\tilde{e}^* m}{1+\tilde{e}^* m}\left(1+\tilde{e}^* m\right)^{-\alpha}\right]\right\}$$

$$= \frac{1}{\alpha} m^{\alpha-1}$$

$$\times \left\{ \left[\alpha^2 \left(\tilde{b}^* m^{1-\alpha^2} \right.\right.\right.$$

$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2} \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\right)^{1/\alpha}$$

$$+ \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.$$

$$\times \left(1+\tilde{e}^* m\right)^{\alpha+\beta}\bigg)^{(1/\alpha)-1}$$

$$\times \left(\left(1-\alpha^2\right) \tilde{b}^* m^{1-\alpha^2}\right.$$

$$+ (\alpha+\beta)\,\tilde{d}^* \frac{\tilde{e}^* m}{1+\tilde{e}^* m} m^{1-\alpha-\beta-\alpha^2}$$

$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}$$
$$+ \left(1 - \alpha - \beta - \alpha^2\right) \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)\Big]$$
$$- \Big[\alpha^2 \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha}$$
$$- \alpha^2 \tilde{c}_* \frac{\tilde{e}^* m}{1 + \tilde{e}^* m}\left(1 + \tilde{e}^* m\right)^{-\alpha}\Big]\Big\}$$
$$= \frac{1}{\alpha} m^{\alpha-1}$$
$$\times \Bigg\{ \left(\tilde{b}^* m^{1-\alpha^2}\right.$$
$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$
$$\times \Big[\alpha^2 \left(\tilde{b}^* m^{1-\alpha^2}\right.$$
$$\left. + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)$$
$$+ \Big(\left(1 - \alpha^2\right) \tilde{b}^* m^{1-\alpha^2}$$
$$+ (\alpha + \beta) \tilde{d}^* \frac{\tilde{e}^* m}{1 + \tilde{e}^* m} m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}$$
$$+ \left(1 - \alpha - \beta - \alpha^2\right) \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)\Big]$$
$$- \Big[\alpha^2 \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha}$$
$$\times \left(1 - \frac{\tilde{e}^* m}{1 + \tilde{e}^* m}\right)\Big]\Big]\Big\}$$
$$= \frac{1}{\alpha} m^{\alpha-1}$$
$$\times \Bigg\{ \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.$$
$$\left. \times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$
$$\times \Big[\left(\alpha^2 \tilde{b}^* m^{1-\alpha^2}\right.$$
$$\left. + \left(1 - \alpha^2\right) \tilde{b}^* m^{1-\alpha^2}\right)$$
$$+ \left(\alpha^2 \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right.$$

$$+ (\alpha + \beta) \tilde{d}^* \frac{\tilde{e}^* m}{1 + \tilde{e}^* m} m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}$$
$$+ \left(1 - \alpha - \beta - \alpha^2\right) \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big)\Big]$$
$$- \alpha^2 \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \frac{1}{1 + \tilde{e}^* m}\Big\}$$
$$= \frac{1}{\alpha} m^{\alpha-1}$$
$$\times \Bigg\{ \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.$$
$$\left. \times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$
$$\times \Big[\tilde{b}^* m^{1-\alpha^2}$$
$$+ \left(\alpha^2 + (\alpha + \beta) \frac{\tilde{e}^* m}{1 + \tilde{e}^* m}\right.$$
$$\left. + \left(1 - \alpha - \beta - \alpha^2\right)\right) \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big]$$
$$- \alpha^2 \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \frac{1}{1 + \tilde{e}^* m}\Big\}$$
$$= \frac{1}{\alpha} m^{\alpha-1}$$
$$\times \Bigg\{ \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.$$
$$\left. \times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$
$$\times \Big[\tilde{b}^* m^{1-\alpha^2}$$
$$+ \left(1 - \frac{\alpha + \beta}{1 + \tilde{e}^* m}\right) \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$
$$\times \left(1 + \tilde{e}^* m\right)^{\alpha+\beta}\Big]$$
$$- \alpha^2 \tilde{c}_* \left(1 + \tilde{e}^* m\right)^{-\alpha} \frac{1}{1 + \tilde{e}^* m}\Big\}$$
$$= \frac{1}{\alpha} \frac{m^{\alpha-1}}{\left(1 + \tilde{e}^* m\right)^{\alpha+1}}$$
$$\times \Bigg\{ \left(\tilde{b}^* m^{1-\alpha^2} + \tilde{d}^* m^{1-\alpha-\beta-\alpha^2}\right.$$

$$\times \left(1 + \widetilde{e}^* m\right)^{\alpha+\beta}\Big)^{(1/\alpha)-1}$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2} \times \left(1 + \widetilde{e}^* m\right)^{\alpha+1}\right.$$

$$+ \left(1 - \alpha - \beta + \widetilde{e}^* m\right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2}$$

$$\left. \times \left(1 + \widetilde{e}^* m\right)^{2\alpha+\beta}\right] - \alpha^2 \widetilde{c}_*\Big\}.$$

