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Research Article

Weighted Morrey Estimates for Multilinear Fourier Multiplier Operators

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The multilinear Fourier multipliers and their commutators with Sobolev regularity are studied. The purpose of this paper is to establish that these operators are bounded on certain product Morrey spaces $L^{p,k}(\mathbb{R}^n)$. Based on the boundedness of these operators from $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ to $L^p(\prod_{j=1}^m \omega^{p/p_j})$, we obtained that they are also bounded from $L^{p_1,k}(\omega_1)\times\cdots\times L^{p_m,k}(\omega_m)$ to $L^{p,k}(\prod_{j=1}^m \omega^{p/p_j})$, with $0<k<1$, $1<p_j<\infty$, $1/p=1/p_1+\cdots+1/p_m$, and $\omega_j\in A_{p_j}$, $j=1,\ldots,m$.

1. Introduction

Recently some authors have taken so much interest in the text of multilinear Fourier multipliers with Sobolev regularity. To state some interesting results, we recall some necessary notations and definitions. Let $\sigma\in L^\infty(\mathbb{R}^{mn})$; the multilinear Fourier multiplier operator T_σ is defined by

$$T_\sigma\left(\vec{f}\right)(x)=\int_{\mathbb{R}^{mn}}\exp\left(2\pi ix\left(\xi_1+\cdots+\xi_m\right)\right)\times\sigma\left(\xi_1,\ldots,\xi_m\right)\widehat{f}_1\left(\xi_1\right)\cdots\widehat{f}_m\left(\xi_m\right)d\vec{\xi} \tag{1}$$

for all $\vec{f}=(f_1,\ldots,f_m)\in\mathcal{S}(\mathbb{R}^n)^m$, where $d\vec{\xi}=d\xi_1\cdots d\xi_m$ and $\widehat{f}$ is the Fourier transform of f. It is well known that [1] the boundedness of T_m from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ holds if $\sigma\in C^s(\mathbb{R}^{mn}\setminus\{0\})$ satisfying the condition

$$\left|\partial_\xi^\alpha\sigma(\xi)\right|\le C_\alpha|\xi|^{-|\alpha|} \tag{2}$$

for all multi-indice $|\alpha|\le s$ with $s\ge 2mn+1$ and all $1<p,p_1,\ldots,p_m<\infty$ with $1/p_1+\cdots+1/p_m=1/p$. Grafakos and Torres [2] improved the multiplier theorem of Coifman and Meyer to the indices $1/m\le p\le 1$ by the multilinear Calderón-Zygmund operator theory in the case of $s\ge mn+1$.

An important progress in this topic was given by Tomita. Let $\Phi\in\mathcal{S}(\mathbb{R}^{mn})$ satisfy

$$\begin{gathered}\operatorname{supp}\Phi\subset\left\{\left(\xi_1,\ldots,\xi_m\right):\frac{1}{2}\le\sum_{k=1}^m\left|\xi_k\right|\le 2\right\};\\ \sum_{l\in\mathbb{Z}}\Phi\left(2^{-l}\xi_1,\ldots,2^{-l}\xi_m\right)=1,\\ \text{for all }\left(\xi_1,\ldots,\xi_m\right)\in\mathbb{R}^{mn}\setminus\{0\}.\end{gathered} \tag{3}$$

Set

$$\begin{gathered}\sigma_l\left(\xi_1,\ldots,\xi_m\right)=\Phi\left(\xi_1,\ldots,\xi_m\right)\sigma\left(2^l\xi_1,\ldots,2^l\xi_m\right),\\ \left\|\sigma_l\right\|_{W^s(\mathbb{R}^{mn})}=\left(\int_{\mathbb{R}^{mn}}\left(1+\left|\xi_1\right|^2+\cdots+\left|\xi_m\right|^2\right)^s\times\left|\widehat{\sigma}\left(\xi_1,\ldots,\xi_m\right)\right|^2d\vec{\xi}\right)^{1/2}.\end{gathered} \tag{4}$$

Tomita [3] proved that if

$$\sup_{l\in\mathbb{Z}}\left\|\sigma_l\right\|_{W^s(\mathbb{R}^{mn})}<\infty, \tag{5}$$

for some $s \in (mn/2, \infty)$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $1 < p, p_1, \ldots, p_m < \infty$ and $1/p = \sum_{k=1}^m 1/p_k$. Grafakos and Si in [4] obtained that T_σ maps from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, if σ satisfies (5) and $1/m \le p \le 1$. Miyachi and Tomita [5] considered the problem to find minimal smoothness condition for multilinear Fourier multiplier. Let

$$\|\sigma_l\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})} = \left(\int_{\mathbb{R}^{2n}} \langle\xi_1\rangle^{2s_1} \cdots \langle\xi_m\rangle^{2s_m} \left|\widehat{\sigma}_l(\xi_1, \ldots, \xi_m)\right|^2 d\vec{\xi}\right)^{1/2}, \tag{6}$$

where $\langle\xi_k\rangle := (1 + |\xi_k|^2)^{1/2}$. Miyachi and Tomita [5] proved that if

$$\sup_{l\in\mathbb{Z}} \|\sigma_l\|_{W^{s_1,s_2}(\mathbb{R}^{2n})} < \infty, \tag{7}$$

for each $s_j \in (n/2, n]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $1 < p_1, p_2 < \infty$, and $p > 2/3$ with $1/p = \sum_{k=1}^2 1/p_k$. Moreover, they also gave minimal smoothness condition for which T_σ is bounded from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Let $mn/2 < s \le mn$, $mn/s < p_1, \ldots, p_m < \infty$, and $1/p_1 + \cdots + 1/p_m = 1/p$. Fujita and Tomita [6] proved the following inequality:

$$\|T_\sigma(f_1, \ldots, f_m)\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{k=1}^m \|f_k\|_{L^{p_k}(\omega_k)}, \tag{8}$$

if $\|\sigma_l\|_{W^{s/m,\ldots,s/m}(\mathbb{R}^{mn})} < \infty$ and $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{p_1 s/(mn)} \times \cdots \times A_{p_m s/(mn)}$, where and in what follows $\nu_{\vec{\omega}} = \prod_{k=1}^m \omega_k^{p/p_k}$. Li et al. [7] obtained the endpoint cases. Hu and Lin [8] also obtained this result from another approach. Replacing $W^{s_1,\ldots,s_m}$ by W^s, Bui and Duong [9] and Li and Sun [10] proved that if $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{(p_1 s/(mn),\ldots,p_m s/(mn))}$, then (8) also holds. Jiao [11] gave a generalization of the above inequality with the class $A_{\vec{P}/\vec{Q}}$, which generalizes the class $A_{\vec{P}}$ introduced by Lerner et al. [12]. Fujita and Tomita showed a counterexample to answer the question whether the inequality (8) holds under the conditions $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{(p_1 s/(mn),\ldots,p_m s/(mn))}$ and $\|\sigma_l\|_{W^{s/m,\ldots,s/m}(\mathbb{R}^{mn})} < \infty$.

