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Research Article

Robust Stability Analysis of Neutral-Type Hybrid Bidirectional Associative Memory Neural Networks with Time-Varying Delays

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The global asymptotic robust stability of equilibrium is considered for neutral-type hybrid bidirectional associative memory neural networks with time-varying delays and parameters uncertainties. The results we obtained in this paper are delay-derivative-dependent and establish various relationships between the network parameters only. Therefore, the results of this paper are applicable to a larger class of neural networks and can be easily verified when compared with the previously reported literature results. Two numerical examples are illustrated to verify our results.

1. Introduction

Stability analysis of neural networks is an issue of both theoretical and practical importance due to the fact that in some applications the designed neural network is required to have a unique and stable equilibrium point [1–3]. Time delays are unavoidably encountered in the implementation of neural networks, which may cause undesirable dynamic network behaviors such as oscillation and instability. On the other hand, in practice, the weight coefficients of the neurons depend on certain resistance and capacitance values which are subject to uncertainties. In the design of neural networks, it is important to ensure that the system is stable with respect to these uncertainties.

It is well known that a series of neural networks related to bidirectional associative memory (BAM) models have been proposed by Kosko [4, 5]. These models generalized the single-layer autoassociative Hebbian correlation to a two-layer pattern-matched heteroassociative circuit. This class of networks has been successfully applied to pattern recognition and artificial intelligence. A great number of results for BAM neural networks concerning the existence of equilibrium point and global asymptotic or robust stability have been derived [6–32].

Moreover, due to the complicated dynamic properties of the neural cells in the real world, the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely. It is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [33, 34]. However, the stability analysis of BAM neural networks of neutral type has been investigated by only a few researchers [18, 35–37].

However, the existing stability results [18, 36, 37] derived for the BAM neural networks can be applicable when only a pure delayed neural network model is considered. Recently, a more general class of BAM neural network models, called the hybrid BAM neural network in which both instantaneous and delayed signaling occur, was considered and some sufficient condition for robust stability of this class of BAM neural networks has been presented [23, 25, 38]. But, up to now, there are few results on stability of neutral-type hybrid BAM neural networks with time-varying delays.

Motivated by the preceding discussion, in this paper, we are going to deal with the problem of global asymptotic robust stability for neutral-type hybrid bidirectional associative memory neural networks with time-varying delays and parameters uncertainties. By constructing a novel Lyapunov functional, novel delay-derivative-dependent criteria are derived. Finally, two examples are provided to demonstrate the effectiveness of the obtained results.

Throughout this paper, we will use the following notations: let $v = (v_1, v_2, \ldots, v_n)^T \in \mathfrak{R}^n$ be a column vector and let $Q = (q_{ij})_{n\times n}$ be a real matrix. The three commonly used vector norms $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$ are defined as

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|, \quad \|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i^2|}, \quad \|v\|_\infty = \max_{1\le i\le n} |v_i|. \tag{1}$$

The three commonly used matrix norms $\|Q\|_1$, $\|Q\|_2$, and $\|Q\|_\infty$ are defined as follows:

$$\|Q\|_1 = \max_{1\le j\le n}\sum_{i=1}^{n} |q_{ij}|, \qquad \|Q\|_2 = \left[\lambda_M(Q^TQ)\right]^{1/2}, \qquad \|Q\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n} |q_{ij}|. \tag{2}$$

For the vector $v = (v_1, v_2, \ldots, v_n)^T$, $|v|$ will denote $v = (|v_1|, |v_2|, \ldots, |v_n|)^T$. For the matrix $Q = (q_{ij})_{n\times n}$, the matrix $|Q|$ will denote $|Q| = (|q_{ij}|)_{n\times n}$ and $\lambda_m(Q)$ and $\lambda_M(Q)$ will denote the minimum and maximum eigenvalues of Q, respectively. If $P = (p_{ij})_{n\times n}$ and $Q = (q_{ij})_{n\times n}$ are two real symmetric matrices, then $P \le Q$ will imply that $p_{ij} \le q_{ij}$, $i, j = 1, 2, \ldots, n$.

2. Problem Formulation

Dynamical behavior of a neutral-type hybrid BAM neural network with time-varying delays is described by the following set of differential equations:

$$\begin{aligned}
\dot{u}_i(t) + \sum_{j=1}^{m} e_{ji}\dot{u}_i(t-h) &= -a_i u_i(t) + \sum_{j=1}^{m} w_{ji}\tilde{f}_j(z_j(t)) + \sum_{j=1}^{m} w_{ji}^{\tau}\tilde{f}_j(z_j(t-\tau(t))) + I_i, \quad i = 1, 2, \ldots, n, \\
\dot{z}_j(t) + \sum_{i=1}^{n} \varepsilon_{ij}\dot{z}_j(t-d) &= -b_j z_j(t) + \sum_{i=1}^{n} v_{ij}\tilde{g}_i(u_i(t)) + \sum_{i=1}^{n} v_{ij}^{\sigma}\tilde{g}_i(u_i(t-\sigma(t))) + J_j, \quad j = 1, 2, \ldots, m,
\end{aligned} \tag{3}$$

in which $u = (u_1, u_2, \ldots, u_n)^T$ and $z = (z_1, z_2, \ldots, z_m)^T$ are the neuron state vectors, a_i and b_j denote the neuron charging time constants and passive decay rates, respectively, w_{ji}, w_{ji}^{τ}, v_{ij}, and v_{ij}^{σ} are the connection weights at the time t, $\tilde{f}_i$ and $\tilde{g}_j$ represent the activation functions of the neurons and the propagational signal functions, respectively, and I_i, J_j, denote the external inputs. $h > 0$ and $d > 0$ are positive constants which correspond to the finite speed of axonal signal transmission.

It will be assumed that a_i, b_j, w_{ji}, w_{ji}^{τ}, v_{ij}, v_{ij}^{σ}, e_{ji}, and ε_{ij} in system (3) are uncertain but bounded and belong to the following intervals:

$$\begin{aligned}
A_I &:= \left\{A = \operatorname{diag}(a_i) : 0 < \underline{A} \le A \le \overline{A}, \text{ i.e., } 0 < \underline{a}_i \le a_i \le \overline{a}_i, i = 1, 2, \ldots, n, \forall A \in A_I\right\}, \\
B_I &:= \left\{B = \operatorname{diag}(b_j) : 0 < \underline{B} \le B \le \overline{B}, \text{ i.e., } 0 < \underline{b}_j \le b_j \le \overline{b}_j, j = 1, 2, \ldots, m, \forall B \in B_I\right\}, \\
W_I &:= \left\{W = (w_{ji})_{m\times n} : \underline{W} \le W \le \overline{W}, \text{ i.e., } \underline{w}_{ji} \le w_{ji} \le \overline{w}_{ji}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall W \in W_I\right\}, \\
V_I &:= \left\{V = (v_{ij})_{n\times m} : \underline{V} \le V \le \overline{V}, \text{ i.e., } \underline{v}_{ij} \le v_{ij} \le \overline{v}_{ij}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall V \in V_I\right\}, \\
W_I^{\tau} &:= \left\{W^{\tau} = (w_{ji}^{\tau})_{m\times n} : \underline{W}^{\tau} \le W \le \overline{W}^{\tau}, \text{ i.e., } \underline{w}_{ji}^{\tau} \le w_{ji}^{\tau} \le \overline{w}_{ji}^{\tau}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall W^{\tau} \in W_I^{\tau}\right\}, \\
V_I^{\sigma} &:= \left\{V^{\sigma} = (v_{ij}^{\sigma})_{n\times m} : \underline{V}^{\sigma} \le V \le \overline{V}^{\sigma}, \text{ i.e., } \underline{v}_{ij}^{\sigma} \le v_{ij}^{\sigma} \le \overline{v}_{ij}^{\sigma}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall V^{\sigma} \in V_I^{\sigma}\right\}, \\
E_I &:= \left\{E = (e_{ji})_{m\times n} : \underline{E} \le E \le \overline{E}, \text{ i.e., } \underline{e}_{ji} \le e_{ji} \le \overline{e}_{ji}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall E \in E_I\right\}, \\
\Sigma_I &:= \left\{\Sigma = (\varepsilon_{ij})_{n\times m} : \underline{\Sigma} \le \Sigma \le \overline{\Sigma}, \text{ i.e., } \underline{\varepsilon}_{ij} \le \varepsilon_{ij} \le \overline{\varepsilon}_{ij}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, \forall \Sigma \in \Sigma_I\right\}.
\end{aligned} \tag{4}$$

(H1) $\tau(t) \ge 0$ and $\sigma(t) \ge 0$ are differentiable functions that satisfy

$$0 \le \tau(t) \le \overline{\tau}, \qquad 0 \le \sigma(t) \le \overline{\sigma},$$

$$\dot{\tau}(t) \le \mu_1 \le 1, \qquad \dot{\sigma}(t) \le \mu_2 \le 1 \tag{5}$$

for all $t \ge 0$ and prescribed scalars $\overline{\tau} > 0$, $\overline{\sigma} > 0$, $\mu_1 > 0$, and $\mu_2 > 0$.

