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Research Article

Dimensions of Fractals Generated by Bi-Lipschitz Maps

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On the class of iterated function systems of bi-Lipschitz mappings that are contractions with respect to some metrics, we introduce a logarithmic distortion property, which is weaker than the well-known bounded distortion property. By assuming this property, we prove the equality of the Hausdorff and box dimensions of the attractor. We also obtain a formula for the dimension of the attractor in terms of certain modified topological pressure functions, without imposing any separation condition. As an application, we prove the equality of Hausdorff and box dimensions for certain iterated function systems consisting of affine maps and nonsmooth maps.

1. Introduction

In the literature on the equality of the Hausdorff and box dimensions of the attractor of an iterated function system (IFS), it is usually assumed that the generating maps are C^1 and the bounded distortion property holds (see [1–3]). For IFSs of conformal contractions, the weak separation condition is also assumed (see [3]). These three conditions are usually imposed in order to obtain a formula for the dimensions of the attractor in terms of topological pressures (see, e.g., [4, 5]). The main goal of this paper is to relax these three conditions.

There are many definitions of dimension for fractal sets. As is well known, the Hausdorff and upper box dimensions may be regarded as the smallest and the greatest values of any reasonable definition of dimension. Fox example, the packing dimension introduced by Tricot Jr. [6] always lies between these two values. Motivated by this observation, McLaughlin [7] and Falconer [1] studied conditions under which the Hausdorff and box dimensions of a fractal set are equal. As an application of the so-called implicit method, Falconer [1, Examples 2 and 3] proved the equality of the Hausdorff and box dimensions for all self-similar sets and a class of graph-directed sets (called recurrent sets), without assuming any separation condition. By assuming the C^1-smoothness of the maps of the IFS, the *bounded distortion property (BDP)*, and the *weak separation condition (WSC)*, Lau et al. [3] proved the equality of the two dimensions for self-conformal sets. Under these conditions, the authors [5] proved that the common dimension is given by the zero of some topological pressure functions. For an infinite iterated function system, by assuming the open set condition, BDP, and that the maps of the IFS are $C^{1+\epsilon}$ smooth, Mauldin and Urbański [4] proved that the Hausdorff dimension of the limit set is given by the zero of some topological pressure function.

The dimensions of self-affine sets have also been studied extensively, since the work of McMullen [8] and Falconer [9]. Our results in this paper allow us to deal with a special class of self-affine sets. A simple example in this class is the self-affine set generated by the affine maps

$$S_1(x) = A^{-1}x, \qquad S_2(x) = A^{-1}\left(x + (1,1)^t\right), \qquad A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \tag{1}$$

which arises in the study of connectedness of self-affine sets in [10] (see also [11] and the references therein). This IFS does not satisfy BDP. There are of course plenty of examples of IFSs that do not satisfy WSC or contain maps that are not C^1. We will study such examples in Section 4. Our work is partly motivated by them.

There are two main goals in this paper. First, we would like to prove the equality of the Hausdorff and box dimensions by assuming a weaker set of conditions. We weaken the C^1-smoothness condition to the bi-Lipschitz condition and replace the bounded distortion property by a weaker logarithmic distortion property. Second, under these conditions, we would like to obtain a formula for the common dimension in terms of the zero of some topological pressure functions, without assuming any separation condition.

As some of the mappings we consider are not necessarily contractive with respect to the Euclidean metric, but contractive with respect to some other metric, for convenience we first introduce the definition of an iterated function system of essential contractions.

Definition 1. Let X be a nonempty compact subset of $\mathbb{R}^d$, equipped with the Euclidean metric, and let $S_i : X \to X$, $i = 1, \ldots, N$, be a finite family of mappings. If there exists a metric ϱ on X such that all the S_i are contractions with respect to ϱ, then one says that $\{S_i\}_{i=1}^N$ are *essential contractions with respect to* ϱ (or simply *essential contractions*). In this case one calls $\{S_i\}_{i=1}^N$ an *iterated function system (IFS) of essential contractions*.

Some IFSs of affine mappings are not necessarily contractions with respect to the Euclidean metric but are essential contractions (see [12]). Some of the IFSs we consider in this paper are defined by matrices that are powers of a single matrix (see Example 23). They are also essential contractions.

In order to state our conditions and results, we first introduce some basic definitions and notations. Let X be a nonempty compact subset of $\mathbb{R}^d$, equipped with the Euclidean metric, and let $S_i : X \to X$, $i = 1, \ldots, N$, be essential contractions with respect to some metric ϱ. It is well known that there exists a unique nonempty compact subset $K \subseteq X$, called the *attractor*, such that

$$K = \bigcup_{i=1}^{N} S_i(K) \tag{2}$$

(see [13, 14]). The set K is independent of the metric ϱ. For such an IFS, we define

$$\Sigma^k := \{1, \ldots, N\}^k \quad \text{for } k \geq 0, \qquad \Sigma^* := \bigcup_{k\geq 0} \Sigma^k, \qquad \Sigma^\infty := \{1, \ldots, N\}^\infty, \tag{3}$$

with $\Sigma^0 := \{\emptyset\}$. For $I = (i_1, \ldots, i_k) \in \Sigma^k$, we denote by $|I| = k$ the *length* of I and write $S_I := S_{i_1} \circ \cdots \circ S_{i_k}$ ($S_\emptyset$ is defined to be the identity). We also denote $I = (i_1, \ldots, i_k)$ simply by $I = i_1 \cdots i_k$ and let $I^- := i_1 \cdots i_{n-1}$ be the word obtained from I by deleting its last alphabet.

Let $|\cdot|$ denote the Euclidean norm. Define

$$\begin{aligned} r_I &:= \inf_{x \neq y \in X} \frac{|S_I(x) - S_I(y)|}{|x-y|}, \\ R_I &:= \sup_{x \neq y \in X} \frac{|S_I(x) - S_I(y)|}{|x-y|}, \\ &\qquad I \in \Sigma^*, \\ r &:= \min_{1\leq i\leq N} r_i, \\ R &:= \max_{1\leq i\leq N} R_i, \end{aligned} \tag{4}$$

$$r_\psi := r_I, \quad R_\psi := R_I, \quad \text{if } \psi = S_I \quad \text{for some } I \in \Sigma^*. \tag{5}$$

For any $I, J \in \Sigma^*$, by writing

$$\frac{|S_{IJ}(x) - S_{IJ}(y)|}{|x-y|} = \frac{|S_{IJ}(x) - S_{IJ}(y)|}{|S_J(x) - S_J(y)|} \cdot \frac{|S_J(x) - S_J(y)|}{|x-y|}, \tag{6}$$

we obtain the following sets of inequalities:

$$R_{IJ} \leq R_I R_J, \qquad r_{IJ} \geq r_I r_J, \tag{7}$$

$$R_{IJ} \geq R_I r_J, \qquad R_{IJ} \geq r_I R_J, \qquad r_{IJ} \leq R_I r_J, \qquad r_{IJ} \leq r_I R_J. \tag{8}$$

These inequalities will be used repeatedly.