We still recall the weighted Morrey spaces which were introduced by Komori and Shirai [13]. A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $1 < p < \infty$; a weight function ω is said to belong to the class A_p, if there is a constant C such that for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} \le C, \tag{9}$$

and ω belongs to the class A_1, if there is a constant C such that, for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(x)\,dx \le C \inf_{x\in Q} \omega(x). \tag{10}$$

We denote $A_\infty = \cup_{p>1} A_p$.

Definition 1 (See [13]). Let $1 \le p < \infty$, let $0 < \kappa < 1$, and let ω be a weight function on $\mathbb{R}^n$. The weighted Morrey space is defined by

$$L^{p,\kappa}(\omega) = \left\{f \in L^p_{\mathrm{loc}} : \|f\|_{L^{p,\kappa}(\omega)} < \infty\right\}, \tag{11}$$

where

$$\|f\|_{L^{p,\kappa}(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)^k}\int_Q |f(x)|^p \omega(x)\,dx\right)^{1/p}. \tag{12}$$

Our main results can be stated as follows.

Theorem 2. *Let σ be a multiplier satisfying*

$$\|\sigma_l\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})} < \infty, \tag{13}$$

for $s_1, \ldots, s_m \in (n/2, n]$ and let T_σ be the operator defined by (1) and $0 < \kappa < 1$. Set $t_j = n/s_j$. If $p_j \in (t_j, \infty)$ and the weight $\omega_j \in A_{p_j/t_j}(\mathbb{R}^n)$ for $1 \le j \le m$ and $p \in [1, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, then

$$\|T_\sigma(f_1, \ldots, f_m)\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j,\kappa}(\omega_j)}, \tag{14}$$

where $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$.

Given a multilinear Fourier multiplier operator T_σ and $\vec{b} = (b_1, \ldots, b_m) \in \mathrm{BMO}(\mathbb{R}^n)^m$, we define the commutators $T_{\sigma,\Sigma\mathbf{b}}(\vec{f})(x)$ to be

$$T_{\sigma,\Sigma\mathbf{b}}(\vec{f})(x) = \sum_{j=1}^m [b_j, T_\sigma]_j (f_1, \ldots, f_m)(x), \tag{15}$$

with

$$\begin{aligned}[b_j, T_\sigma]_j (f_1, \ldots, f_m)(x) &= b_j(x) T_\sigma(f_1, \ldots, f_j, \ldots, f_m)(x) \\ &\quad - T_\sigma(f_1, \ldots, b_j f_j, \ldots, f_m)(x).\end{aligned} \tag{16}$$

Theorem 3. *Let σ be a multiplier satisfying*

$$\|\sigma_l\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})} < \infty, \tag{17}$$

for $s_1, \ldots, s_m \in (n/2, n]$ and let T_σ be the operator defined by (1) and $0 < \kappa < 1$. Set $t_j = n/s_j$. If $p_j \in (t_j, \infty)$ and the weight $\omega_j \in A_{p_j/t_j}(\mathbb{R}^n)$ for $1 \le j \le m$ and $p \in [1, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, then for any $b_1, \ldots, b_m \in BMO(\mathbb{R}^n)$,

$$\begin{aligned}&\|T_{\sigma,\Sigma\mathbf{b}}(f_1, \ldots, f_m)\|_{L^{p,\kappa}(\mathbb{R}^n,\nu_{\vec{\omega}})} \\ &\quad \le C\|\vec{b}\|_{BMO^m} \prod_{j=1}^m \|f_j\|_{L^{p_j,\kappa}(\mathbb{R}^n,\omega_j)},\end{aligned} \tag{18}$$

where $\|\vec{b}\|_{BMO^m} = \prod_{j=1}^m \|b_j\|_{BMO}$ and $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$.

Because the regularity condition $\|\sigma_l\|_{W^s(\mathbb{R}^{mn})} < \infty$ is stronger than that of $\|\sigma_l\|_{W^{s_1,\dots,s_m}(\mathbb{R}^{mn})} < \infty$, we have the following corollaries.

Corollary 4. *Let σ be a multiplier satisfying*

$$\|\sigma_l\|_{W^s(\mathbb{R}^{mn})} < \infty, \tag{19}$$

for $s \in (mn/2, mn]$ and let T_σ be the operator defined by (1) *and $0 < \kappa < 1$. Set $r = mn/s$. If $p_j \in (mn/s, \infty)$ and the weight $\omega_j \in A_{p_j/r}(\mathbb{R}^n)$ for $1 \le j \le m$ and $p \in [1, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, then*

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\kappa}(\omega_j)}, \tag{20}$$

where $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}$.

Corollary 5. *Let σ be a multiplier satisfying*

$$\|\sigma_l\|_{W^s(\mathbb{R}^{mn})} < \infty, \tag{21}$$

for $s \in (mn/2, mn]$ and let T_σ be the operator defined by (1) *and $0 < \kappa < 1$. Set $r = mn/s$. If $p_j \in (mn/s, \infty)$ and the weight $\omega_j \in A_{p_j/r}(\mathbb{R}^n)$ for $1 \le j \le m$ and $p \in [1, \infty)$ such that $1/p = 1/p_1 + \cdots + 1/p_m$, then, for any $b_1, \dots, b_m \in BMO(\mathbb{R}^n)$,*

$$\|T_{\sigma,\Sigma\mathbf{b}}(f_1, \dots, f_m)\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C \|\vec{b}\|_{BMO^m} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\kappa}(\omega_j)}, \tag{22}$$

where $\|\vec{b}\|_{BMO^m} = \prod_{j=1}^{m} \|b_j\|_{BMO}$ and $\nu_{\vec{\omega}} = \prod_{j=1}^{m} \omega_j^{p/p_j}$.