The activation functions satisfy the following properties.

(H2) There exist some positive constants ℓ_i, $i = 1, 2, \ldots, n$, and κ_j, $j = 1, 2, \ldots, m$, such that

$$0 \le \frac{\tilde{f}_i(\overline{x}) - \tilde{f}_i(\overline{y})}{\overline{x} - \overline{y}} \le \ell_i, \qquad 0 \le \frac{\tilde{g}_j(\hat{x}) - \tilde{g}_j(\hat{y})}{\hat{x} - \hat{y}} \le \kappa_j \tag{6}$$

for all $\overline{x}, \overline{y}, \hat{x}, \hat{y} \in \mathfrak{R}$.

(H3) There exist positive constants M_i, $i = 1, 2, \ldots, n$, and L_j, $j = 1, 2, \ldots, m$, such that $|\tilde{g}_i(u)| \le M_i$ and $|\tilde{g}_j(z)| \le L_i$ for all $u, z \in \mathfrak{R}$. Note that this assumption implies that the activation functions are bounded.

Assume that $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ and $z^* = (z_1^*, z_2^*, \ldots, z_m^*)^T$ are the equilibrium points of the system. In order to simplify our analysis, we transform the equilibrium points to the origin by the relationship

$$x_i(t) = u_i(t) - u_i^*, \qquad y_i(t) = z_j(t) - z_j^*. \tag{7}$$

Then, the transformed system is as follows:

$$\begin{aligned} &\dot{x}(t) + E\dot{x}(t-h) \\ &\quad = -Ax(t) + Wf(y(t)) + W^{\tau} f(y(t-\tau(t))), \\ &\qquad\qquad i = 1, 2, \ldots, n, \\ &\dot{y}(t) + \Sigma\dot{y}(t-d) \\ &\quad = -By(t) + Vg(x(t)) + V^{\sigma} g(x(t-\sigma(t))), \\ &\qquad\qquad j = 1, 2, \ldots, m, \end{aligned} \tag{8}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$, $f(y(t)) = (f_1(y_1(t)), f_2(y_2(t)), \ldots, f_n(y_n(t)))^T$, $g(x(t-\sigma(t))) = (g_1(x_1(t-\sigma(t)), g_2(x_2(t-\sigma(t))), \ldots, g_n(x_n(t-\sigma(t))))^T$, $f(y(t-\tau)) = (f_1(y_1(t-\tau(t))), f_2(y_2(t-\tau(t))), \ldots, f_m(y_m(t-\tau(t))))^T$. The functions $g_i(x_i)$, and $f_j(y_j)$ are of the form

$$\begin{aligned} g_i(x_i(\cdot)) &= \tilde{g}_i(x_i(\cdot) + u_i^*) - \tilde{g}_i(u_i^*), \quad i = 1, 2, \ldots, n, \\ f(y_j(\cdot)) &= \tilde{f}_j(y_j(\cdot) + z_j^*) - \tilde{f}_j(z_j^*), \quad j = 1, 2, \ldots, m. \end{aligned} \tag{9}$$

It can be verified that the functions g_i and f_j satisfy the assumptions on g_i and f_j; that is, $g_i \in \beta$, $f_j \in \kappa$, and $g_i \in \beta$, $f_j \in \beta$ implies that $g_i \in \beta$ and $f_j \in \beta$, respectively. We also note that $g_i(0) = 0$ and $f_j(0) = 0$, $i = 1, 2, \ldots, n$.

By assumption (H2) and the above equations, we can have

$$0 \le \frac{f(y)}{y} \le \ell_i, \qquad 0 \le \frac{g(x)}{x} \le \kappa_i. \tag{10}$$

3. Preliminaries

In this paper, we will assume that the norms of the matrices $A, B, W = (W_{ji}), W^{\tau} = (W_{ji}^{\tau}), V = (V_{ij})$, and $V^{\sigma} = (W_{ij}^{\sigma})$ are bounded. Based on this property, we can directly observe the following facts.

Fact 1. If $A, B, W = (W_{ji}), W^{\tau} = (W_{ji}^{\tau}), V = (V_{ij})$, and $V^{\sigma} = (W_{ij}^{\sigma})$ satisfy the parameter ranges defined by (4) and have bounded norms, then there exist some positive constants $\sigma(W), \sigma(W^{\tau}), \sigma(V)$, and $\sigma(V^{\sigma})$:

$$\begin{gathered} \|A\|_2 \le \sigma(A), \qquad \|B\|_2 \le \sigma(B), \qquad \|W\|_2 \le \sigma(W), \\ \|W^{\tau}\|_2 \le \sigma(W^{\tau}), \qquad \|V\|_2 \le \sigma(V), \\ \|V^{\sigma}\|_2 \le \sigma(V^{\sigma}). \end{gathered} \tag{11}$$

Lemma 1 (Faydasicok and Arik [39]). *For $W \in W_I := \{W = (w_{ij}) : \underline{W} \le W \le \overline{W}$, i.e., $\underline{w}_{ij} \le w_{ij} \le \overline{w}_{ij}$, $i, j = 1, 2, \ldots, n\}$, the following equation holds:*

$$\sigma_1(W) = \sqrt{\left\| |W^{*T}W^*| + 2|W^{*T}|W_* + W_*^T W_* \right\|_2}, \tag{12}$$

where $W^* = (1/2)(\overline{W} + \underline{W})$ and $W_* = (1/2)(\overline{W} - \underline{W})$.

Lemma 2 (Cao et al. [40]). *For $W \in W_I := \{W = (w_{ij}) : \underline{W} \le W \le \overline{W}$, i.e., $\underline{w}_{ij} \le w_{ij} \le \overline{w}_{ij}$, $i, j = 1, 2, \ldots, n\}$, the following equation holds:*

$$\sigma_2(W) = \|W^*\|_2 + \|W_*\|_2, \tag{13}$$

where $W^* = (1/2)(\overline{W} + \underline{W})$ and $W_* = (1/2)(\overline{W} - \underline{W})$.

Lemma 3 (Ensari and Arik [41]). *For $W \in W_I := \{W = (w_{ij}) : \underline{W} \le W \le \overline{W}$, i.e., $\underline{w}_{ij} \le w_{ij} \le \overline{w}_{ij}$, $i, j = 1, 2, \ldots, n\}$, the following equation holds:*

$$\sigma_3(W) = \sqrt{\|W^*\|_2^2 + \|W_*\|_2^2 + 2\|W_*^T |W^*|\|_2}, \tag{14}$$

where $W^* = (1/2)(\overline{W} + \underline{W})$ and $W_* = (1/2)(\overline{W} - \underline{W})$.

Lemma 4 (Singh [42]). *For $W \in W_I := \{W = (w_{ij}) : \underline{W} \le W \le \overline{W}$, that is, $\underline{w}_{ij} \le w_{ij} \le \overline{w}_{ij}$, $i, j = 1, 2, \ldots, n\}$, the following equation holds:*

$$\sigma_4(W) = \|\widehat{W}\|_2, \tag{15}$$

where $\widehat{W} = (\widehat{w}_{ij})_{n\times n}$ with $\widehat{w}_{ij} = \max\{|\underline{w}_{ij}|, |\overline{w}_{ij}|\}$.