Assumption 2. Throughout this paper we assume that $r > 0$; equivalently, S_i, $i = 1, \ldots, N$, are bi-Lipschitz.

Remark 3. It is possible that $R \geq 1$. Since all S_i, $1 \leq i \leq N$, are essential contractions, R_I converges uniformly to 0 as $|I|$ tends to infinity. As a consequence, we also have $r < 1$.

For any $E \subseteq \mathbb{R}^d$, we let $\dim_H(E)$, $\dim_P(E)$, $\dim_B(E)$, $\mathscr{H}^s(E)$, $\mathscr{L}^d(E)$, $|E|$, and E° denote, respectively, the Hausdorff dimension, packing dimension, box dimension, s-dimensional Hausdorff measure, d-dimensional Lebesgue measure, Euclidean diameter, and interior of E. Given an IFS $\{S_i\}_{i=1}^N$ on X, a nonempty set $U \subseteq X$ (not necessarily open) is said to be *invariant* if $\bigcup_{i=1}^N S_i(U) \subseteq U$. We say that $U \subseteq X$ is *open* if it is open in the relative Euclidean topology of X.

Fix an invariant set $U \subseteq X$ and let $0 < b < 1$. Define

$$\begin{aligned} \mathscr{I}_b &:= \{I = (i_1, \ldots, i_n) \in \Sigma^* : R_I \leq b < R_{I^-}\}, \\ \mathscr{I}_b^*(U) &:= \{I = (i_1, \ldots, i_n) \in \Sigma^* : \\ &\qquad \mathscr{L}^d(S_I(U)) \leq b^d \mathscr{L}^d(U) < \mathscr{L}^d(S_{I^-}(U))\}, \\ \mathscr{A}_b &:= \{S_I : I \in \mathscr{I}_b\}, \qquad \mathscr{A}_b^*(U) := \{S_I : I \in \mathscr{I}_b^*(U)\}. \end{aligned} \tag{9}$$

We make a few remarks concerning these sets of indices or mappings. First, since R can be greater than 1, for $(i_1, i_2, \ldots) \in \Sigma^\infty$, it is possible that there are more than one prefix $I = (i_1, \ldots, i_n) \in \Sigma^*$ such that $I \in \mathcal{I}_b$. However, in view of Remark 3, the number of such prefixes must be finite. Second, it is possible that $S_I = S_{I'}$ for distinct $I, I' \in \Sigma^*$; we identify such S_I and $S_{I'}$. Last, for IFSs of contractive similitudes, $\mathcal{I}_b = \mathcal{I}_b^*(U)$ and so $\mathcal{A}_b = \mathcal{A}_b^*(U)$. In general, however, they need not be the same.

Definition 4. Let $X \subset \mathbb{R}^d$ be a compact subset with $X^\circ \neq \emptyset$ and let $S_i : X \to X$, $i = 1, \ldots, N$, be bi-Lipschitz essential contractions. We say that $\{S_i\}_{i=1}^N$ has the *logarithmic distortion property (LDP)* if there is a constant $\sigma > 0$ such that

$$\lim_{b \to 0^+} \sup_{I \in \mathcal{I}_b} \frac{b}{r_I |\ln b|^\sigma} = 0. \tag{10}$$

Remark 5. In the above definition, we do not assume that the maps of the IFS are differentiable. Besides this, if $\{S_i\}_{i=1}^N$ satisfies BDP, then there is a constant $c > 0$ such that $b/r_I \leq c$ for all $b \in (0,1)$ and $I \in \mathcal{I}_b$. Thus LDP holds. Hence LDP is an extension of BDP. Examples of IFSs satisfying LDP but not BDP will be given in Section 4.

Definition 6. Let $X, \{S_i\}_{i=1}^N, U$ satisfy the hypotheses of Definition 4, $U \subseteq X$ be a bounded invariant set that is open in the relative topology of X with $\mathcal{L}^d(U) > 0$, and Φ be a finite subset of $\{S_I : I \in \Sigma^*\}$. One calls a finite subcollection $\{\varphi_1, \ldots, \varphi_k\} \subseteq \Phi$ a *packing family* for Φ with respect to U if the following conditions are satisfied:

(i) $\varphi_1(U), \ldots, \varphi_k(U)$ are pairwise disjoint;

(ii) for any $\varphi \in \Phi$, $\varphi(U)$ intersects at least one $\varphi_j(U)$.

Denote the class of all packing families of $\mathcal{A}_b$ with respect to U by $\mathcal{P}_U(b)$, and denote the class of all packing families of $\mathcal{A}_b^*(U)$ by $\mathcal{P}_U^*(b)$.

Example 7. Let $U = (0,3)$, $S_i(x) = (1/2)(x+i)$, $i = 0,1,2,3$, and $\Phi = \{S_i\}_{i=0}^3$. Then $\{S_0, S_3\}$, $\{S_1\}$, and $\{S_2\}$ are three packing families of Φ.

Definition 8. Let X, $\{S_i\}_{i=1}^N$, U satisfy the hypotheses of Definition 4 and fix $\lambda \in (0,1)$. Define

$$\underline{Q}_\lambda(s) := \underline{\lim}_{n\to\infty} \frac{1}{n} \ln\left(\inf_{\Phi \in \mathcal{P}_U(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s \right), \quad s \in \mathbb{R},$$
$$\overline{Q}_\lambda(s) := \overline{\lim}_{n\to\infty} \frac{1}{n} \ln\left(\sup_{\Phi \in \mathcal{P}_U(\lambda^n)} \sum_{\varphi \in \Phi} R_\varphi^s \right), \quad s \in \mathbb{R}. \tag{11}$$

We call $\underline{Q}_\lambda$ (resp., $\overline{Q}_\lambda$) the *lower (resp., upper) topological pressure function (with scale λ)*. If $\underline{Q}_\lambda = \overline{Q}_\lambda$, we denote the common function by Q_λ and call it a *topological pressure function (with scale λ)*. Note that λ is fixed and s is the variable of the functions $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$.

Remark 9. The above $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$ are similar to those in [5], but they are different, since packing families are used here.

The functions $\underline{Q}_\lambda$, $\overline{Q}_\lambda$, and Q_λ depend on λ. However, they have a common zero (independent of λ), as is shown in the following main theorem.