(53)

Since the function

$$\Phi(m)$$

$$=: \left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{\alpha+1}\right.$$

$$\left. + \left(1 - \alpha - \beta + \widetilde{e}^* m\right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{2\alpha+\beta}\right]$$

(54)

is increasing on $m \in [0, +\infty)$ and $\Phi(0) = 0$, and $\lim_{m \to +\infty} \Phi(m) = +\infty$, the equation $\Phi(m) = \alpha^2 \widetilde{c}_*$ has a unique solution $m_0 \in (0, +\infty)$. Thus the function F attains a minimum at $m = m_0$, which provides a suitable m satisfying (51).

Theorem 7. *Suppose that $0 < \alpha$, $\beta < 1$, $1 - \alpha - \alpha^2 - \beta > 0$, and there exist continuous and nonnegative functions b, c, d on $[0, 1]$ such that $b \geq c > 0$ and*

$$c(t) x^{-\alpha} \leq p(t) x^{-\alpha} + q(t) x^\beta \leq b(t) x^{-\alpha} + d(t) x^\beta,$$

$$t \in [0, 1], \quad x \in (0, +\infty).$$

(55)

If $\widetilde{e}_* < 0 < \widetilde{e}^*$ and

$$\widetilde{e}_* \geq m_0^\alpha \left[\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{1/\alpha}\right.$$

$$\left. - \widetilde{c}_*\left(1 + \widetilde{e}^* m_0\right)^{-\alpha}\right],$$

(56)

where m_0 is the unique positive solution of the equation

$$\left(\widetilde{b}^* m^{1-\alpha^2} + \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$

$$\times \left[\widetilde{b}^* m^{1-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{\alpha+1}\right.$$

$$\left. + \left(1 - \alpha - \beta + \widetilde{e}^* m\right) \widetilde{d}^* m^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m\right)^{2\alpha+\beta}\right]$$

$$= \alpha^2 \widetilde{c}_*,$$

(57)

then (4) has at least one positive periodic solution.

Proof. By similar discussion in Theorem 5, we have that condition (H_1) of Theorem 3 is satisfied under the assumption in this theorem. Set $R = (1/m) + \widetilde{e}^*$. Assumption (56) shows

$$\widetilde{e}_* \geq \left[m\left(\widetilde{b}^* + \widetilde{d}^*\left(\frac{1 + \widetilde{e}^* m}{m}\right)^{\alpha+\beta}\right)\right]^{1/\alpha}$$

$$- \frac{\widetilde{c}_* m^\alpha}{\left(1 + \widetilde{e}^* m\right)^\alpha} = \left(\frac{\widetilde{b}^* + \widetilde{d}^* R^{\alpha+\beta}}{R - \widetilde{e}^*}\right)^{1/\alpha} - \frac{\widetilde{c}_*}{R^\alpha}.$$

(58)

To verify that R satisfies

$$R > R^{-\alpha}\left(\widetilde{b}^* + R^{\alpha+\beta}\widetilde{d}^*\right) + \widetilde{e}^*,$$

(59)

we note that, by (57), there holds

$$\alpha^2 \widetilde{c}_* \geq \left(\widetilde{b}^* m_0^{1-\alpha^2}\right)^{(1/\alpha)-1}\left(\widetilde{b}^* m_0^{1-\alpha^2}\right) = \left(\widetilde{b}^* m_0^{1-\alpha^2}\right)^{1/\alpha},$$

(60)

which leads to

$$m_0 \leq \left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}}\right)^{\alpha/(1-\alpha^2)}.$$

(61)

Furthermore, we also have

$$\alpha^2 \widetilde{c}_* \geq \left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{(1/\alpha)-1}$$

$$\times \left[\left(1 - \alpha - \beta\right)\widetilde{b}^* m_0^{1-\alpha^2}\right.$$

$$\left. + \left(1 - \alpha - \beta\right)\widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right]$$

(62)

$$= \left(1 - \alpha - \beta\right)$$

$$\times \left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{1/\alpha};$$

that is,

$$\frac{\alpha^2 \widetilde{c}_*}{\left(1 - \alpha - \beta\right)} \geq \left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{1/\alpha}.$$

(63)

Combining (61) with (63), we get

$$m_0^{\alpha+1}\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{1/\alpha}$$

$$< \left(\left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}}\right)^{\alpha/(1-\alpha^2)}\right)^{\alpha+1} \frac{\alpha^2 \widetilde{c}_*}{\left(1 - \alpha - \beta\right)}$$

$$= \left(\frac{\alpha^2 \widetilde{c}_*}{\widetilde{b}^{*(1/\alpha)}}\right)^{\alpha/(1-\alpha)} \frac{\alpha^2 \widetilde{c}_*}{\left(1 - \alpha - \beta\right)}$$

$$= \alpha^{2\alpha/(1-\alpha)}\left(\frac{\widetilde{c}_*}{\widetilde{b}^*}\right)^{\alpha/(1-\alpha)} \frac{\alpha^2}{\left(1 - \alpha - \beta\right)} < 1 < 1 + \widetilde{e}^* m_0.$$

(64)

Substituting $R = (1/m) + \widetilde{e}^*$ into the above inequality, we get (59), which completes the proof. □

5. Example

In this section, we apply the results established in this paper to two second-order singular equations.