Lemma 5. *For any two vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ and $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)^T$, the following inequality holds:*

$$2\omega^T \upsilon = 2\upsilon^T \omega \le \gamma\omega^T \varpi + \frac{1}{\gamma}\upsilon^T \upsilon, \tag{16}$$

where γ is any positive constant.

4. Global Robust Stability Results

Note that the equilibrium point of system (3) is globally asymptotically stable, if the origin of system (8) is a globally asymptotically stable equilibrium point. Therefore, in order to prove the global asymptotic stability of the equilibrium point of system (3), it will be sufficient to prove the global asymptotic stability of the origin of system (8). We can now proceed with the following result.

Theorem 6. *For given scalars $0 \leq \mu_1 \leq 1$ and $0 \leq \mu_2 \leq 1$, let the activation functions satisfy assumptions (H2) and (H3) and let the network parameters satisfy (4). Then, the origin of neural network model (8) is globally asymptotically stable, if there exist positive diagonal matrices $H_1 = \operatorname{diag}(h_{1i} > 0)$ and $H_2 = \operatorname{diag}(h_{2j} > 0)$, positive definite matrices R, and T, and four positive scalars α, β, χ, and δ, such that*

$$\begin{gathered}
\theta_1 = \|\underline{A}\|_2 - \|H_1\|_2 - \frac{1}{\alpha} - \frac{1}{\beta(1-\mu_1)} - \sigma^2(A)\left\|R^{-1}\right\|_2 > 0, \\
\theta_2 = \|H_1\|_2 - 3\sigma^2(E)\|R\|_2 > 0, \\
\theta_3 = \|\underline{A}\|_2\left\|K^{-2}\right\|_2 - \chi\sigma^2(V) - \sigma^2(V)\left\|T^{-1}\right\|_2 \\
- \frac{1}{(1-\mu_2)}\sigma^2(V^\sigma)\left\|T^{-1}\right\|_2 - \delta\sigma^2(V^\sigma) > 0, \\
\theta_4 = \|\underline{B}\|_2 - \|H_2\|_2 - \frac{1}{\chi} - \frac{1}{\delta(1-\mu_2)} - \sigma^2(B)\left\|T^{-1}\right\|_2 > 0, \\
\theta_5 = \|H_2\|_2 - 3\sigma^2(\Sigma)\|T\|_2 > 0, \\
\theta_6 = \|\underline{B}\|_2\left\|L^{-2}\right\|_2 - \alpha\sigma^2(W) - \sigma^2(W)\left\|R^{-1}\right\|_2 \\
- \frac{1}{(1-\mu_1)}\sigma^2(W^\tau)\left\|R^{-1}\right\|_2 - \beta\sigma^2(W^\tau) > 0,
\end{gathered} \tag{17}$$

where

$$\begin{aligned}
\sigma(A) = \min\Bigg\{ & \sqrt{\left\| \left|A^{*T}A^*\right| + 2\left|A^{*T}\right|A_* + A_*^T A_*\right\|_2}, \\
& \|A^*\|_2 + \|A_*\|_2, \\
& \sqrt{\|A^*\|_2^2 + \|A_*\|_2^2 + 2\left\|A_*^T|A^*|\right\|_2}, \left\|\widehat{A}\right\|_2 \Bigg\},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\sigma(B) = \min\Bigg\{ & \sqrt{\left\| \left|B^{*T}B^*\right| + 2\left|B^{*T}\right|B_* + B_*^T B_*\right\|_2}, \\
& \|B^*\|_2 + \|B_*\|_2, \\
& \sqrt{\|B^*\|_2^2 + \|B_*\|_2^2 + 2\left\|B_*^T|B^*|\right\|_2}, \left\|\widehat{B}\right\|_2 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\sigma(W) = \min\Bigg\{ & \sqrt{\left\| \left|W^{*T}W^*\right| + 2\left|W^{*T}\right|W_* + W_*^T W_*\right\|_2}, \\
& \|W^*\|_2 + \|W_*\|_2, \\
& \sqrt{\|W^*\|_2^2 + \|W_*\|_2^2 + 2\left\|W_*^T|W^*|\right\|_2}, \left\|\widehat{W}\right\|_2 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\sigma(V) & = \min\Bigg\{ \sqrt{\left\| \left|V^{*T}V^*\right| + 2\left|V^{*T}\right|V_* + V_*^T V_*\right\|_2}, \\
& \|V^*\|_2 + \|V_*\|_2, \\
& \sqrt{\|V^*\|_2^2 + \|V_*\|_2^2 + 2\left\|V_*^T|V^*|\right\|_2}, \left\|\widehat{V}\right\|_2 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\sigma(W^\tau) & = \min\Bigg\{ \sqrt{\left\| \left|W^{\tau*T}W^{\tau*}\right| + 2\left|W^{\tau*T}\right|W_*^\tau + W_*^{\tau T} W_*^\tau\right\|_2}, \\
& \|W^{\tau*}\|_2 + \|W_*^\tau\|_2, \\
& \sqrt{\|W^{\tau*}\|_2^2 + \|W_*^\tau\|_2^2 + 2\left\|W_*^{\tau T}|W^{\tau*}|\right\|_2}, \left\|\widehat{W}^\tau\right\|_2 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\sigma(V^\sigma) & = \min\Bigg\{ \sqrt{\left\| \left|V^{\sigma*T}V^{\sigma*}\right| + 2\left|V^{\sigma*T}\right|V_*^\sigma + V_*^{\sigma T} V_*^\sigma\right\|_2}, \\
& \|V^{\sigma*}\|_2 + \|V_*^\sigma\|_2, \\
& \sqrt{\|V^{\sigma*}\|_2^2 + \|V_*^\sigma\|_2^2 + 2\left\|V_*^{\sigma T}|V^{\sigma*}|\right\|_2}, \left\|\widehat{V}^\sigma\right\|_2 \Bigg\},
\end{aligned}$$

$$\begin{gathered}
A^* = \frac{1}{2}\left(\overline{A} + \underline{A}\right), \qquad A_* = \frac{1}{2}\left(\overline{A} - \underline{A}\right), \\
B^* = \frac{1}{2}\left(\overline{B} + \underline{B}\right), \qquad B_* = \frac{1}{2}\left(\overline{B} - \underline{B}\right), \\
W^* = \frac{1}{2}\left(\overline{W} + \underline{W}\right), \qquad W_* = \frac{1}{2}\left(\overline{W} - \underline{W}\right), \\
W^{\tau*} = \frac{1}{2}\left(\overline{W}^\tau + \underline{W}^\tau\right), \qquad W_*^\tau = \frac{1}{2}\left(\overline{W}^\tau - \underline{W}^\tau\right), \\
V^* = \frac{1}{2}\left(\overline{V} + \underline{V}\right), \qquad V_* = \frac{1}{2}\left(\overline{V} - \underline{V}\right), \\
V^{\sigma*} = \frac{1}{2}\left(\overline{V}^\sigma + \underline{V}^\sigma\right), \qquad V_*^\sigma = \frac{1}{2}\left(\overline{V}^\sigma - \underline{V}^\sigma\right), \\
\widehat{A} = \operatorname{diag}(\widehat{a}_i) \quad \text{with} \quad \widehat{a}_i = \max\left\{|\underline{a}_i|, |\overline{a}_i|\right\}, \\
\widehat{B} = \operatorname{diag}(\widehat{b}_j) \quad \text{with} \quad \widehat{b}_j = \max\left\{\left|\underline{b}_j\right|, \left|\overline{b}_j\right|\right\}, \\
\widehat{W} = \left(\widehat{w}_{ij}\right)_{n\times n} \quad \text{with} \quad \widehat{w}_{ij} = \max\left\{\left|\underline{w}_{ij}\right|, \left|\overline{w}_{ij}\right|\right\}, \\
\widehat{W}^\tau = \left(\widehat{w}_{ij}^\tau\right)_{n\times n} \quad \text{with} \quad \widehat{w}_{ij}^\tau = \max\left\{\left|\underline{w}_{ij}^\tau\right|, \left|\overline{w}_{ij}^\tau\right|\right\}, \\
K = \operatorname{diag}\left(\kappa_j > 0\right), \\
\widehat{V} = \left(\widehat{v}_{ij}\right)_{n\times n} \quad \text{with} \quad \widehat{v}_{ij} = \max\left\{\left|\underline{v}_{ij}\right|, \left|\overline{v}_{ij}\right|\right\}, \\
\widehat{V}^\sigma = \left(\widehat{v}_{ij}^\sigma\right)_{n\times n} \quad \text{with} \quad \widehat{v}_{ij}^\sigma = \max\left\{\left|\underline{v}_{ij}^\sigma\right|, \left|\overline{v}_{ij}^\sigma\right|\right\}, \\
L = \operatorname{diag}\left(\ell_i > 0\right).
\end{gathered} \tag{19}$$