Theorem 10. *Let X, $\{S_i\}_{i=1}^N$, U satisfy the hypotheses of Definition 6. Fix any $\lambda \in (0,1)$ and any sequence of packing families $\{S_{I_{n,1}}, \ldots, S_{I_{n,k_n}}\} \in \mathcal{P}_U(\lambda^n)$, where $n \in \mathbb{N}$. Then*

(a) $\underline{Q}_\lambda(s) = \overline{Q}_\lambda(s) = \lim_{n\to\infty}(1/n)\ln(\sum_{j=1}^{k_n} R_{I_{n,j}}^s) = (s - \dim_H K)\ln\lambda$ *for all $s \in \mathbb{R}$;*

(b) $\dim_H K = \dim_P K = \dim_B K = \lim_{n\to\infty} \ln k_n / (-n \ln \lambda)$.

For some applications, it is easier to treat $\mathcal{L}^d(S_I(U))$ than R_I and r_I. Similar to Definition 8, we define

$$\underline{Q}_\lambda^*(s) := \underline{\lim}_{n\to\infty} \frac{1}{n} \ln\left(\inf_{\Psi \in \mathcal{P}_U^*(\lambda^n)} \sum_{\varphi \in \Psi} \left[\mathcal{L}^d(\varphi(U))\right]^{s/d} \right), \quad s \in \mathbb{R},$$
$$\overline{Q_\lambda^*}(s) := \overline{\lim}_{n\to\infty} \frac{1}{n} \ln\left(\sup_{\Psi \in \mathcal{P}_U^*(\lambda^n)} \sum_{\varphi \in \Psi} \left[\mathcal{L}^d(\varphi(U))\right]^{s/d} \right), \quad s \in \mathbb{R}. \tag{12}$$

We have the following theorem.

Theorem 11. *Let X, $\{S_i\}_{i=1}^N$, U satisfy the hypotheses of Definition 6. Fix any $\lambda \in (0,1)$ and any sequence of packing families $\{S_{I_{n,1}}, \ldots, S_{I_{n,k_n}}\} \in \mathcal{P}_U^*(\lambda^n)$, where $n \in \mathbb{N}$. Then*

$$\underline{Q}_\lambda^*(s) = \overline{Q_\lambda^*}(s) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\sum_{j=1}^{k_n} \left[\mathcal{L}^d\left(S_{I_{n,j}}(U)\right)\right]^{s/d} \right) = Q_\lambda(s) = \left(s - \dim_H K\right)\ln\lambda. \tag{13}$$

A key in the proof of Theorem 11 is to use the volume estimates in (15).

In the following example, Theorem 11 is used in computing the dimension of the attractor. Although the dimension of the self-affine set can also be computed by the method by Bárány [11], the method we use appears to be simpler (see Section 4).

Example 12. Let K be the self-affine set defined by the IFS in (1) (see Figure 1). Then $\dim_H K = \dim_P K = \dim_B K = 1$.

Remark 13. Theorem 11 makes dimension computation easier. The computation would be very complicated if we use Theorem 10 or the definitions of the Hausdorff or box dimensions.

The rest of this paper is organized as follows. In Section 2 we establish some basic properties of the topological pressure functions. Section 3 is devoted to the proof of the main theorems. In Section 4 we illustrate our main results by some examples.

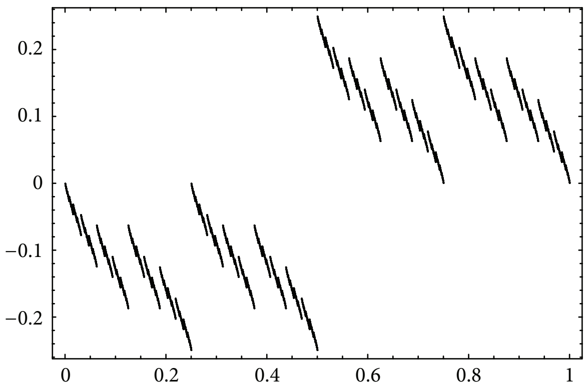


Figure 1: The self-affine set in Example 12.

2. Properties of Topological Pressures

In this section we prove some basic properties of the topological pressure functions. Let $\{S_i\}_{i=1}^N$ be an IFS of bi-Lipschitz essential contractions on a compact subset $X \subset \mathbb{R}^d$. The following inequalities will be used repeatedly, for any $E \subseteq X$, and any $I \in \Sigma^*$:

$$r_I|E| \le |S_I(E)| \le R_I|E|, \tag{14}$$

$$r_I\left(\mathscr{L}^d(E)\right)^{1/d} \le \left(\mathscr{L}^d(S_I(E))\right)^{1/d} \le R_I\left(\mathscr{L}^d(E)\right)^{1/d}. \tag{15}$$

We first state some basic properties of the topological pressures, without assuming LDP. The proof of the following proposition is similar to that of [5, Proposition 2.3]; we will only give an outline.

Proposition 14. *Let X, $\{S_i\}_{i=1}^N$, and U satisfy the hypotheses of Theorem 10 and let $\lambda \in (0,1)$. Then both $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$ are real-valued, strictly decreasing, and continuous functions on $\mathbb{R}$ that tend to $-\infty$ and ∞ as s tends to ∞ and $-\infty$, respectively. Moreover, $\overline{Q}_\lambda(0) \ge \underline{Q}_\lambda(0) \ge 0$ and $\overline{Q}_\lambda(s)$ is convex on $\mathbb{R}$.*

Proof. Since $R_I \to 0$ uniformly as $|I| \to \infty$, there is an integer $k_0 > 0$ such that $R_I \le r$ for all $I \in \Sigma^*$ such that $|I| \ge k_0$. Let $C := \max\{R_I : |I| < k_0\}$, $n \in \mathbb{N}$, and $\varphi = S_{i_1\cdots i_k} \in \Phi \in \mathscr{P}_U(\lambda^n)$. Write $k-1 = \ell k_0 + m$ with $0 \le m < k_0$. Then we have $R_{i_1\cdots i_{k-1}} \le Cr^\ell \le C' r^{k/k_0}$ for some constant C'. Hence (7) implies

$$r^k \le r_\varphi \le R_\varphi \le \lambda^n < R_{i_1\cdots i_{k-1}} \le C' r^{k/k_0}. \tag{16}$$

It follows that

$$n\log_r\lambda \le k < nk_0\log_r\lambda - k_0\log_r C', \tag{17}$$

and thus

$$\#\Phi \le N^{nk_0\log_r\lambda - k_0\log_r C'}, \qquad R_\varphi \ge r^{nk_0\log_r\lambda - k_0\log_r C'}. \tag{18}$$

Using (16)–(18) and a similar derivation as that in [5, Proposition 2.3] gives

$$\begin{aligned} s\left(k_0\log_r\lambda\right)\ln r \le \underline{Q}_\lambda(s) \le \overline{Q}_\lambda(s) & \\ \le \left(k_0\log_r\lambda\right)\ln N + s\ln\lambda, & \quad \text{if } s \ge 0, \\ s\ln\lambda \le \underline{Q}_\lambda(s) \le \overline{Q}_\lambda(s) & \\ \le \left(k_0\log_r\lambda\right)\ln N + s\left(k_0\log_r\lambda\right)\ln r, & \\ & \quad \text{if } s < 0. \end{aligned} \tag{19}$$

Hence $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$ are real-valued, $\overline{Q}_\lambda(0) \ge \underline{Q}_\lambda(0) \ge 0$. Moreover, since $0 < \lambda < 1$, we have $\lim_{s\to\infty}\underline{Q}_\lambda(s) = \lim_{s\to\infty}\overline{Q}_\lambda(s) = -\infty$ and $\lim_{s\to-\infty}\underline{Q}_\lambda(s) = \lim_{s\to-\infty}\overline{Q}_\lambda(s) = \infty$.