Remark 6. For $m = 1$ and $\omega \in A_p$, we also extend Hörmander's theorem [14] to the weighted Morrey spaces.

2. Some Notations and Lemmas

We begin with the definitions of Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \tag{23}$$

and of the sharp maximal function,

$$M^\sharp(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c|\, dy. \tag{24}$$

For $\delta > 0$, we also define the following maximal functions $M_\delta(f) = M(|f|^\delta)^{1/\delta}$ and $M_\delta^\sharp(f) = M^\sharp(|f|^\delta)^{1/\delta}$. The following classical result belongs to Fefferman and Stein [15].

Lemma 7. *Let $0 < p, \delta < \infty$, and $\omega \in A_\infty$. Then there exists some constant $C_{n,p,\delta,\omega}$ such that*

$$\|M_\delta(f)\|_{L^p(\omega)} \le C_{n,p,\delta,\omega} \|M_\delta^\sharp(f)\|_{L^p(\omega)}. \tag{25}$$

Similarly, we have the responding lemma on weighted Morrey spaces as a consequent result.

Lemma 8. *Let $0 < \kappa < 1$, $0 < p, \delta < \infty$, and $\omega \in A_\infty$. Then there exists some constant $C_{n,p,\delta,\omega}$ such that*

$$\|M_\delta(f)\|_{L^{p,\kappa}(\omega)} \le C_{n,p,\delta,\omega} \|M_\delta^\sharp(f)\|_{L^{p,\kappa}(\omega)}. \tag{26}$$

For $\vec{f} = (f_1, \dots, f_m)$, $r_i > 0$, $i = 1, \dots, m$, and set $\vec{r} = (r_1, \dots, r_m)$, we define

$$\mathscr{M}_{\vec{r}}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \left(\frac{1}{|Q|} \int_Q |f_l(y_j)|^{r_j} dy_j \right)^{1/r_j}. \tag{27}$$

This maximal function is the generalization of $\mathscr{M}$ which is introduced by Lerner et al. [12], we refer to [11] for some properties of it. The following lemma is the special example of [11, Theorem 2.1].

Lemma 9. *Let $p_1, \dots, p_m, p \in (0, \infty)$, $r_j \in (0, p_j)$, and $\omega_j \in A_{p_j/r_j}$ for $1 \le j \le m$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Then we have*

$$\|\mathscr{M}_{\vec{r}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)}, \tag{28}$$

and if at least one $r_l = p_l$, then

$$\|\mathscr{M}_{\vec{r}}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)}, \tag{29}$$

where $\nu_{\vec{\omega}} = \prod_{l=1}^{m} \omega_l^{p/p_l}$.

Lemma 10. *Let $\kappa \in (0,1)$, $p_1, \dots, p_m, p \in (0, \infty)$, $r_j \in (0, p_l)$, and $\omega_j \in A_{p_j/r_j}$ for $1 \le j \le m$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Then we have*

$$\|\mathscr{M}_{\vec{r}}(\vec{f})\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\kappa}(\omega_j)}. \tag{30}$$

Proof. From [11], there exists some $q \in (0, 1)$ such that

$$\mathscr{M}_{\vec{r}}(\vec{f})(x) \le C \prod_{j=1}^{m} \left\{ M_{\nu_{\vec{\omega}}}^c \left(\left(\frac{|f_j|^{p_j} \omega_j}{\nu_{\vec{\omega}}} \right)^q \right)(x) \right\}^{1/(qp_j)}, \tag{31}$$

where $M^c_{\nu_{\vec{\omega}}}$ is the weighted centered maximal operator. Then by the Hölder inequality and [13, Theorem 3.1], we get

$$\begin{aligned}
&\left\|\mathscr{M}_{\vec{r}}(\vec{f})(x)\right\|_{L^{p,k}(\nu_{\vec{\omega}})} \\
&\quad\le C\left\|\prod_{j=1}^{m}\left\{M^c_{\nu_{\vec{\omega}}}\left(\left[\frac{|f_j|^{p_j}\omega_j}{\nu_{\vec{\omega}}}\right]^q\right)\right\}^{1/(qp_j)}\right\|_{L^{p,k}(\nu_{\vec{\omega}})} \\
&\quad\le C\prod_{j=1}^{m}\left\|\left\{M^c_{\nu_{\vec{\omega}}}\left(\left[\frac{|f_j|^{p_j}\omega_j}{\nu_{\vec{\omega}}}\right]^q\right)\right\}^{1/(qp_j)}\right\|_{L^{p_j,k}(\nu_{\vec{\omega}})} \\
&\quad\le C\prod_{j=1}^{m}\left\|M^c_{\nu_{\vec{\omega}}}\left(\left[\frac{|f_j|^{p_j}\omega_j}{\nu_{\vec{\omega}}}\right]^q\right)\right\|^{1/(qp_j)}_{L^{1/q,k}(\nu_{\vec{\omega}})} \\
&\quad\le C\prod_{j=1}^{m}\left\|\left(\frac{|f_j|^{p_j}\omega_j}{\nu_{\vec{\omega}}}\right)^q\right\|^{1/(qp_j)}_{L^{1/q,k}(\nu_{\vec{\omega}})} \\
&\quad\le C\prod_{j=1}^{m}\left\|f_j\right\|_{L^{p_j,k}(\omega_j)}.
\end{aligned} \tag{32}$$