Proof. Define the following positive definite Lyapunov functional:

$$
\begin{aligned}
V&(x(t), y(t))\\
&= [x(t) + Ex(t-h)]^T [x(t) + Ex(t-h)]\\
&\quad + \sum_{i=1}^{n} h_{1i} \int_{t-h}^{t} x_i^2(s)\, ds\\
&\quad + [y(t) + \Sigma y(t-d)]^T [y(t) + \Sigma y(t-d)]\\
&\quad + \sum_{j=1}^{m} h_{2j} \int_{t-d}^{t} y_j^2(s)\, ds\\
&\quad + (\gamma_1 + \beta_1) \sum_{j=1}^{m} \int_{t-\tau(t)}^{t} f_j(y_j(\xi))\, d\xi\\
&\quad + (\gamma_2 + \beta_2) \sum_{i=1}^{n} \int_{t-\sigma(t)}^{t} g_i(x_i(\xi))\, d\xi.
\end{aligned}
\tag{20}
$$

The derivative of $V(x(t), y(t))$ along the trajectories of the system is obtained as follows:

$$
\begin{aligned}
\dot{V}&(x(t), y(t))\\
&= -2x^T(t) Ax(t) + 2x^T(t) Wf(y(t))\\
&\quad + 2x^T(t) W^\tau f(y(t-\tau(t)))\\
&\quad - 2x^T(t-h) E^T Ax(t) + 2x^T(t-h) E^T Wf(y(t))\\
&\quad + 2x^T(t-h) E^T W^\tau f(y(t-\tau(t)))\\
&\quad + x^T(t) H_1 x(t) - x^T(t-h) H_1 x(t-h)\\
&\quad - 2y^T(t) By(t) + 2y^T(t) Vg(x(t))\\
&\quad + 2y^T(t) V^\sigma g(x(t-\sigma(t)))\\
&\quad - 2y^T(t-d) \Sigma^T By(t) + 2y^T(t-d) \Sigma^T Vg(x(t))\\
&\quad + 2y^T(t-d) \Sigma^T V^\sigma g(x(t-\sigma(t)))\\
&\quad + y^T(t) H_2 y(t) - y^T(t-d) H_2 y(t-d)\\
&\quad + \gamma_1 \|f(y(t))\|_2^2 - \gamma_1(1-\mu_1) \|f(y(t-\tau(t)))\|_2^2\\
&\quad + \beta_1 \|f(y(t))\|_2^2 - \beta_1(1-\mu_1) \|f(y(t-\tau(t)))\|_2^2\\
&\quad + \gamma_2 \|g(x(t))\|_2^2 - \gamma_2(1-\mu_2) \|g(x(t-\sigma(t)))\|_2^2\\
&\quad + \beta_2 \|g(x(t))\|_2^2 - \beta_2(1-\mu_2) \|g(x(t-\sigma(t)))\|_2^2.
\end{aligned}
\tag{21}
$$

We can write the following inequalities as follows:

$$
\begin{aligned}
&2x^T(t) Wf(y(t))\\
&\le \frac{1}{\alpha} x^T(t) x(t) + \alpha f^T(y(t)) W^T Wf(y(t))\\
&\le \frac{1}{\alpha} \|x(t)\|_2^2 + \alpha \|W\|_2^2 \|f(y(t))\|_2^2,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&2x^T(t) W^\tau f(y(t-\tau))\\
&\quad \le \frac{1}{\beta(1-\mu_1)} x^T(t) x(t)\\
&\qquad + \beta(1-\mu_1) f^T(y(t-\tau(t))) W^{\tau T}\\
&\qquad \times W^\tau f(y(t-\tau(t)))\\
&\quad \le \frac{1}{\beta(1-\mu_1)} \|x(t)\|_2^2\\
&\qquad + \beta(1-\mu_1) \|W^\tau\|_2^2\\
&\qquad \times \|f(y(t-\tau(t)))\|_2^2,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&-2x^T(t-h) E^T Ax(t)\\
&\quad \le x^T(t-h) E^T REx(t-h)\\
&\qquad + x^T(t) A^T R^{-1} Ax(t)\\
&\quad \le \|E\|_2^2 \|R\|_2 \|x(t-h)\|_2^2\\
&\qquad + \|A\|_2^2 \left\|R^{-1}\right\|_2 \|x(t)\|_2^2,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&2x^T(t-h) E^T Wf(y(t))\\
&\quad \le x^T(t-h) E^T REx(t-h)\\
&\qquad + f(y(t)) W^T R^{-1} Wf(y(t))\\
&\quad \le \|E\|_2^2 \|R\|_2 \|x(t-h)\|_2^2\\
&\qquad + \|W\|_2^2 \left\|R^{-1}\right\|_2 \|f(y(t))\|_2^2,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&2x^T(t-h) E^T W^\tau f(y(t-\tau))\\
&\quad \le x^T(t-h) E^T REx(t-h)\\
&\qquad + f(y(t-\tau(t))) W^{\tau T}\\
&\qquad \times R^{-1} W^\tau f(y(t-\tau(t)))\\
&\quad \le \|E\|_2^2 \|R\|_2 \|x(t-h)\|_2^2\\
&\qquad + \|W^\tau\|_2^2 \left\|R^{-1}\right\|_2 \|f(y(t-\tau(t)))\|_2^2,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&2y^T(t) Vg(x(t))\\
&\quad \le \frac{1}{\chi} y^T(t) y(t) + \chi g^T(x(t))
\end{aligned}
$$

$$\times V^T V g(x(t)) \leq \frac{1}{\chi}\|y(t)\|_2^2 + \chi\|V\|_2^2\|g(x(t))\|_2^2, \tag{27}$$

$$\begin{aligned} &2y^T(t)V^\sigma g(x(t-\sigma)) \\ &\quad \leq \frac{1}{\delta(1-\mu_2)} y^T(t)y(t) + \delta(1-\mu_2) g^T \\ &\qquad \times (x(t-\sigma(t))) V^{\sigma T} V^\sigma g(x(t-\sigma(t))) \\ &\quad \leq \frac{1}{\delta(1-\mu_2)}\|y(t)\|_2^2 \\ &\qquad + \delta(1-\mu_2)\|V^\sigma\|_2^2\|g(x(t-\sigma(t)))\|_2^2, \end{aligned} \tag{28}$$

$$\begin{aligned} &-2y^T(t-d)\Sigma^T B y(t) \\ &\quad \leq y^T(t-d)\Sigma^T T\Sigma y(t-d) \\ &\qquad + y^T(t) B^T T^{-1} B y(t) \\ &\quad \leq \|\Sigma\|_2^2\|T\|_2\|y(t-d)\|_2^2 \\ &\qquad + \|B\|_2^2\|T^{-1}\|_2\|y(t)\|_2^2, \end{aligned} \tag{29}$$

$$\begin{aligned} &2y^T(t-d)\Sigma^T V g(x(t)) \\ &\quad \leq y^T(t-d)\Sigma^T T\Sigma y(t-d) \\ &\qquad + g^T(x(t)) V^T T^{-1} V g(x(t)) \\ &\quad \leq \|\Sigma\|_2^2\|T\|_2\|y(t-d)\|_2^2 \\ &\qquad + \|V\|_2^2\|T^{-1}\|_2\|g(x(t))\|_2^2, \end{aligned} \tag{30}$$

$$\begin{aligned} &2y^T(t-d)\Sigma^T V^\sigma g(x(t-\sigma)) \\ &\quad \leq y^T(t-d)\Sigma^T T\Sigma y(t-d) \\ &\qquad + g^T(x(t-\sigma(t))) V^{\sigma T} \\ &\qquad \times T^{-1} D g(x(t-\sigma(t))) \\ &\quad \leq \|\Sigma\|_2^2\|T\|_2\|y(t-d)\|_2^2 \\ &\qquad + \|V^\sigma\|_2^2\|T^{-1}\|_2\|g(x(t-\sigma(t)))\|_2^2. \end{aligned} \tag{31}$$