Example 1. Consider the periodic problem for a second-order singular equation

$$x'' + x = \frac{\sin^2{(2\pi tx)} + 1}{x^{1/2}} + x^{1/8} - 0.03. \tag{65}$$

We will make use of Theorem 5 to show that this equation has at least one positive solution. We first note that (65) is nonresonant and the associated Green function G is nonnegative. In fact, we have

$$G(t, s)$$

$$= \frac{1}{2(1 - \cos 1)} \begin{cases} \sin{(t - s)} + \sin{(1 - t + s)}, & 0 \le s \le t \le 1; \\ \sin{(s - t)} + \sin{(1 - s + t)}, & 0 \le t \le s \le 1. \end{cases} \tag{66}$$

Furthermore, there also holds $\int_0^1 G(t, s) ds = 1$. In order to verify the conditions in Theorem 5, we set for any $t \in [0, 1]$ and any $x \in (0, +\infty)$

$$a(t) = 1, \qquad e(t) = -0.03,$$

$$f(t, x) = \frac{\sin^2{(2\pi tx)} + 1}{x^{1/2}} + x^{1/8},$$

$$b(t) = 2, \qquad c(t) = d(t) = 1, \qquad \alpha = \frac{1}{2}, \qquad \beta = \frac{1}{8}. \tag{67}$$

It is easy to see that there hold $1 - \alpha - \alpha^2 - \beta = (1/8) > 0$ and, for any $t \in [0, 1]$ and $x \in (0, +\infty)$,

$$c(t) x^{-\alpha} < f(t, x) < b(t) x^{-\alpha} + d(t) x^\beta. \tag{68}$$

To verify (29), we point out that

$$\tilde{c}(t) = \tilde{d}(t) = \frac{1}{2}\tilde{b}(t) = \int_0^1 G(t, s) ds = 1,$$

$$\tilde{e}(t) = \int_0^1 G(t, s) e(s) ds = -0.03, \tag{69}$$

which lead to $\tilde{b}^* = 2$, $\tilde{d}^* = \tilde{c}_* = 1$, and $\tilde{e}_* = \tilde{e}^* = -0.03$. By numerical calculation, we get the unique positive solution of equation

$$\left(2m^{3/4} + m^{1/8}\right)\left(2m^{3/4} + \frac{3}{8}m^{1/8}\right) = \frac{1}{4} \tag{70}$$

as $m_0 \doteq 0.02123$. Hence, we have the desired inequalities

$$m_0^\alpha \left[\left(\tilde{b}^* m_0^{1-\alpha^2} + \tilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\right)^{1/\alpha} - \tilde{c}_*\right] \tag{71}$$

$$= m_0^{1/2}\left(2m_0^{3/4} + m_0^{1/8}\right) \doteq -0.03948 < \tilde{e}_* < 0.$$

Since all the conditions are satisfied, Theorem 5 guarantees that (65) has a positive solution x (see Figure 1, numerical simulation for $e(t) = -0.03$).
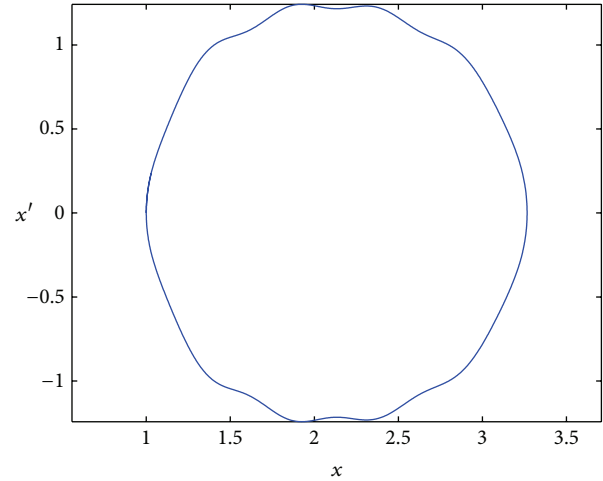

FIGURE 1: Numerical simulation for $e(t) = -0.03$.