Next, for any $\delta > 0$, by using (16)–(18), we get

$$\begin{aligned} \underline{Q}_\lambda(s) + \delta\left(k_0\log_r\lambda\right)\ln r \le \underline{Q}_\lambda(s+\delta) & \\ \le \underline{Q}_\lambda(s) + \delta\ln\lambda < \underline{Q}_\lambda(s), & \\ \underline{Q}_\lambda(s) \le \underline{Q}_\lambda(s-\delta) \le \underline{Q}_\lambda(s) - \delta\left(k_0\log_r\lambda\right)\ln r. & \end{aligned} \tag{20}$$

Exactly the same inequalities hold for $\overline{Q}_\lambda$. Therefore, $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$ are strictly decreasing and continuous on $\mathbb{R}$. The convexity of $\overline{Q}_\lambda$ follows from Hölder's inequality. □

By using the inequalities in (15), we can prove the following proposition in the same way.

Proposition 15. *Under the same assumptions of Proposition 14, both $\underline{Q}_\lambda^*(s)$ and $\overline{Q_\lambda^*}(s)$ are real-valued, strictly decreasing, and continuous functions on $\mathbb{R}$ that tend to $-\infty$ and ∞ as s tends to ∞ and $-\infty$, respectively. Moreover, $\overline{Q_\lambda^*}(0) \ge \underline{Q}_\lambda^*(0) \ge 0$ and $\overline{Q_\lambda^*}(s)$ is convex on $\mathbb{R}$.*

We now state some simple consequences of LDP.

Lemma 16. *Assume the same hypotheses on X, $\{S_i\}_{i=1}^N$, and U as in Theorem 10. Let $0 < b_0 < 1$, R_I and let r_I be defined as in (4), and let $\sigma > 0$ be defined as in Definition 4. The following hold.*

(a) *There is a constant $c_1 > 0$ such that*

$$\frac{b}{c_1|\ln b|^\sigma} \le r_I \le R_I \le b, \quad \forall I \in \mathscr{I}_b,\ b \in (0, b_0). \tag{21}$$

(b) *There exists a constant $c_2 > 0$ such that*

$$\frac{b}{c_2|\ln b|^\sigma} \le r_J \le R_J \le c_2|\ln b|^\sigma b, \quad \forall J \in \mathscr{I}_b^*(U),\ b \in (0, b_0). \tag{22}$$

Proof. (a) By Definition 4, we have

$$0 < c_1 := \sup_{b\in(0,b_0)}\ \sup_{I\in\mathscr{I}_b} \frac{b}{r_I|\ln b|^\sigma} < \infty. \tag{23}$$

Hence $b/(c_1|\ln b|^\sigma) \le r_I$, and the conclusion follows.

(b) As mentioned in Remark 3, the IFS is not necessarily contractive in the Euclidean metric. Nevertheless, since $R_J \to 0$ as $|J| \to \infty$, there exists some $k_0 > 0$ such that $R_I \le r$ when $|I| \ge k_0$. For any $J \in \Sigma^*$ with $|J| = n \ge k_0$, let $n = \ell k_0 + t$ with $0 \le t < k_0$; that is, J can be decomposed into ℓ parts, with $\ell - 1$ of them having length k_0 and one of them having length $k_0 + t$. Hence $r^n \le r_J \le R_J \le r^\ell$. Taking logarithm, we have

$$\begin{aligned} n \ln r^{-1} &\ge \ln r_J^{-1} \ge \ln R_J^{-1} \ge \ell \ln r^{-1} \\ &\ge \left(\frac{n}{k_0} - 1\right) \ln r^{-1}, \quad |J| > k_0. \end{aligned} \tag{24}$$

Hence the set $\{(\ln r_J^{-1})/(\ln R_J^{-1}) : J \in \Sigma^*, |J| > k_0\}$ is bounded. Let $c > 0$ be a constant such that

$$\frac{\ln r_J^{-1}}{\ln R_J^{-1}} \le c, \quad \forall J \in \Sigma^* \text{ with } |J| > k_0. \tag{25}$$

Let $J \in \mathcal{I}_b^*(U)$. The definition of k_0 shows that there is a decomposition $J = J_1 J_2$ with $|J_2| \le k_0$ so that $J_1 \in \mathcal{I}_{R_J}(U)$ (since it is possible that $R_{IJ} > R_I$ for some $I, J \in \Sigma^*$). Substituting $b = R_J$ and $I = J_1$ into (21) yields

$$\frac{R_J}{r_{J_1} \left(\ln R_J^{-1}\right)^\sigma} \le c_1. \tag{26}$$

We need only prove (22) for small $b > 0$, since for any given $b_1 > 0$ the set $\{(b/r_J |\ln b|^\sigma) : J \in \mathcal{I}_b^*(U), b_1 \le b < b_0\}$ is finite. Without loss of generality, we can assume $|J| > k_0$ for any $J \in \mathcal{I}_b^*(U)$. Using (8) and the facts that $J_1 \in \mathcal{I}_{R_J}(U)$ and $|J_2| \le k_0$, we have $r^{k_0} r_{J_1} \le r_{J_1} r_{J_2} \le r_J \le r_{J_1} R^{k_0}$. As $J \in \mathcal{I}_b^*(U)$, we have

$$\left(\frac{\mathcal{L}^d\left(S_J(U)\right)}{\mathcal{L}^d(U)}\right)^{1/d} \le b < \left(\frac{\mathcal{L}^d\left(S_{J^-}^{-1}(U)\right)}{\mathcal{L}^d(U)}\right)^{1/d} \tag{27}$$

and thus $r_J \le b < R_{J^-}$. As $R_J \ge R_{J^-} r$, we get

$$r_J \le b < r^{-1} R_J. \tag{28}$$

From (25) and (28), we see that there exists some constant $\tilde{c} \ge 1$ such that $\tilde{c}^{-1} \le \ln R_J / \ln b \le \tilde{c}$. Combining the above estimates, we get

$$\frac{b}{r_J |\ln b|^\sigma} \le \frac{r^{-1} R_J \tilde{c}^\sigma}{r^{k_0} r_{J_1} \left|\ln R_J\right|^\sigma} \le \frac{\tilde{c}^\sigma c_1}{r^{k_0+1}} =: c_2 < \infty. \tag{29}$$

That is, $b/(c_2 |\ln b|^\sigma) \le r_J$. Similarly, we can show that $R_J \le c_2 |\ln b|^\sigma b$. Thus, (22) holds and this completes the proof. □

For IFSs satisfying LDP, the definitions of the topological pressures are independent of the choice of the invariant open set U and the packing families. To see this we need the following lemma.