□

Lemma 11 (See [6]). *Let $1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Suppose that $\sigma \in L^\infty(\mathbb{R}^{mn})$ satisfies*

$$\|\sigma_l\|_{W^{s_1,\ldots,s_m}(\mathbb{R}^{mn})} < \infty. \tag{33}$$

Then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

For $q_1, \ldots, q_m \in (0,\infty)$ and $s_1, \ldots, s_m \in \mathbb{R}$, the weighted Lebesgue space of mixed type $L^{(q_1,\ldots,q_m)}(\omega_{(s_1,\ldots,s_m)})$ is defined by the norm

$$\begin{aligned}
&\|F\|_{L^{(q_1,\ldots,q_m)}(\omega_{(s_1,\ldots,s_m)})} \\
&\quad=\left[\int_{\mathbb{R}^n}\cdots\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x)|^{q_1}\langle x_1\rangle^{s_1}dx_1\right)^{q_2/q_1}\right.\right. \\
&\qquad\left.\left.\times\langle x_2\rangle^{s_2}dx_2\right\}^{q_3/q_2}\cdots\langle x_m\rangle^{s_m}dx_m\right]^{1/q_m}.
\end{aligned} \tag{34}$$

Lemma 12 (See [6]). *Let $r > 0$, $2 \le q_j < \infty$, and $s_j \ge 0$ for $1 \le j \le m$. Then there exists a constant $C > 0$ such that*

$$\left\|\widehat{F}\right\|_{L^{(q_1,\ldots,q_m)}(\omega_{(s_1,\ldots,s_m)})} \le C\|F\|_{W^{s_1/q_1,\ldots,s_m/q_m}}, \tag{35}$$

for all $F \in W^{s_1/q_1,\ldots,s_m/q_m}(\mathbb{R}^{mn})$ with supp F $\subset \{|x_1|^2\cdots + |x_m|^2 \le r\}$.

By the reverse Hölder inequality, we have the following lemma.

Lemma 13. *Assume that $\vec{\omega} \in \prod_{j=1}^m A_{p_j}$, with $1 < p_1, \ldots, p_m < \infty$. Let $n/2 < s_j \le n$; then there exist constants $1 < \epsilon_j < \min\{p_j, s_j/(s_j-1), 2s_j/n\}$ such that $\omega_j \in A_{p_j/\epsilon_j}$.*

The following lemma is the key to our main results.

Lemma 14. *Let "σ" be a multplier satisfying*

$$\|\sigma_l\|_{W^{(s_1,\ldots,s_m)}(\mathbb{R}^n)} < \infty, \tag{36}$$

for $s_1, \ldots, s_m \in (n/2, n]$ and let T_σ be the operator defined by (1). If $1 < p_j < \infty$, $t_j = n/s_j$ and $0 < \delta < r/m$, where $1/r = 1/r_1+\cdots+1/r_m$, $r_j = \epsilon_j t_j$ and $1 < \epsilon_j < \min\{p_j, s_j/(s_j-1), 2s_j/n\}$. Then for all $\vec{f} \in L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ with $r_j \le p_j < \infty$ for $1 \le j \le m$,

$$M^\sharp_\delta\left(T_\sigma\left(\vec{f}\right)\right)(x) \le C\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x), \tag{37}$$

where $\vec{r} = (r_1, \ldots, r_m)$.

Proof. By Lemma 13, $1 < t_j\epsilon_j \le 2$; then $r_j/m \le 1$. Fix a point x and a cube Q such that $x \in Q$. It suffices to prove that

$$\left(\frac{1}{|Q|}\int_Q\left|T_\sigma\left(\vec{f}\right)(z) - c_Q\right|^\delta dz\right)^{1/\delta} \le C\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x), \tag{38}$$

for some constant c_Q. We decompose $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j\chi_{Q^*}$ for all $j = 1, \ldots, m$ and $Q^* = 4\sqrt{n}Q$. Then

$$\begin{aligned}
\prod_{j=1}^m f_j\left(y_j\right) &= \prod_{j=1}^m\left(f_j^0\left(y_j\right) + f_j^\infty\left(y_j\right)\right) \\
&= \sum_{\alpha_1,\ldots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}\left(y_1\right)\cdots f_m^{\alpha_m}\left(y_m\right) \\
&= \prod_{j=1}^m f_j^0\left(y_j\right) + \sum_{\alpha_1,\ldots,\alpha_m\in\mathscr{I}} f_1^{\alpha_1}\left(y_1\right)\cdots f_m^{\alpha_m}\left(y_m\right),
\end{aligned} \tag{39}$$

where $\mathscr{I} = \{\alpha_1, \ldots, \alpha_m$: there is at least one $\alpha_j \ne 0\}$. Then we can write

$$\begin{aligned}
T_\sigma\left(\vec{f}\right)(z) &= T_\sigma\left(\vec{f}^0\right)(z) + \sum_{\alpha_1,\ldots,\alpha_m\in\mathscr{I}} T_\sigma\left(f_1^{\alpha_1}\cdots f_m^{\alpha_m}\right)(z) \\
&:= I + II.
\end{aligned} \tag{40}$$

Applying Kolmogorov's inequality to I, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\left|T_\sigma\left(\vec{f}^0\right)(z)\right|^\delta dz\right)^{1/\delta} \\
&\quad\le C\left\|T_\sigma\left(\vec{f}^0\right)\right\|_{L^{r,\infty}(Q,dx/|Q|)} \\
&\quad\le C\prod_{j=1}^m\left(\frac{1}{|Q^*|}\int_{Q^*}\left|f_j\left(y_j\right)\right|^{r_l}dy_j\right)^{r_j} \\
&\quad\le C\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x),
\end{aligned} \tag{41}$$

since T_σ is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ to L^r by Lemma 11.