Combining (22)–(31) into (21) and considering

$$\begin{aligned} x^T(t)x(t) &\geq g^T(x(t))K^{-2}g(x(t)), \\ y^T(t)y(t) &\geq f^T(y(t))L^{-2}f(y(t)), \end{aligned} \tag{32}$$

we have

$$\begin{aligned} &\dot{V}(x(t), y(t)) \\ &\quad \leq -\left(\|\underline{A}\|_2 - \|H_1\|_2 - \frac{1}{\alpha} - \frac{1}{\beta(1-\mu_1)} - \|A\|_2^2\|R^{-1}\|_2\right) \\ &\qquad \times \|x(t)\|_2^2 \\ &\qquad - \left(\|H_1\|_2 - 3\|E\|_2^2\|R\|_2\right)\|x(t-h)\|_2^2 \\ &\qquad - \left(\|\underline{A}\|_2\|K^{-2}\|_2 - \chi\|V\|_2^2 - \|V\|_2^2\|T^{-1}\|_2 - \gamma_2 - \beta_2\right) \\ &\qquad \times \|g(x(t))\|_2^2 \\ &\qquad - \left(\beta_2(1-\mu_2) + \gamma_2(1-\mu_2)\right. \\ &\qquad\quad \left. -\delta(1-\mu_2)\|V^\sigma\|_2^2 - \|V^\sigma\|_2^2\|T^{-1}\|_2\right) \\ &\qquad \times \|g(x(t-\sigma(t)))\|_2^2 \\ &\qquad - \left(\|\underline{B}\|_2 - \|H_2\|_2 - \frac{1}{\chi} - \frac{1}{\delta(1-\mu_2)} - \|B\|_2^2\|T^{-1}\|_2\right) \\ &\qquad \times \|y(t)\|_2^2 - \left(\|H_2\|_2 - 3\|\Sigma\|_2^2\|T\|_2\right)\|y(t-d)\|_2^2 \\ &\qquad - \left(\|\underline{B}\|_2\|L^{-2}\|_2 - \alpha\|W\|_2^2 - \|W\|_2^2\|R^{-1}\|_2 - \gamma_1 - \beta_1\right) \\ &\qquad \times \|f(y(t))\|_2^2 \\ &\qquad - \left(\beta_1(1-\mu_1) + \gamma_1(1-\mu_1) - \beta(1-\mu_1)\|W^\tau\|_2^2\right. \\ &\qquad\quad \left. -\|W^\tau\|_2^2\|R^{-1}\|_2\right) \\ &\qquad \times \|f(y(t-\tau(t)))\|_2^2. \end{aligned} \tag{33}$$

Let $\beta_1 = \beta\|W^\tau\|_2^2$, $\gamma_1 = (1/(1-\mu_1))\|W^\tau\|_2^2\|R^{-1}\|_2$, $\beta_2 = \delta\|V^\sigma\|_2^2$, and $\gamma_2 = (1/(1-\mu_2))\|V^\sigma\|_2^2\|T^{-1}\|_2$, $\dot{V}(x(t), y(t))$ can be written in the form

$$\begin{aligned} &\dot{V}(x(t), y(t)) \\ &\quad \leq -\left(\|\underline{A}\|_2 - \|H_1\|_2 - \frac{1}{\alpha} - \frac{1}{\beta(1-\mu_1)} - \|A\|_2^2\|R^{-1}\|_2\right) \\ &\qquad \times \|x(t)\|_2^2 - \left(\|H_1\|_2 - 3\|E\|_2^2\|R\|_2\right)\|x(t-h)\|_2^2 \\ &\qquad - \left(\|\underline{A}\|_2\|K^{-2}\|_2 - \chi\|V\|_2^2 - \|V\|_2^2\|T^{-1}\|_2\right. \\ &\qquad\quad \left. -\frac{1}{(1-\mu_2)}\|V^\sigma\|_2^2\|T^{-1}\|_2 - \delta\|V^\sigma\|_2^2\right)\|g(x(t))\|_2^2 \end{aligned}$$

$$
\begin{aligned}
&-\left(\|\underline{B}\|_2-\|H_2\|_2-\frac{1}{\chi}-\frac{1}{\delta(1-\mu_2)}-\|B\|_2^2\left\|T^{-1}\right\|_2\right)\\
&\times\|y(t)\|_2^2-\left(\|H_2\|_2-3\|\Sigma\|_2^2\|T\|_2\right)\|y(t-d)\|_2^2\\
&-\left(\|\underline{B}\|_2\left\|L^{-2}\right\|_2-\alpha\|W\|_2^2-\|W\|_2^2\left\|R^{-1}\right\|_2\right.\\
&\left.\quad-\frac{1}{(1-\mu_1)}\|W^{\tau}\|_2^2\left\|R^{-1}\right\|_2-\beta\|W^{\tau}\|_2^2\right)\|f(y(t))\|_2^2.
\end{aligned}
\tag{34}
$$

By Fact 1, $\|A\|_2 \le \sigma(A)$, $\|B\|_2 \le \sigma(B)$, $\|W\|_2 \le \sigma(W)$, $\|W^{\tau}\|_2 \le \sigma(W^{\tau})$, $\|V\|_2 \le \sigma(V)$, and $\|V^{\sigma}\|_2 \le \sigma(V^{\sigma})$, one can have

$$
\begin{aligned}
&\dot{V}(x(t),y(t))\\
&\quad\le-\left(\|\underline{A}\|_2-\|H_1\|_2-\frac{1}{\alpha}-\frac{1}{\beta(1-\mu_1)}-\sigma^2(A)\left\|R^{-1}\right\|_2\right)\\
&\qquad\times\|x(t)\|_2^2-\left(\|H_1\|_2-3\sigma^2(E)\|R\|_2\right)\|x(t-h)\|_2^2\\
&\qquad-\left(\|\underline{A}\|_2\left\|K^{-2}\right\|_2-\chi\sigma^2(V)-\sigma^2(V)\left\|T^{-1}\right\|_2\right.\\
&\qquad\quad\left.-\frac{1}{(1-\mu_2)}\sigma^2(V^{\sigma})\left\|T^{-1}\right\|_2-\delta\sigma^2(V^{\sigma})\right)\\
&\qquad\times\|g(x(t))\|_2^2\\
&\qquad-\left(\|\underline{B}\|_2-\|H_2\|_2-\frac{1}{\chi}-\frac{1}{\delta(1-\mu_2)}-\sigma^2(B)\left\|T^{-1}\right\|_2\right)\\
&\qquad\times\|y(t)\|_2^2-\left(\|H_2\|_2-3\sigma^2(\Sigma)\|T\|_2\right)\|y(t-d)\|_2^2\\
&\qquad-\left({}_2\left\|L^{-2}\right\|_2-\alpha\sigma^2(W)-\sigma^2(W)\left\|R^{-1}\right\|_2\right.\\
&\qquad\quad\left.-\frac{1}{(1-\mu_1)}\sigma^2(W^{\tau})\left\|R^{-1}\right\|_2-\beta\sigma^2(W^{\tau})\right)\\
&\qquad\times\|f(y(t))\|_2^2,
\end{aligned}
\tag{35}
$$

or equivalently

$$
\begin{aligned}
\dot{V}(x(t),y(t))\le&-\theta_1\|x(t)\|_2^2-\theta_2\|x(t-h)\|_2^2\\
&-\theta_3\|g(x(t))\|_2^2-\theta_4\|y(t)\|_2^2\\
&-\theta_5\|y(t-d)\|_2^2-\theta_6\|f(y(t))\|_2^2.
\end{aligned}
\tag{36}
$$

Clearly, $\theta_i > 0$ and $i = 1,2,\ldots,6$, imply that $\dot{V}(x(t),y(t)) < 0$. On the other hand, $V(x(t),y(t)) \to \infty$ as $x(t) \to \infty$, $y(t) \to \infty$, meaning that the Lyapunov functional used for the stability analysis is radially unbounded. Then, by the standard Lyapunov functional theory, it is concluded that system (8) or equivalently the equilibrium point of system (3) is globally asymptotically stable. This completes the proof of Theorem 6. □

Remark 7. The stability results presented [18, 36, 37] considered a pure delayed neural network mode and are expressed in the linear matrix inequality (LMI) forms. The LMI approach to the stability problem of neutral-type neural networks involves some difficulties with determining the constraint conditions on the network parameters as it requires testing positive definiteness of high dimensional matrices. However, Theorem 6 considers hybrid BAM neural networks and establishes various relationships between the network parameters only. Therefore, the results of this paper are applicable to a larger class of neural networks and can be easily verified when compared with the previously reported literature results.