Lemma 17. *Assume that $\{S_i\}_{i=1}^N$ and X satisfy the hypotheses of Theorem 10, $b_0 \in (0,1)$ are fixed, and U, V are nonempty invariant open subsets of X with $\mathcal{L}^d(U) > 0$ and $\mathcal{L}^d(V) > 0$. Then there is a constant $c_2 > 0$, depending only on U, V, and s, such that for any $b \in (0, b_0)$ and any two packing families $\{S_{I_1}, \dots, S_{I_k}\} \in \mathcal{P}_U(b)$ and $\{S_{J_1}, \dots, S_{J_n}\} \in \mathcal{P}_V(b)$,*

$$\left(c_2 |\ln b|^{\sigma d + \sigma |s|}\right)^{-1} \sum_{j=1}^n R_{J_j}^s \le \sum_{i=1}^k R_{I_i}^s \le c_2 |\ln b|^{\sigma d + \sigma |s|} \sum_{j=1}^n R_{J_j}^s. \tag{30}$$

Proof. Let $\Phi_i := \{S_{J_\ell} : 1 \le \ell \le n, S_{J_\ell}(U) \cap S_{I_i}(U) \ne \emptyset\}$, $i = 1, \dots, k$. By using the definition of r_{J_j}, the disjointness of $S_{J_1}(V), \dots, S_{J_n}(V)$, and the equality $\{S_{J_1}, \dots, S_{J_n}\} = \bigcup_{i=1}^k \Phi_i$, we get

$$\begin{aligned} \sum_{j=1}^n r_{J_j}^d \mathcal{L}^d(V) &\le \sum_{j=1}^n \mathcal{L}^d\left(S_{J_j}(V)\right) \\ &\le \sum_{i=1}^k \mathcal{L}^d\left(\bigcup_{\varphi \in \Phi_i} \varphi(V)\right). \end{aligned} \tag{31}$$

Let $B_\gamma(x_0) \subset U$ be a ball with radius $\gamma > 0$ and center x_0. Then $S_{I_i}(U)$ contains a ball $B_{r_{I_i}\gamma}(x_i)$ with radius $r_{I_i}\gamma > 0$ and center $x_i = S_{I_i}(x_0)$. For each $S_{J_\ell} \in \Phi_i$, $S_{J_\ell}(U) \cap S_{I_i}(U) \ne \emptyset$, and both $S_{J_j}(V)$ and $S_{I_i}(V)$ have diameters bounded above by $b|V|$. Let $\tau = 2(|U| + |V|)$. Then $S_{J_\ell}(U) \cup S_{J_\ell}(V) \subset B_{\tau b}(x_i)$ for $S_{J_\ell} \in \Phi_i$. Therefore, (31) implies

$$\sum_{j=1}^n r_{J_j}^d \mathcal{L}^d(V) \le \sum_{i=1}^k \mathcal{L}^d\left(B_{\tau b}(x_i)\right) \le k b^d \mathcal{L}^d\left(B_\tau(0)\right). \tag{32}$$

By using the inequality $r_{J_j} \ge b/(c_1 |\ln b|^\sigma)$ from Lemma 16, we get

$$n \left(\frac{b}{c_1 |\ln b|^\sigma}\right)^d \mathcal{L}^d(V) \le k b^d \mathcal{L}^d\left(B_\tau(0)\right), \tag{33}$$

and so $n \le ck |\ln b|^{\sigma d}$ with $c = c_1^d \mathcal{L}^d(B_\tau(0))(\mathcal{L}^d(V))^{-1}$. Interchanging the roles of the two packing families and using the same argument, we get $k \le cn |\ln b|^{\sigma d}$. Hence

$$c^{-1} \left|\ln b\right|^{-\sigma d} k \le n \le ck \left|\ln b\right|^{\sigma d}. \tag{34}$$

Also, by Lemma 16, we have $b \le c_1 |\ln b|^\sigma r_{I_i}$ and $b \le c_1 |\ln b|^\sigma r_{J_j}$. Hence for all $s \ge 0$,

$$\begin{aligned} \sum_{j=1}^n R_{J_j}^s &\le n b^s \le ck |\ln b|^{\sigma d} b^s \\ &\le c c_1^s |\ln b|^{\sigma d + \sigma s} \sum_{i=1}^k r_{I_i}^s \le c c_1^s |\ln b|^{\sigma d + \sigma s} \sum_{i=1}^k R_{I_i}^s. \end{aligned} \tag{35}$$

By symmetry, $\sum_{i=1}^{k} R_{I_i}^s \le cc_1^s |\ln b|^{\sigma d+\sigma s} \sum_{j=1}^{n} R_{J_j}^s$. Therefore,

$$\left(cc_1^s|\ln b|^{\sigma d+\sigma s}\right)^{-1}\sum_{j=1}^{n}R_{J_j}^s \le \sum_{i=1}^{k}R_{I_i}^s \le cc_1^s|\ln b|^{\sigma d+\sigma s}\sum_{j=1}^{n}R_{J_j}^s, \quad \forall s \ge 0. \tag{36}$$

The conclusion for the case $s \ge 0$ follows by letting $c_2 = cc_1^s$. The proof for the case $s < 0$ is similar; we omit the details. □

The following proposition follows easily from Lemma 17 and its proof.