Taking

$$c_Q=\sum_{\alpha_1,\dots,\alpha_m\in\mathcal{I}} T_\sigma\left(f_1^{\alpha_1}\cdots f_m^{\alpha_m}\right)(x), \tag{42}$$

we claim that, for any $z\in Q$,

$$\sum_{\alpha_1,\dots,\alpha_m\in\mathcal{I}}\left|T_\sigma\left(f_1^{\alpha_1}\cdots f_m^{\alpha_m}\right)(z)-T_\sigma\left(f_1^{\alpha_1}\cdots f_m^{\alpha_m}\right)(x)\right| \le C\mathcal{M}_{\vec{r}}\left(\vec{f}\right)(x). \tag{43}$$

Let

$$W_l\left(x,z;y_1,\dots,y_m\right)=\check{\sigma}_l\left(x-y_1,\dots,x-y_m\right)-\check{\sigma}_l\left(z-y_1,\dots,z-y_m\right). \tag{44}$$

At first we consider the case $\alpha_1=\cdots=\alpha_m$,

$$\begin{aligned}
&\left|T_\sigma\left(f_1^\infty\cdots f_m^\infty\right)(z)-T_\sigma\left(f_1^\infty\cdots f_m^\infty\right)(x)\right|\\
&\le\sum_{l\in\mathbb{Z}}\left|T_{\sigma_l}\left(f_1^\infty\cdots f_m^\infty\right)(z)-T_{\sigma_l}\left(f_1^\infty\cdots f_m^\infty\right)(x)\right|\le\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^{mn}\setminus(Q^*)^m}\left|W_l\left(x,z;y_1,\dots,y_m\right)\right|\prod_{j=1}^m f_j\left(y_j\right)d\vec{y}\\
&\le\sum_{l\in\mathbb{Z}}\sum_{k=0}^\infty\int_{(2^{k+1}Q^*\setminus 2^kQ^*)^m}\left|W_l\left(x,z;y_1,\dots,y_m\right)\right|\prod_{j=1}^m f_j\left(y_j\right)d\vec{y}\\
&\le\sum_{k=0}^\infty\sum_{l\in\mathbb{Z}}\prod_{j=1}^m\left(\int_{2^{k+1}Q^*}\left|f_j\left(y_j\right)\right|^{r_j}dy_j\right)^{1/r_j}\\
&\quad\times\left(\int_{(2^{k+1}Q^*)\setminus(2^kQ^*)}\left(\int_{(2^{k+1}Q^*)\setminus(2^kQ^*)}\cdots\left(\int_{(2^{k+1}Q^*)\setminus(2^kQ^*)}\left|W_l\left(x,z;y_1,\dots,y_m\right)\right|^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}dy_m\right)^{1/r_m'}\\
&:=\sum_{k=0}^\infty\sum_{l\in\mathbb{Z}}\prod_{j=1}^m\left(\int_{2^{k+1}Q^*}\left|f_j\left(y_j\right)\right|^{r_j}dy_j\right)^{1/r_j}II_{k,l}^{\infty,\dots,\infty}.
\end{aligned} \tag{45}$$

Denote $h=z-x$ and $\widetilde{Q}=x-Q^*$; it follows from Lemma 12 that

$$\begin{aligned}
&II_{k,l}^{\infty,\dots,\infty}\\
&=\left(\int_{(2^{k+1}\widetilde{Q})\setminus(2^k\widetilde{Q})}\left(\int_{(2^{k+1}\widetilde{Q})\setminus(2^k\widetilde{Q})}\cdots\left(\int_{(2^{k+1}\widetilde{Q})\setminus(2^k\widetilde{Q})}\left|\check{\sigma}_l\left(h+y_1,\dots,h+y_m\right)-\check{\sigma}_l\left(y_1,\dots,y_m\right)\right|^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}dy_m\right)^{1/r_m'}\\
&\le 2\left(\int_{c_12^k\ell(Q)\le|y_m|<c_22^{k+1}\ell(Q)}\left(\int_{c_12^k\ell(Q)\le|y_{m-1}|<c_22^{k+1}\ell(Q)}\cdots\left(\int_{c_12^k\ell(Q)\le|y_1|<c_22^{k+1}\ell(Q)}\left|\check{\sigma}_l\left(y_1,\dots,y_m\right)\right|^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}dy_m\right)^{1/r_m'}\\
&\le C\left(2^kl(Q)\right)^{-(s_1+\cdots+s_m)}\\
&\quad\times\left(\int_{c_12^k\ell(Q)\le|y_m|<c_22^{k+1}\ell(Q)}\left(\int_{c_12^k\ell(Q)\le|y_{m-1}|<c_22^{k+1}\ell(Q)}\cdots\left(\int_{c_12^k\ell(Q)\le|y_1|<c_22^{k+1}\ell(Q)}\left|\check{\sigma}_l\left(y_1,\dots,y_m\right)\right|^{r_1'}\langle y_1\rangle^{s_1r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}\right.\\
&\qquad\left.\times\langle y_m\rangle^{s_mr_m'}dy_m\right)^{1/r_m'}
\end{aligned}$$

$$\begin{aligned}
&\leq C\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} 2^{l(s_1+\cdots+s_m)} \\
&\quad\times\Bigg(\int_{c_1 2^k \ell(Q)\leq|y_m|<c_2 2^{k+1}\ell(Q)} \\
&\qquad\left(\int_{c_1 2^k \ell(Q)\leq|y_{m-1}|<c_2 2^{k+1}\ell(Q)}\cdots\left(\int_{c_1 2^k \ell(Q)\leq|y_1|<c_2 2^{k+1}\ell(Q)}\left|2^{-lmn}\check{\sigma}_l\left(2^{-l}y_1,\ldots,2^{-l}y_m\right)\right|^{r_1'}\left\langle 2^{-l}y_1\right\rangle^{s_1 r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'} \\
&\qquad\times\left\langle 2^{-l}y_m\right\rangle^{s_m r_m'}dy_m\Bigg)^{1/r_m'} \\
&\leq C\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} 2^{l(s_1+\cdots+s_m)} 2^{-lmn} 2^{-l(n/r_1'+\cdots+n/r_m')} \\
&\quad\times\Bigg(\int_{c_1 2^k \ell(Q)\leq|y_m|<c_2 2^{k+1}\ell(Q)}\left(\int_{c_1 2^k \ell(Q)\leq|y_{m-1}|<c_2 2^{k+1}\ell(Q)}\cdots\left(\int_{c_1 2^k \ell(Q)\leq|y_1|<c_2 2^{k+1}\ell(Q)}\left|\check{\sigma}_l\left(z_1,\ldots,z_m\right)\right|^{r_1'}\left\langle z_1\right\rangle^{s_1 r_1'}dz_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'} \\
&\qquad\times\left\langle z_m\right\rangle^{s_m r_m'}dz_m\Bigg)^{1/r_m'} \\
&\leq C\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} 2^{-l(n/r_1+\cdots+n/r_m-s_1-\cdots-s_m)}\left\|\sigma_l\right\|_{W^{s_1,\ldots,s_m}}.
\end{aligned} \tag{46}$$