Choosing H_1, H_2, R, and T in the conditions of Theorem 6 as $H_1 = h_1 I$, $H_2 = h_2 I$, $R = rI$, and $T = tI$, we can express some special cases of Theorem 6 as follows.

Corollary 8. *For given scalars $0 \le \mu_1 \le 1$ and $0 \le \mu_2 \le 1$, let the activation functions satisfy assumptions (H2) and (H3) and let the network parameters satisfy* (4). *Then, the origin of neural network model* (8) *is globally asymptotically stable, if there exist eight positive scalars α, β, χ, δ, h_1, h_2, r, and t, such that*

$$
\begin{aligned}
\theta_1^*&=\|\underline{A}\|_2-h_1-\frac{1}{\alpha}-\frac{1}{\beta(1-\mu_1)}-\sigma^2(A)\frac{1}{r}>0,\\
\theta_2^*&=h_1-3\sigma^2(E)r>0,\\
\theta_3^*&=\|\underline{A}\|_2\left\|K^{-2}\right\|_2-\chi\sigma^2(V)-\sigma^2(V)\frac{1}{t}\\
&\quad-\frac{1}{(1-\mu_2)}\sigma^2(V^{\sigma})\frac{1}{t}-\delta\sigma^2(V^{\sigma})>0,\\
\theta_4^*&=\|\underline{B}\|_2-h_2-\frac{1}{\chi}-\frac{1}{\delta(1-\mu_2)}-\sigma^2(B)\frac{1}{t}>0,\\
\theta_5^*&=h_2-3\sigma^2(\Sigma)t>0,\\
\theta_6^*&=\|\underline{B}\|_2\left\|L^{-2}\right\|_2-\alpha\sigma^2(W)-\sigma^2(W)\frac{1}{r}\\
&\quad-\frac{1}{(1-\mu_1)}\sigma^2(W^{\tau})\frac{1}{r}-\beta\sigma^2(W^{\tau})>0,
\end{aligned}
\tag{37}
$$

and the other parameters are defined in Theorem 6.

By setting $\mu_1 = \mu_2 = 0$, the stability criterion for hybrid BAM neural network with constant time delays is established from Theorem 6.

Corollary 9. *Let the activation functions satisfy assumptions (H2) and (H3) and let the network parameters satisfy* (4). *Then, the origin of neural network model* (8) *is globally* *asymptotically stable, if there exists eight positive scalars α, β, χ, δ, h_1, h_2, r, and t, such that*

$$\begin{aligned}
\theta_1^{**} &= \|\underline{A}\|_2 - h_1 - \frac{1}{\alpha} - \frac{1}{\beta} - \sigma^2(A)\frac{1}{r} > 0,\\
\theta_2^{**} &= h_1 - 3\sigma^2(E)\, r > 0,\\
\theta_3^{**} &= \|\underline{A}\|_2\left\|K^{-2}\right\|_2 - \chi\sigma^2(V) - \sigma^2(V)\frac{1}{t}\\
&\quad - \sigma^2(V^\sigma)\frac{1}{t} - \delta\sigma^2(V^\sigma) > 0,\\
\theta_4^{**} &= \|\underline{B}\|_2 - h_2 - \frac{1}{\chi} - \frac{1}{\delta} - \sigma^2(B)\frac{1}{t} > 0,\\
\theta_5^{**} &= h_2 - 3\sigma^2(\Sigma)\, t > 0,\\
\theta_6^{**} &= \|\underline{B}\|_2\left\|L^{-2}\right\|_2 - \alpha\sigma^2(W) - \sigma^2(W)\frac{1}{r}\\
&\quad - \frac{1}{(1-\mu_1)}\sigma^2(W^\tau)\frac{1}{r} - \beta\sigma^2(W^\tau) > 0,
\end{aligned} \tag{38}$$

and the other parameters are defined in Theorem 6.

Assume that there are no neutral terms and the system of BAM neural networks is described as

$$\begin{aligned}
\dot{x}(t) &= -Ax(t) + Wf(y(t)) + W^\tau f(y(t-\tau(t))),\\
&\qquad i = 1, 2, \ldots, n,\\
\dot{y}(t) &= -By(t) + Vg(x(t)) + V^\sigma g(x(t-\sigma(t))),\\
&\qquad j = 1, 2, \ldots, m.
\end{aligned} \tag{39}$$

Define the following positive definite Lyapunov functional:

$$\begin{aligned}
V(x(t), y(t)) &= x^T(t)\, x(t)\, y(t)^T y(t) + \sum_{i=1}^{n} h_{1i}\int_{t-h}^{t} x_i^2(s)\, ds\\
&\quad + \sum_{j=1}^{m} h_{2j}\int_{t-d}^{t} y_j^2(s)\, ds\\
&\quad + \beta_1\sum_{j=1}^{m}\int_{t-\tau(t)}^{t} f_j(y_j(\xi))\, d\xi\\
&\quad + \beta_2\sum_{i=1}^{n}\int_{t-\sigma(t)}^{t} g_i(x_i(\xi))\, d\xi.
\end{aligned} \tag{40}$$

Following the similar line of the proof of Theorem 6, Corollary 10 is derived as follows.

Corollary 10. *For given scalars $0 \le \mu_1 \le 1$, and $0 \le \mu_2 \le 1$, let the activation functions satisfy assumptions (H2) and (H3) and let the network parameters satisfy (4). Then, the origin of neural network model (8) is globally asymptotically stable, if there exist four positive scalars α, β, χ, and δ, such that*

$$\begin{aligned}
\eta_1 &= \|\underline{A}\|_2 - \frac{1}{\alpha} - \frac{1}{\beta(1-\mu_1)} > 0,\\
\eta_2 &= \|\underline{A}\|_2\left\|K^{-2}\right\|_2 - \chi\sigma^2(V) - \delta\sigma^2(V^\sigma) > 0,\\
\eta_3 &= \|\underline{B}\|_2 - \frac{1}{\chi} - \frac{1}{\delta(1-\mu_2)} > 0,\\
\eta_4 &= \|\underline{B}\|_2\left\|L^{-2}\right\|_2 - \alpha\sigma^2(W) - \beta\sigma^2(W^\tau) > 0,
\end{aligned} \tag{41}$$

and the other parameters are defined in Theorem 6.

5. Comparative Numerical Examples

We will now give the following examples to demonstrate the applicability and advantages of our results.