Proposition 18. *Let X and $\{S_i\}_{i=1}^N$ satisfy the hypotheses of Theorem 10. Then for any nonempty invariant open set $U \subset X$ with $\mathscr{L}^d(U) > 0$, and any sequence of packing families $\{S_{I_{n,j}}\}_{j=1}^{k_n}$ of $\mathscr{A}_{\lambda^n}$, one has*

$$\underline{Q}_\lambda(s) = \varliminf_{n\to\infty}\frac{1}{n}\ln\sum_{j=1}^{k_n}R_{I_{n,j}}^s = \varliminf_{n\to\infty}\frac{1}{n}\ln\sum_{j=1}^{k_n}r_{I_{n,j}}^s, \quad s\in\mathbb{R}, \tag{37}$$

$$\overline{Q}_\lambda(s) = \varlimsup_{n\to\infty}\frac{1}{n}\ln\sum_{j=1}^{k_n}R_{I_{n,j}}^s = \varlimsup_{n\to\infty}\frac{1}{n}\ln\sum_{j=1}^{k_n}r_{I_{n,j}}^s, \quad s\in\mathbb{R}. \tag{38}$$

Thus, the definitions of $\underline{Q}_\lambda$, $\overline{Q}_\lambda$, and Q_λ are independent of the choices of the invariant open set $U \subset X$ and the packing families. Furthermore, in Definition 8, R_φ can be replaced by r_φ.

In the following, the open set U will not be mentioned unless it is necessary.

In order to obtain a lower estimate for the Hausdorff dimension in Theorem 10, we need the *mass distribution principle* (Lemma 19) and Proposition 20 below.

Lemma 19 (see, e.g., [13, Theorem 4.9]). *Let $K \subseteq \mathbb{R}^n$, μ a positive Borel measure on K with $0 < \mu(K) < \infty$, and $s \ge 0$. If there is a constant $c > 0$ such that $\varlimsup_{r\to 0^+}(\mu(B_r(x))/r^s) \le c$ for any $x \in K$, then $\dim_H K \ge s$.*

Recall that an IFS $\{S_i\}_{i=1}^N$ on X satisfies the *open set condition* (OSC) if there exists a nonempty bounded invariant open (in the relative topology of X) set $O \subset X$, called an *OSC set*, such that $\bigcup_{i=1}^N S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i \ne j$.

The following result is similar to that of [15, Theorem 10.3] where the strong separation condition is used; we include a proof for convenience.

Proposition 20. *Let K be the attractor of an IFS $\{S_i\}_{i=1}^N$ satisfying the hypotheses of Theorem 10. If OSC holds with an OSC set U satisfying $U \supseteq K$ and $\sum_{j=1}^N r_j^s > 1$, then $\dim_H K \ge s$.*

Proof. Choose $t > s$ such that $\sum_{j=1}^N r_j^t = 1$. Let $p_j = r_j^t$ and let μ be the invariant probability measure associated with the weights $\{p_i\}_{i=1}^N$ (see [14]); that is, $\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1}$.

For any $x \in K$ and sufficiently small $b > 0$, let

$$\{I_1,\dots,I_k\} = \{I \in \mathscr{I}_b : S_I(U) \cap B_b(x) \ne \emptyset\}. \tag{39}$$

Then OSC implies that $S_{I_1}(U),\dots,S_{I_k}(U)$ are disjoint and the fact that $I_j \in \mathscr{I}_b$ implies that $|S_{I_j}(U)| \le b|U|$. The definition of r_I implies that $\mathscr{L}^d(S_{I_j}(U)) \ge r_{I_j}^d \mathscr{L}^d(U)$ (see (15) also), and Lemma 16 implies $b/(c_1|\ln b|^\sigma) \le r_{I_j} \le R_{I_j} \le b$. Hence

$$\begin{aligned} b^d\mathscr{L}^d(B_{1+|U|}(x)) &= \mathscr{L}^d(B_{b+b|U|}(x)) \ge \mathscr{L}^d\left(\bigcup_{j=1}^k S_{I_j}(U)\right) \\ &= \sum_{j=1}^k \mathscr{L}^d\left(S_{I_j}(U)\right) \ge k\left(\frac{b}{c_1|\ln b|^\sigma}\right)^d \mathscr{L}^d(U). \end{aligned} \tag{40}$$

Thus there is a constant $c_4 > 0$ such that

$$k \le c_4|\ln b|^{\sigma d}. \tag{41}$$

Combining OSC and the fact that $U \supseteq K$ gives

$$\mu\left(S_{i_1\cdots i_n}(U)\right) = p_{i_1}\cdots p_{i_n} = r_{i_1}^t\cdots r_{i_n}^t \le r_{i_1\cdots i_n}^t. \tag{42}$$

Using $U \supseteq K$, (41), together with the fact that $r_{I_i} \le b$, we get

$$\mu(B_b(x)) \le \sum_{j=1}^k r_{I_j}^t \le c_4 b^t |\ln b|^{\sigma d}. \tag{43}$$

Since $t > s$, $\lim_{b\to 0^+} b^{t-s}|\ln b|^{\sigma d} = 0$. Hence inequality (43) implies $\lim_{b\to 0^+}\mu(B_b(x))/b^s = 0$ for all $x \in K$. The conclusion follows by using Lemma 19. □

3. Proof of the Main Theorems

This section is devoted to the proofs of the main theorems.

Proof of Theorem 10. In order to apply Proposition 20, we first use Proposition 18 to require, in addition, that $U \supseteq K$. Let α and β be the zeroes of $\underline{Q}_\lambda(s)$ and $\overline{Q}_\lambda(s)$, respectively. By Proposition 18, α and β are independent of the choice of the packing family. Proposition 14 implies that both $\underline{Q}_\lambda(0)$ and $\overline{Q}_\lambda(0)$ are real numbers.

We first prove

$$\begin{aligned} \underline{Q}_\lambda(s) &= \underline{Q}_\lambda(0) + s\ln\lambda = (s-\alpha)\ln\lambda, \\ \overline{Q}_\lambda(s) &= \overline{Q}_\lambda(0) + s\ln\lambda = (s-\beta)\ln\lambda. \end{aligned} \tag{44}$$

Substituting $b = \lambda^n$ and $I = I_{n,j}$ into (21) gives

$$\frac{\lambda^n}{c_1(n|\ln\lambda|)^\sigma} \le r_{I_{n,j}} \le R_{I_{n,j}} \le \lambda^n, \quad n = 1,2,\dots,\ i = 1,\dots,k. \tag{45}$$

Hence

$$\lambda^{ns}k_n \geq \sum_{i=1}^{k_n} R_{I_{n,j}}^s \geq \left(\frac{1}{c_1(n|\ln\lambda|)^\sigma}\right)^s k_n\lambda^{ns}, \quad \text{if } s \geq 0,$$
$$\lambda^{ns}k_n \leq \sum_{i=1}^{k_n} R_{I_{n,j}}^s \leq \left(\frac{1}{c_1(n|\ln\lambda|)^\sigma}\right)^s k_n\lambda^{ns}, \quad \text{if } s < 0. \tag{46}$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{c_1(n|\ln\lambda|)^\sigma}\right)^s = 0, \tag{47}$$

by using **Proposition 18** and the fact that $\underline{Q}_\lambda(0)$ and $\overline{Q}_\lambda(0)$ are real numbers, we have

$$\begin{aligned}\underline{Q}_\lambda(s) &= \varliminf_{n\to\infty}\frac{1}{n}\ln\left(\sum_{i=1}^{k_n} R_{I_{n,j}}^s\right)\\ &= \varliminf_{n\to\infty}\frac{1}{n}\ln\left(\lambda^{ns}k_n\right) = \underline{Q}_\lambda(0) + s\ln\lambda\\ \overline{Q}_\lambda(s) &= \varlimsup_{n\to\infty}\frac{1}{n}\ln\left(\sum_{i=1}^{k_n} R_{I_{n,j}}^s\right)\\ &= \varlimsup_{n\to\infty}\frac{1}{n}\ln\left(\lambda^{ns}k_n\right) = \overline{Q}_\lambda(0) + s\ln\lambda.\end{aligned} \tag{48}$$