Example 2. Consider the periodic problem for a second-order singular equation

$$x'' + x = \frac{\sin^2{(2\pi(t + x))} + 1}{x^{1/2}} + x^{1/8}$$

$$+ 0.03\left(\sin{(2\pi t)} - \max{\{p(t)\}}\right), \tag{72}$$

where

$$p(t) = \int_0^1 G(t, s) \sin{(2\pi s)} ds = (1 - \cos 1)^{-1}\left(4\pi^2 - 1\right)^{-1}$$

$$\times \left[2\pi(1 - \cos{(2\pi t)}\cos{(1 - t)})\sin t\right.$$

$$\left. + \sin{(2\pi t)}(\cos{(1 - t)}\cos t - 1)\right]. \tag{73}$$

In this example, we note that there hold

$$a(t) = 1, \qquad e(t) = 0.03\left(\sin{(2\pi t)} - \max{\{p(t)\}}\right),$$

$$f(t, x) = \frac{\sin^2{(2\pi(t + x))} + 1}{x^{1/2}} + x^{1/8}, \tag{74}$$

$$t \in [0, 1], \quad x \in (0, +\infty).$$

To meet the conditions in Theorem 7, we also set for any $t \in [0, 1]$ and any $x \in (0, +\infty)$

$$b(t) = 2, \qquad c(t) = d(t) = 1, \qquad \alpha = \frac{1}{2}, \qquad \beta = \frac{1}{8}. \tag{75}$$

It is clear that there hold $1 - \alpha - \alpha^2 - \beta = (1/8) > 0$ and, for any $t \in [0, 1]$ and $x \in (0, +\infty)$,

$$c(t) x^{-\alpha} < f(t, x) < b(t) x^{-\alpha} + d(t) x^\beta. \tag{76}$$

By direct calculation, we have $\tilde{b}^* = 2$, $\tilde{d}^* = \tilde{c}_* = 1$, and

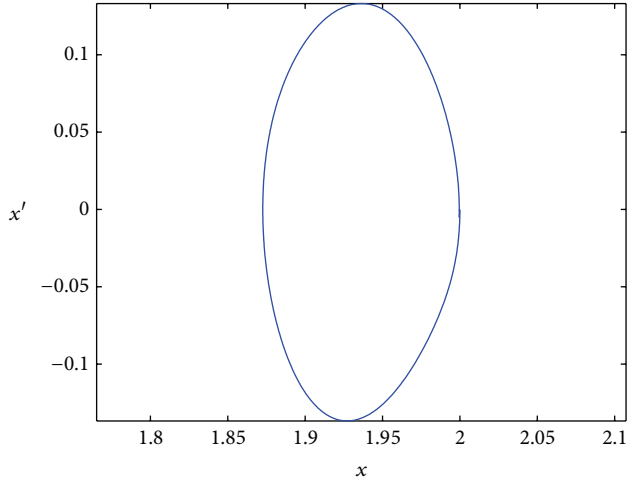$$-\tilde{e}_* = \tilde{e}^* = 0.015 \max{\{p(t)\}} \doteq 0.0053706, \tag{77}$$

Figure 2: Numerical simulation for $e(t) = 0.03 \times (\sin(2\pi t) - 0.179025044320086)$.

which leads us to consider the equation

$$
\left(2m^{3/4} + m^{1/8}\left(1 + \widetilde{e}^* m\right)^{9/8}\right)
$$

$$
\times \left(2m^{3/4}\left(1 + \widetilde{e}^* m\right)^{\alpha+1} + \left(\frac{3}{8} + \widetilde{e}^* m\right)m^{1/8}\left(1 + \widetilde{e}^* m\right)^{9/8}\right)
$$

$$
= \frac{1}{4}. \tag{78}
$$

By numerical calculation, we get the unique positive solution of the above equation as $m_0 = 0.017206$. Thus, we get

$$
m_0^\alpha \left[\left(\widetilde{b}^* m_0^{1-\alpha^2} + \widetilde{d}^* m_0^{1-\alpha-\beta-\alpha^2}\left(1 + \widetilde{e}^* m_0\right)^{\alpha+\beta}\right)^{1/\alpha}\right.
$$

$$
\left. - \widetilde{c}_*\left(1 + \widetilde{e}^* m_0\right)^{-\alpha}\right]
$$

$$
= m_0^{1/2}\left[\left(2m_0^{3/4} + m_0^{1/8}\left(1 + \widetilde{e}^* m_0\right)^{5/8}\right)^2\right. \tag{79}
$$

$$
\left. - \left(1 + \widetilde{e}^* m_0\right)^{-1/2}\right] \doteq -0.06804 < \widetilde{e}_*.
$$

Together with the fact that $\widetilde{e}_* < 0 < \widetilde{e}^*$, the above inequality shows that all the conditions of Theorem 7 are satisfied. Consequently, Theorem 7 guarantees that (72) has a positive solution x (see Figure 2, numerical simulation).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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