Example 11. Assume that the network parameters of neural system (8) are given as follows:

$$\begin{aligned}
\underline{W} = \underline{W}^\tau = \underline{V} = \underline{V}^\sigma &= \begin{bmatrix} 3\lambda & 3\lambda & 4\lambda \\ 6\lambda & 2\lambda & 7\lambda \\ -7\lambda & 7\lambda & -6\lambda \end{bmatrix},\\
\overline{W} = \overline{W}^\tau = \overline{V} = \overline{V}^\sigma &= \begin{bmatrix} 7\lambda & 5\lambda & 4\lambda \\ 8\lambda & 4\lambda & 9\lambda \\ -3\lambda & 7\lambda & -2\lambda \end{bmatrix},\\
\underline{A} = A = \overline{A} = \underline{B} = B = \overline{B} &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},\\
K = L &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{42}$$

where $\lambda = 0$ is real number. We can conclude that the matrices W^*, W_*, $W^{\tau *}$, W_*^τ, V^*, V_*, $V^{\sigma *}$, and V_*^σ are in the forms

$$\begin{gathered}
\|A\|_2 = \|B\|_2 = 3,\\
W^* = W^{\tau *} = V^* = V^{\sigma *} = \begin{bmatrix} 5\lambda & 4\lambda & 4\lambda \\ 7\lambda & 3\lambda & 8\lambda \\ -5\lambda & 7\lambda & -4\lambda \end{bmatrix},\\
W_* = W_*^\tau = V_* = V_*^\sigma = \begin{bmatrix} 2\lambda & \lambda & 0 \\ \lambda & \lambda & \lambda \\ 2\lambda & 0 & 0 \end{bmatrix},\\
\tau = 2, \qquad \sigma = 3, \qquad h = 1,\\
d = 2, \qquad \mu_1 = 0, \qquad \mu_2 = 0.
\end{gathered} \tag{43}$$

Then we obtain

$$\begin{aligned}
\sigma_1(V^\sigma) &= \sigma_1(W^\tau) = \sigma_1(V) = \sigma_1(W) \\
&= \sqrt{\left\| |W^{*T}W^*| + 2|W^{*T}|W_* + W_*^T W_* \right\|_2} \\
&= \sqrt{272.7882\lambda^2} = 16.5163\lambda, \\
\sigma_2(V^\sigma) &= \sigma_2(W^\tau) = \sigma_2(V) = \sigma_2(W) \\
&= \|W^*\|_2 + \|W_*\|_2 = 17.5942\lambda. \\
\sigma_3(V^\sigma) &= \sigma_3(W^\tau) = \sigma_3(V) = \sigma_3(W) \\
&= \sqrt{\|W^*\|_2^2 + \|W_*\|_2^2 + 2\|W_*^T|W^*|\|_2} \\
&= \sqrt{308.2903\lambda^2} = 17.5582\lambda. \\
\sigma_4(V^\sigma) &= \sigma_4(W^\tau) = \sigma_4(V) \\
&= \sigma_4(W) = \|\widehat{W}\|_2 \\
&= 19.2861\lambda.
\end{aligned} \tag{44}$$

Since $\min\{\sigma_1(W), \sigma_2(W), \sigma_3(W), \sigma_4(W)\} = 16.5163\lambda$, we obtain $\sigma(V) = \sigma(W) = \sigma(V^\sigma) = \sigma(W^\tau) = 16.5163\lambda$.

For the sufficiently small values of $\|E\|_2$, $\|\Sigma\|_2$, h_1, and h_2 and sufficiently large value of r, t, and $\alpha = \beta$, $\chi = \delta$, the conditions of Corollary 9 can be approximately stated as follows: $\theta_6^{**} = 3 - 2\alpha \times 272.7882\lambda^2 > 0$, $\theta_6^{**} \cong 3 - 2\alpha \times 272.7882\lambda^2 > 0$, and

$$\begin{aligned}
\theta_1^{**} &\cong 3 - \frac{2}{\alpha} > 0, \\
\theta_2^{**} &\cong h_1 - 3\sigma^2(E)\, r > 0, \\
\theta_3^{**} &\cong 3 - 2\chi \times 272.7882\lambda^2 > 0, \\
\theta_4^{**} &\cong 3 - \frac{2}{\chi} > 0, \\
\theta_5^{**} &\cong h_2 - 3\sigma^2(F)\, t > 0, \\
\theta_6^{**} &\cong 3 - 2\alpha \times 272.7882\lambda^2 > 0.
\end{aligned} \tag{45}$$

The four required conditions for stability are $\alpha > 2/3$, $\chi > 2/3$ and $\lambda^2 < 3/(2\chi \times 272.7882)$, $\lambda^2 < 3/(2\alpha \times 272.7882)$, implying that $\lambda < 0.0908$. Hence, if $\lambda < 0.0908$ holds, then the conditions of Corollary 9 are satisfied which indicates that the BAM neural network is global asymptotic robust stable.

In what follows, we consider a special model in this example and give simulation results for the sake of verification of our proposed results. We choose $\lambda = 0.06$ that satisfies the condition $\lambda < 0.0908$. For this example, the Matlab simulation results are presented in Figure 1.

Example 12. Assume that the network parameters of neural system (8) are given as follows:

$$\begin{gathered}
\underline{A} = A = \overline{A} = \underline{B} = B = \overline{B} = I, \\
\underline{W} = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0.2 \end{bmatrix}, \qquad \overline{W} = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}, \\
\underline{W}^\tau = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}, \qquad \overline{W}^\tau = \begin{bmatrix} 0.3 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}, \\
\underline{V} = \begin{bmatrix} 0.1 & 0.1 \\ -0.3 & 0.1 \end{bmatrix}, \qquad \overline{V} = \begin{bmatrix} 0.3 & 0.3 \\ 0.1 & 0.3 \end{bmatrix}, \\
\underline{V}^\sigma = \begin{bmatrix} 0.1\lambda & 0.1\lambda \\ 0.1\lambda & 0.2\lambda \end{bmatrix}, \qquad \underline{V}^\sigma = \begin{bmatrix} 0.3\lambda & 0.3\lambda \\ 0.3\lambda & 0.4\lambda \end{bmatrix}, \\
\underline{E} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \qquad \overline{E} = \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 0.05 \end{bmatrix}, \\
\underline{\Sigma} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \qquad \overline{\Sigma} = \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 0.05 \end{bmatrix}, \\
K = \Sigma = 0.5I, \qquad \tau(t) = 0.5\sin t + 0.1, \\
\sigma(t) = 0.5\sin t + 0.2, \qquad \mu_1 = \mu_2 = 0.5,
\end{gathered} \tag{46}$$

where $\lambda > 0$ is real number. We can obtain

$$\begin{gathered}
\|A\|_2 = \|B\|_2 = 1, \qquad \sigma(A) = \sigma(B) = 1, \\
W^* = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \qquad W_* = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\
W^{\tau *} = \begin{bmatrix} 0.2 & 0.3 \\ 0.25 & 0.2 \end{bmatrix}, \qquad W_*^\sigma = \begin{bmatrix} 0.1 & 0.1 \\ 0.05 & 0.2 \end{bmatrix}, \\
V^* = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \qquad V_* = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\
V^{\sigma *} = \begin{bmatrix} 0.4\lambda & 0.3\lambda \\ 0.25\lambda & 0.2\lambda \end{bmatrix}, \qquad V_*^\sigma = \begin{bmatrix} 0.3\lambda & 0.1\lambda \\ 0.05\lambda & 0.2\lambda \end{bmatrix}, \\
E^* = \begin{bmatrix} 0.03 & 0.03 \\ 0.03 & 0.03 \end{bmatrix}, \qquad E_* = \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix}, \\
\Sigma^* = \begin{bmatrix} 0.03 & 0.03 \\ 0.03 & 0.03 \end{bmatrix}, \qquad \Sigma_* = \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix}.
\end{gathered} \tag{47}$$

By Lemmas 1–4, we can calculate

$$\begin{gathered}
\sigma_1(W) = \sqrt{\left\| |W^{*T}W^*| + 2|W^{*T}|W_* + W_*^T W_* \right\|_2} \\
= \sqrt{0.2546}, \quad \sigma_2(W) = \|W^*\|_2 + \|W_*\|_2 = 0.5159, \\
\sigma_3(W) = \sqrt{\|W^*\|_2^2 + \|W_*\|_2^2 + 2\|W_*^T|W^*|\|_2} = 0.5, \\
\sigma_4(W) = \|\widehat{W}\|_2 = 0.2656.
\end{gathered} \tag{48}$$