Equation (44) now follows from the equalities $\underline{Q}_\lambda(0) = -\alpha\ln\lambda$ and $\overline{Q}_\lambda(0) = -\beta\ln\lambda$.

Next, we prove

$$\beta \leq \dim_H K. \tag{49}$$

Suppose, on the contrary, $\beta > \dim_H K$. Then there exists s such that $\dim_H K < s < \beta$. We will derive a contradiction.

By (44), $\overline{Q}_\lambda(s) = (s-\beta)\ln\lambda > 0$. Choose a sequence of packing families $\{S_{I_{n,j}}\}_{j=1}^{k_n}$ of $\mathscr{A}_{\lambda^n}$ with respect to U, where $n > 0$. Then by using (38), there exists an integer $n > 0$ such that

$$\frac{1}{n}\ln\left(\sum_{j=1}^{k_n} r_{I_{n,j}}^s\right) \geq \frac{\overline{Q}_\lambda(s)}{2} > 0. \tag{50}$$

Denote the new IFS $\{S_{I_{n,j}}\}_{j=1}^{k_n}$ by $\{f_j\}_{j=1}^{k_n}$ and let K_n be its attractor. Then this IFS satisfies OSC with U being an OSC set. Since $U \supseteq K$, by applying **Proposition 20** to the new IFS $\{f_j\}_{j=1}^{k_n}$ and noticing that $K_n \subseteq K$, we get $\dim_H K \geq \dim_H(K_n) \geq s$, a contradiction. Thus $\dim_H K \geq \beta$.

Now, we prove

$$\underline{Q}_\lambda(s) = \overline{Q}_\lambda(s) = (s - \dim_H K)\ln\lambda. \tag{51}$$

To this end we first prove $\alpha \geq \dim_H K$. Let $s > \alpha$. Then $\underline{Q}_\lambda(s) < 0$ and $s > 0$ by the fact that $\overline{Q}_\lambda(0) \geq 0$ (**Proposition 14**). For every integer $n > 0$, choose a sequence of packing families $\{S_{I_{n,j}}\}_{j=1}^{k_n}$ of $\mathscr{A}_{\lambda^n}$ with respect to U.

For any $\varphi \in \mathscr{A}_{\lambda^n}$, there is at least one j such that $\varphi(U) \cap S_{I_{n,j}}(U) \neq \emptyset$. Choose $x_{n,j} \in S_{I_{n,j}}(U)$. Since (14) implies $|\varphi(U)|$, $|S_{I_{n,j}}(U)| \leq \lambda^n|U|$, $\varphi(U)$ is contained in the ball $B_{2\lambda^n|U|}(x_{n,j})$ with radius $2\lambda^n|U|$. Since $K \subset U$, we have

$$K = \bigcup_{\varphi\in\mathscr{A}_{\lambda^n}}\varphi(K) \subset \bigcup_{\varphi\in\mathscr{A}_{\lambda^n}}\varphi(U) \subset \bigcup_{j=1}^{k_n} B_{2\lambda^n|U|}\left(x_{n,j}\right). \tag{52}$$

Hence $\{B_{2\lambda^n|U|}(x_{n,j})\}_{j=1}^{k_n}$ is a $\delta_n := 4\lambda^n|U|$-cover of K. By (48),

$$\varliminf_{n\to\infty}\frac{1}{n}\ln\left[k_n(4\lambda^n|U|)^s\right] = \underline{Q}_\lambda(0) + s\ln\lambda = \underline{Q}_\lambda(s) < 0. \tag{53}$$

Hence, $k_n(4\lambda^n|U|)^s < 1$ for infinitely many integers n. Therefore,

$$\mathscr{H}^s(K) = \lim_{n\to\infty}\mathscr{H}^s_{\delta_n}(K) \leq \varliminf_{n\to\infty} k_n(4\lambda^n|U|)^s < \infty, \tag{54}$$

and thus $s \geq \dim_H K$. Since $s > \alpha$ is arbitrary, we conclude that $\alpha \geq \dim_H K$.

Since $\alpha \leq \beta$, by combining this with (49), we get $\alpha = \beta = \dim_H K$. Equation (51) now follows by substituting this into (44).

Finally, we prove $\dim_H K = \dim_B K$. Let U be as above. For any $n > 0$ and $\lambda \in (0,1)$, choose a packing family $\{S_{I_{n,j}}\}_{j=1}^{k_n}$ of $\mathscr{A}_{\lambda^n}$ with respect to U. Let

$$\begin{aligned}\mathscr{B}_n := \Bigg\{&\prod_{i=1}^d\left[(m_i-1)\lambda^n, m_i\lambda^n\right) : m_i \in \mathbb{Z},\\ &K \cap \prod_{i=1}^d\left[(m_i-1)\lambda^n, m_i\lambda^n\right) \neq \emptyset\Bigg\}\end{aligned} \tag{55}$$

and let $N_n := \#\mathscr{B}_n$, the cardinality of $\mathscr{B}_n$. According to (52), we define

$$\begin{aligned}\mathscr{B}_{n,j} = \Bigg\{&\prod_{i=1}^d\left[(m_i-1)\lambda^n, m_i\lambda^n\right) \in \mathscr{B}_n : m_i \in \mathbb{Z},\\ &B_{2\lambda^n|U|}\left(x_{n,j}\right) \cap \prod_{i=1}^d\left[(m_i-1)\lambda^n, m_i\lambda^n\right) \neq \emptyset\Bigg\}.\end{aligned} \tag{56}$$