Given that $2^{l_0}\leq l(Q)\leq 2^{l_0+1}$, then we have that

$$\begin{aligned}
&\sum_{l<l_0} II_{k,l}^{\infty,\ldots,\infty} \\
&\quad\leq C\sup_l\left\|\sigma_l\right\|_{W^{s_1,\ldots,s_m}}\sum_{l<l_0}\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} \\
&\qquad\times 2^{-l(n/r_1+\cdots+n/r_m-s_1-\cdots-s_m)} \\
&\quad\leq C\sup_l\left\|\sigma_l\right\|_{W^{s_1,\ldots,s_m}} 2^{-k(s_1+\cdots+s_m)} l(Q)^{-(n/r_1+\cdots+n/r_m)}.
\end{aligned} \tag{47}$$

On the other hand, a similar process follows in [10]; we get that

$$\begin{aligned}
II_{k,l}^{\infty,\ldots,\infty} &= \left(\int_{(2^{k+1}\tilde{Q})\setminus(2^k\tilde{Q})}\left(\int_{(2^{k+1}\tilde{Q})\setminus(2^k\tilde{Q})}\cdots\left(\int_{(2^{k+1}\tilde{Q})\setminus(2^k\tilde{Q})}\left|\check{\sigma}_l(h+y_1,\ldots,h+y_m)-\check{\sigma}_l(y_1,\ldots,y_m)\right|^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}dy_m\right)^{1/r_m'} \\
&\leq \Bigg(\int_{c_1 2^k \ell(Q)\leq|y_m|<c_2 2^{k+1}\ell(Q)}\Bigg(\int_{c_1 2^k \ell(Q)\leq|y_{m-1}|<c_2 2^{k+1}\ell(Q)}\cdots \\
&\qquad\left(\int_{c_1 2^k \ell(Q)\leq|y_1|<c_2 2^{k+1}\ell(Q)}\left(\int_0^1\left|\vec{h}\cdot\nabla\check{\sigma}_l\left(y_1+\theta h,\ldots,y_m+\theta h\right)\right|d\theta\right)^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots\Bigg)^{r_m'/r_{m-1}'}dy_m\Bigg)^{1/r_m'}
\end{aligned}$$

$$\leq \int_0^1 \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \left| \vec{h} \cdot \nabla \check{\sigma}_l \left(y_1 + \theta h, \ldots, y_m + \theta h \right) \right|^{r_1'} dy_1 \right)^{r_2'/r_1'} \cdots \right)^{r_m'/r_{m-1}'} dy_m \right)^{1/r_m'} d\theta$$

$$\leq \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \left| \vec{h} \cdot \nabla \check{\sigma}_l \left(y_1, \ldots, y_m \right) \right|^{r_1'} dy_1 \right)^{r_2'/r_1'} \cdots \right)^{r_m'/r_{m-1}'} dy_m \right)^{1/r_m'}, \tag{48}$$

where $\vec{h} = (h, \ldots, h) \in \mathbb{R}^{mn}$. Since

$$\vec{h} \cdot \nabla \check{\sigma}_l \left(y_1, \ldots, y_m \right) = \sum_{j=1}^{m} h_j \partial_j \nabla \check{\sigma}_l \left(y_1, \ldots, y_m \right), \tag{49}$$

we have

$$II_{k,l}^{\infty,\ldots,\infty}$$

$$\leq \sum_{j=1}^{m} \ell(Q) \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \left| \partial_j \cdot \check{\sigma}_l \left(y_1, \ldots, y_m \right) \right|^{r_1'} dy_1 \right)^{r_2'/r_1'} \cdots \right)^{r_m'/r_{m-1}'} dy_m \right)^{1/r_m'}$$

$$\leq \sum_{j=1}^{m} \ell(Q) \left(2^k l(Q) \right)^{-(s_1 + \cdots + s_m)}$$

$$\times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \left| \partial_j \check{\sigma}_l \left(y_1, \ldots, y_m \right) \right|^{r_1'} \langle y_1 \rangle^{s_1 r_1'} dy_1 \right)^{r_2'/r_1'} \cdots \right)^{r_m'/r_{m-1}'} \right.$$

$$\left. \times \langle y_m \rangle^{s_m r_m'} dy_m \right)^{1/r_m'}$$

$$\leq C \left(2^k l(Q) \right)^{-(s_1 + \cdots + s_m)} 2^{l(s_1 + \cdots + s_m)}$$

$$\times \left(\int_{c_1 2^k \ell(Q) \leq |y_m| < c_2 2^{k+1} \ell(Q)} \left(\int_{c_1 2^k \ell(Q) \leq |y_{m-1}| < c_2 2^{k+1} \ell(Q)} \cdots \left(\int_{c_1 2^k \ell(Q) \leq |y_1| < c_2 2^{k+1} \ell(Q)} \left| 2^{-lmn} \cdot \partial_j \check{\sigma}_l \left(2^{-l} y_1, \ldots, 2^{-l} y_m \right) \right|^{r_1'} \left\langle 2^{-l} y_1 \right\rangle^{s_1 r_1'} dy_1 \right)^{r_2'/r_1'} \cdots \right)^{r_m'/r_{m-1}'} \right.$$

$$\left. \times \left\langle 2^{-l} y_m \right\rangle^{s_m r_m'} dy_m \right)^{1/r_m'}$$

$$\le C\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} 2^{l(s_1+\cdots+s_m)} 2^{-lmn} 2^{-l(n/r_1'+\cdots+n/r_m')}$$
$$\times\left(\int_{c_1 2^k \ell(Q)\le|y_m|<c_2 2^{k+1}\ell(Q)}\left(\int_{c_1 2^k \ell(Q)\le|y_{m-1}|<c_2 2^{k+1}\ell(Q)}\cdots\left(\int_{c_1 2^k \ell(Q)\le|y_1|<c_2 2^{k+1}\ell(Q)}\left|\partial_j\check{\sigma}_l(z_1,\ldots,z_m)\right|^{r_1'}\langle z_1\rangle^{s_1 r_1'}dz_1\right)^{r_2'/r_1'}\cdots\right)^{r_m'/r_{m-1}'}\right.$$
$$\left.\times\langle z_m\rangle^{s_m r_m'}dz_m\right)^{1/r_m'}$$
$$\le C\left(2^k l(Q)\right)^{-(s_1+\cdots+s_m)} 2^{-l(n/r_1+\cdots+n/r_m+1-s_1-\cdots-s_m)}\left\|\sigma_l\right\|_{W^{s_1,\ldots,s_m}}. \tag{50}$$