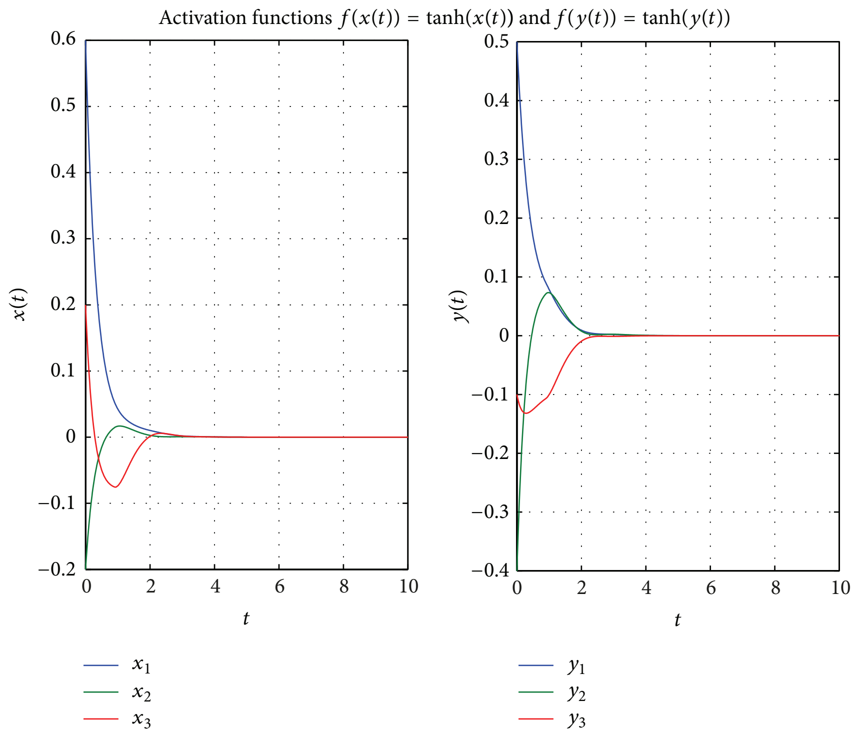

FIGURE 1: Trajectories of $x(t)$ and $y(t)$ of system (8) for the initial states $x(0) = [0.6 \ \ -0.2 \ \ 0.2]$ and $y(0) = [0.5 \ \ -0.4 \ \ -0.1]$.

Since $\sigma(W) = \min\{\sigma_1(W), \sigma_2(W), \sigma_3(W), \sigma_4(W)\}$, we obtain $\sigma(W) = 0.5$, Similarly, we obtain

$$\begin{aligned}
\sigma_1(W^\tau) &= \sqrt{0.5018}, \qquad \sigma_2(W^\tau) = 0.7188,\\
\sigma_3(W^\tau) &= \sqrt{0.4967}, \qquad \sigma_4(W^\tau) = 0.6085,\\
\sigma_1(V) &= \sqrt{0.3225}, \qquad \sigma_2(V) = 0.5618,\\
\sigma_3(V) &= \sqrt{0.3440}, \qquad \sigma_4(V) = 0.6,\\
\sigma_1(V^\sigma) &= \sqrt{0.4281}\lambda, \qquad \sigma_2(V^\sigma) = 0.6562\lambda,\\
\sigma_3(V^\sigma) &= \sqrt{0.4292}\lambda, \qquad \sigma_4(V^\sigma) = 0.6541\lambda,\\
\sigma_1(E) &= \sigma_1(\Sigma) = \sigma_2(E) = \sigma_2(\Sigma)\\
&= \sigma_3(E) = \sigma_3(\Sigma) = \sigma_4(E) = \sigma_4(\Sigma) = 0.1.
\end{aligned} \tag{49}$$

Thus we have $\sigma(W^\tau) = 0.6085$, $\sigma(V) = 0.5618$, $\sigma(V^\sigma) = 0.6541\lambda$, and $\sigma(E) = \sigma(\Sigma) = 0.1$.

Let $\alpha = \beta = 6$, $\chi = \delta = 6$, and $r = t = 6$; the conditions of Corollary 8 can be stated as follows:

$$\begin{aligned}
\theta_1^* &= 1 - h_1 - \frac{1}{6} - \frac{1}{6(1-0.5)} - \frac{1}{6} > 0,\\
\theta_2^* &= h_1 - 3 \times 0.01 \times 6 > 0,\\
\theta_3^* &= 4 - 6 \times 0.3156 - \frac{0.3156}{6}\\
&\quad - \frac{1}{(1-0.5)} 0.6541^2\lambda^2 \times \frac{1}{6} - 6 \times 0.6541^2\lambda^2 > 0,\\
\theta_4^* &= 1 - h_2 - \frac{1}{4} - \frac{1}{4(1-0.5)} - \frac{1}{6} > 0,\\
\theta_5^* &= h_2 - 3 \times 0.01 \times 6 > 0,\\
\theta_6^* &= 4 - 6 \times 0.25 - 0.25 \times \frac{1}{6} - \frac{1}{(1-0.5)}\\
&\quad \times 0.3703 \times \frac{1}{6} - 6 \times 0.3703 = 0.1131 > 0,
\end{aligned} \tag{50}$$

in which $\lambda < 0.8706$ implies that the conditions of Corollary 8 are satisfied which indicates that the network is global asymptotic robust stable.

For the neural network parameters given in Example 12, we choose $\lambda = 0.6$ that satisfies the condition $\lambda < 0.8706$. For this example, the Matlab simulation results are presented in Figure 2.

6. Conclusions

In this paper, we have obtained new sufficient conditions for the global asymptotic robust stability of the equilibrium point for the class of neutral-type hybrid bidirectional associative memory neural networks with time-varying delays

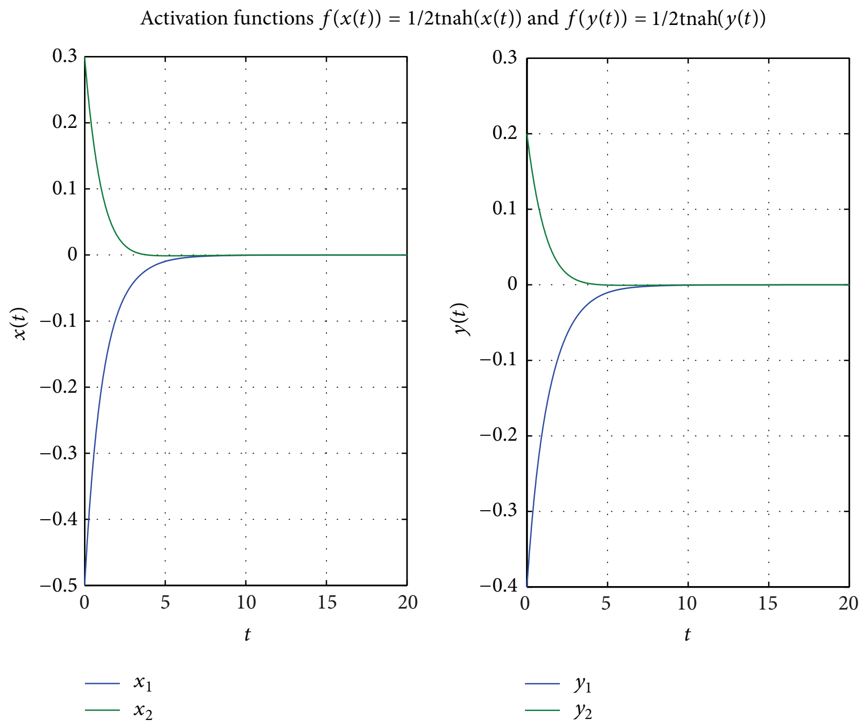

FIGURE 2: Trajectories of $x(t)$ and $y(t)$ of system (8) for the initial states $x(0) = [-0.5 \quad 0.3]$ and $y(0) = [-0.4 \quad 0.2]$.

and parameters uncertainties. Some new delay-derivative-dependent stability criteria are derived to ascertain the global asymptotic stability of the BAM neural networks. To obtain less conservative stability criterion, some new upper bound norms for the interconnection matrices of the neural networks are used. The obtained results can be easily verified as they can be expressed in terms of the network parameters only. Two illustrative examples are given to show the effectiveness of the proposed results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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