From Lemma 13, $n/r_1+\cdots+n/r_m > s_1+\cdots+s_m-1$, it deduces that

$$\sum_{l\ge l_0} II_{k,l}^{\infty,\ldots,\infty} \le C\sup_l\left\|\sigma_l\right\|_{W^{s_1,\ldots,s_m}} 2^{-k(s_1+\cdots+s_m)} l(Q)^{-(n/r_1+\cdots+n/r_m)}. \tag{51}$$

So

$$\left|T_\sigma\left(f_1^\infty\cdots f_m^\infty\right)(z)-T_\sigma\left(f_1^\infty\cdots f_m^\infty\right)(x)\right|$$
$$\le C\sum_{k=0}^{\infty}2^{-k(s_1+\cdots+s_m-n/r_1-\cdots-n/r_m)}\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x)$$
$$\le C\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x). \tag{52}$$

It remains to consider the case that there exists a proper subset $\{j_1,\ldots,j_\gamma\}$ of $\{1,\ldots,m\}$, $1\le\gamma<m$, such that $\alpha_{j_1}=\cdots=\alpha_{j_\gamma}=0$. Without loss of generality, we write, for the case $\{j_1,\ldots,j_\gamma\}=\{1,\ldots,\gamma\}$,

$$\left|T_\sigma\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(z)-T_\sigma\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right|$$
$$\le\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}}\prod_{j=1}^{m}\left(\int_{2^{k+1}Q^*}\left|f_j\left(y_j\right)\right|^{r_j}d\vec{y}\right)^{1/r_j}$$
$$\times\left(\int_{2^{k+1}Q^*\setminus 2^kQ^*}\cdots\left(\int_{2^{k+1}Q^*\setminus 2^kQ^*}\left(\int_{Q^*}\cdots\left(\int_{Q^*}\left|W_l(x,z;y_1,\ldots,y_m)\right|^{r_1'}dy_1\right)^{r_2'/r_1'}\cdots dy_\gamma\right)^{r_{\gamma+1}'/r_\gamma'}dy_{\gamma+1}\right)^{r_{\gamma+2}'/r_{\gamma+1}'}\cdots dy_m\right)^{1/r_m'}. \tag{53}$$

The same argument as the case $\alpha_1=\cdots=\alpha_m=\infty$ computes that

$$\left|T_\sigma\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(z)-T_\sigma\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right| \le C\mathscr{M}_{\vec{r}}\left(\vec{f}\right)(x). \tag{54}$$

This completes the proof. □

Lemma 15. *"Let σ" be a multplier satisfying*

$$\left\|\sigma_l\right\|_{W^{(s_1,\cdots,s_m)}(\mathbb{R}^n)}<\infty, \tag{55}$$

for $s_1,\ldots,s_m\in(n/2,n]$ and let T_σ be the operator defined by (1). *If $1<p_j<\infty$, $t_j=n/s_j$ and $0<\delta<\varepsilon<r/m$, where $1/r=1/r1+\ldots+1/rm$, $r_j=\epsilon_j t_j$ and $1<\epsilon_j<\min\{p_j,s_j/(s_j-1),2s_j/n\}$, and let $\vec{b}\in BMO^m$. Then for any $\vec{\gamma}>\vec{r}$, that is, $\gamma_j>r_j$, $j=1,\ldots,m$, there exists some constant $C>0$ such that*

$$M_\delta^\sharp\left(T_{\sigma,\Sigma\mathbf{b}}\left(\vec{f}\right)\right)(x) \le C\left\|\vec{b}\right\|_{BMO^m}\left(M_\varepsilon\left(T_\sigma\left(\vec{f}\right)\right)(x)+\mathscr{M}_{\vec{\gamma}}\left(\vec{f}\right)(x)\right), \tag{56}$$

for all m-tuples $\vec{f}=(f_1,\ldots,f_m)$ of bounded measurable functions with compact support.

Proof. By linearity it is sufficient to consider the particular case when $\vec{b}=b\in BMO$. Fix $b\in$ BMO and consider the operator

$$T_{\sigma,b}\left(\vec{f}\right)(x)=b(x)T_\sigma\left(\vec{f}\right)(x)-T_\sigma\left(bf_1,f_2,\ldots,f_m\right)(x). \tag{57}$$

Fix $x \in \mathbb{R}^n$, for any cube Q with center at x; set $\lambda = b_{Q^*}$, where $Q^* = 4\sqrt{n}Q$. We have

$$\begin{aligned} T_{\sigma,b}\left(\vec{f}\right)(x) \\ = (b(x) - \lambda) T\left(\vec{f}\right)(x) - T_\sigma\left((b-\lambda) f_1, f_2, \ldots, f_m\right)(x). \end{aligned} \tag{58}$$

Since $0 < \delta < r/m < 1$,

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|\left|T_{\sigma,b}\left(\vec{f}\right)(z)\right|^\delta - |c|^\delta\right| dz\right)^{1/\delta} \\ &\quad \le \left(\frac{1}{|Q|}\int_Q \left|T_{\sigma,b}\left(\vec{f}\right)(z) - c\right|^\delta dz\right)^{1/\delta} \\ &\quad \le \left(\frac{C}{|Q|}\int_Q \left|(b(z) - \lambda) T_\sigma\left(\vec{f}\right)(z)\right|^\delta dz\right)^{1/\delta} \\ &\qquad + \left(\frac{C}{|Q|}\int_Q \left|T_\sigma\left((b-\lambda) f_1, \ldots, f_m\right)(z) - c\right|^\delta dz\right)^{1/\delta} \\ &\quad := A + B. \end{aligned} \tag{59}$$

By the John-Nirenberg inequality and Hölder inequality, one has, for $1 < q < \epsilon/\delta$ such that $q'\delta > 1$,

$$\begin{aligned} A &\le C\left(\frac{1}{|Q|}\int_Q |b(z) - \lambda|^{q'\delta} dz\right)^{1/q'\delta} \\ &\quad \times \left(\frac{1}{|Q|}\int_Q \left|T_\sigma\left(\vec{f}\right)(z)\right|^{q\delta} dz\right)^{1/q\delta} \\ &\le C\|b\|_{\mathrm{BMO}} M_{q\delta}\left(T_\sigma\left(\vec{f}\right)\right)(x) \\ &\le C\|b\|_{\mathrm{BMO}} M_{\epsilon}\left(T_\sigma\left(\vec{f}\right)\right)(x). \end{aligned} \tag{60}$$

To estimate term B, we split each function f_j as $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j \chi_{Q^*}$ for $j = 1, \ldots, m$. We also have the same decomposition,

$$\begin{aligned} &\prod_{j=1}^m f_j\left(y_j\right) \\ &\quad = \prod_{j=1}^m \left(f_j^0\left(y_j\right) + f_j^\infty\left(y_j\right)\right) \\ &\quad = \sum_{\alpha_1,\ldots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m) \\ &\quad = \prod_{j=1}^m f_j^0\left(y_j\right) + \sum_{\alpha_1,\ldots,\alpha_m \in \mathcal{I}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m), \end{aligned} \tag{61}$$

where $\mathcal{I} = \{\alpha_1, \ldots, \alpha_m : \text{there is atleast one } \alpha_j \ne 0\}$.

Taking $c = \sum_{\alpha_1,\ldots,\alpha_m \in \mathcal{I}} T_\sigma((b-\lambda) f_1^{\alpha_1} \cdots f_m^{\alpha_m})$, we have

$$\begin{aligned} B &\le C\left\{\left(\frac{1}{|Q|}\int_Q \left|T_\sigma\left((b-\lambda) f_1^0, \ldots, f_m^0\right)(z)\right|^\delta dz\right)^{1/\delta}\right. \\ &\quad + \sum_{\alpha_1,\ldots,\alpha_m \in \mathcal{I}} \left(\frac{1}{|Q|}\int_Q \left|T_\sigma\left((b-\lambda) f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right)(z)\right.\right. \\ &\qquad - T_\sigma\left((b-\lambda) f_1^{\alpha_1}, \ldots, f_m^{\alpha_m}\right) \\ &\qquad \left.\left. \times (x) dz\right|^\delta\right)^{1/\delta}\Bigg\} \\ &:= B_1 + B_2. \end{aligned} \tag{62}$$

By using Kolmogorov's inequality and Hölder's inequality, one has

$$\begin{aligned} B_1 &\le C\left\|T_\sigma((b-\lambda) f_1^0, \ldots, f_m^0)(z)\right\|_{L^{\gamma,\infty}(Q, dx/|Q|)} \\ &\le C\left(\frac{1}{|Q^*|}\int_{Q^*} \left|(b-\lambda) f_1^0(z)\right|^{r_1} d\right)^{1/r_1} \\ &\quad \times \prod_{j=2}^m \left(\frac{1}{|Q^*|}\int_{Q^*} \left|f_j^0(z)\right|^{r_j} dz\right)^{1/r_j} \\ &\le C\|b\|_{\mathrm{BMO}} \mathcal{M}_{\vec{\gamma}}\left(\vec{f}\right). \end{aligned} \tag{63}$$

By the same argument in the proof of Lemma 14, we have the following estimate:

$$\begin{aligned} &\sum_{\alpha_1,\ldots,\alpha_m \in \mathcal{I}} \left|T_\sigma\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)(z) - T_\sigma\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)(x)\right| \\ &\quad \le C\sum_{k=0}^\infty 2^{-k(s_1+\cdots+s_m-n/r_1-\cdots-n/r_m)} \\ &\qquad \times \left(\frac{1}{|2^{k+1}Q^*|}\int_{2^{k+1}Q^*} \left|(b-\lambda) f_1^{\alpha_1}(z)\right|^{r_1} dz\right)^{1/r_1} \\ &\qquad \times \prod_{j=2}^m \left(\frac{1}{|2^{k+1}Q^*|}\int_{2^{k+1}Q^*} \left|f_j^{\alpha_j}(z)\right|^{r_j} dz\right)^{1/r_j} \\ &\qquad \le C\mathcal{M}_{\vec{r}}\left(\vec{f}\right)(x). \end{aligned} \tag{64}$$

This completes the proof. □

3. Proof of Theorems

Proof of Theorem 2. By Lemmas 8 and 14, we have

$$\begin{aligned}\left\|T_\sigma\left(\vec{f}\right)\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} &\le \left\|M_\delta\left(T_\sigma\left(\vec{f}\right)\right)\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \\ &\le C\left\|M_\delta^\sharp(T_\sigma(\vec{f}))\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \\ &\le C\left\|\mathscr{M}_{\vec{r}}\left(\vec{f}\right)\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \\ &\le C\prod_{j=1}^m\left\|f_j\right\|_{L^{p_j,\kappa}}.\end{aligned} \tag{65}$$

□

Proof of Theorem 3. By Lemma 13, there are $\epsilon_j' < p_j/r_j$ such that $\omega_j \in A_{p_j/(r_j\epsilon_j')}$. Let $\gamma_j = r_j\epsilon_j'$; by Lemmas 8 and 15, one has

$$\left\|\mathscr{M}_{\vec{\gamma}}(\vec{f})\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m\left\|f_j\right\|_{L^{p_j,\kappa}}\left(\omega_j\right), \tag{66}$$

and then

$$\left\|T_{\sigma,\Sigma\mathbf{b}}(\vec{f})\right\|_{L^{p,\kappa}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^m\left\|f_j\right\|_{L^{p_j,\kappa}(\omega_j)}. \tag{67}$$

□

Conflict of Interests

The authors declare that they have no conflict of interests.

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