Then (52) implies $\mathscr{B}_n = \bigcup_{j=1}^{k_n}\mathscr{B}_{n,j}$. If $B_{2\lambda^n|U|}(x_{n,j}) \cap \prod_{i=1}^d[(m_i-1)\lambda^n, m_i\lambda^n) \neq \emptyset$, then $\prod_{i=1}^d[(m_i-1)\lambda^n, m_i\lambda^n)$ is contained in $B_{2\lambda^n|U|+\sqrt{d}\lambda^n}(x_{n,j})$. Since $\{\prod_{i=1}^d[(m_i-1)\lambda^n, m_i\lambda^n)\}$ are disjoint, we have

$$\begin{aligned}\#\mathscr{B}_{n,j} &\leq \frac{\mathscr{L}^d\left(B_{2\lambda^n|U|+\sqrt{d}\lambda^n}\left(x_{n,j}\right)\right)}{\mathscr{L}^d\left(\prod_{i=1}^d\left[(m_i-1)\lambda^n, m_i\lambda^n\right)\right)}\\ &= \mathscr{L}^d\left(B_{2|U|+\sqrt{d}}(0)\right).\end{aligned} \tag{57}$$

Therefore,

$$N_n \leq \sum_{j=1}^{k_n}\#\mathscr{B}_{n,j} \leq k_n\mathscr{L}^d\left(B_{2|U|+\sqrt{d}}(0)\right). \tag{58}$$

Hence

$$\overline{\dim}_B K = \overline{\lim_{n\to\infty}} \frac{\ln N_n}{-\ln \lambda^n} \le \overline{\lim_{n\to\infty}} \frac{\ln k_n}{-n\ln\lambda} = \frac{\overline{Q}_\lambda(0)}{-\ln\lambda} = \beta = \dim_H K. \tag{59}$$

It follows immediately that $\overline{\dim}_B K = \underline{\dim}_B K = \beta = \dim_H K$. Since $\dim_H K \le \dim_P K \le \overline{\dim}_B K$, the proof is complete. □

A similar argument shows the following corollary.

Corollary 21. *Assume the same hypotheses of Lemma 17. Also, for any given $\lambda \in (0,1)$, $L \in \mathbb{N}$ and $\{S_{I_{n,j}}\}_{j=1}^{m_n} \subset \mathscr{A}_{\lambda^n}$ for $n \in \mathbb{N}$, assume that the following conditions hold.*

(1) *For any $\varphi \in \mathscr{A}_{\lambda^n}$, there is at least one $j \in \{1,\ldots,m_n\}$ such that $\varphi(U)$ intersects $S_{I_{n,j}}(U)$.*

(2) *For any $S_{I_{n,\ell}}$, there are at most L maps $S_{I_{n,j}}$ such that $S_{I_{n,\ell}}(U) \cap S_{I_{n,j}}(U) \ne \emptyset$.*

Then Theorem 10 holds by replacing the packing families with $\{S_{I_{n,j}}\}_{j=1}^{m_n}$, $n \in \mathbb{N}$.

Remark 22. For IFSs consisting of C^1 conformal contractions and satisfying BDP and WSC (see [3]), Theorem 1.1 of [5] gives a method for computing $\dim_H K$ by solving the equation $P(s) = 0$. We remark that, in computing the function $P(s)$, the sum in the definition of $P(s)$ is over distinct maps, and thus in numerical computations the following two types of mistakes may occur:

(a) $S_I \ne S_J$, but numerical approximations show $S_I = S_J$;

(b) $S_I = S_J$, but numerical approximations show $S_I \ne S_J$.

In view of Corollary 21 and the definition of packing families, the formula $\dim_H K = \lim_{n\to\infty}(\ln k_n/n\ln\lambda)$ is numerically much more stable.

Proof of Theorem 11. In view of (15), we have

$$\begin{aligned}\sum_{j=1}^{k_n} r_{I_{n,j}}^s &\le \sum_{j=1}^{k_n} \frac{\left[\mathscr{L}^d\left(S_{I_{n,j}}(U)\right)\right]^{s/d}}{\left[\mathscr{L}^d(U)\right]^{s/d}} \\ &\le \sum_{j=1}^{k_n} R_{I_{n,j}}^s \le \sup_{\Psi\in\mathscr{P}_U^*(\lambda^n)} \sum_{\psi\in\Psi} R_\psi^s, \quad s \ge 0, \\ \sum_{j=1}^{k_n} r_{I_{n,j}}^s &\ge \sum_{j=1}^{k_n} \frac{\left[\mathscr{L}^d\left(S_{I_{n,j}}(U)\right)\right]^{s/d}}{\left[\mathscr{L}^d(U)\right]^{s/d}} \\ &\ge \sum_{j=1}^{k_n} R_{I_{n,j}}^s \ge \inf_{\Psi\in\mathscr{P}_U^*(\lambda^n)} \sum_{\psi\in\Psi} R_\psi^s, \quad s < 0.\end{aligned} \tag{60}$$

Thus, by using (37) and (38) we need only prove

$$\begin{aligned}Q_\lambda(s) &= \lim_{n\to\infty} \frac{1}{n} \ln\left(\sup_{\Psi\in\mathscr{P}_U^*(\lambda^n)} \sum_{\psi\in\Psi} R_\psi^s\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \ln\left(\inf_{\Psi\in\mathscr{P}_U^*(\lambda^n)} \sum_{\psi\in\Psi} R_\psi^s\right), \quad s \in \mathbb{R}.\end{aligned} \tag{61}$$

For any $b \in (0,\lambda]$ and any two packing families $\{S_{I_1},\ldots,S_{I_k}\} \in \mathscr{P}_U(b)$ and $\{S_{J_1},\ldots,S_{J_m}\} \in \mathscr{P}_U^*(b)$. Similar to the proof of Lemma 17, let $\Psi^i = \{S_{J_j} : S_{J_j}(U) \cap S_{I_i}(U) \ne \emptyset\}$ and $\Phi^j = \{S_{I_i} : S_{J_j}(U) \cap S_{I_i}(U) \ne \emptyset\}$.

Using LDP, we have

$$\begin{aligned}\left(\frac{R_{J_j}}{c_1 c_2^2 |\ln b|^{3\sigma}}\right)^d \mathscr{L}^d(U) &\le \left(\frac{r_{J_j}}{c_1 |\ln b|^\sigma}\right)^d \mathscr{L}^d(U) \\ &\le \left(\frac{1}{c_1|\ln b|^\sigma}\right)^d \mathscr{L}^d\left(S_{J_j}(U)\right) \\ &\le \left(\frac{b}{c_1|\ln b|^\sigma}\right)^d \mathscr{L}^d(U) \\ &\le r_{I_i}^d \mathscr{L}^d(U) \le \mathscr{L}^d\left(S_{I_i}(U)\right) \\ &\le R_{I_i}^d \mathscr{L}^d(U),\end{aligned} \tag{62}$$

where the first and fourth inequalities follow from (22) and (21), respectively, the second, fifth, and last ones follow from (15), and the third one follows from the definition of $\mathscr{I}_b^*(U)$. We assume, without loss of generality, that $c_2 \ge c_1$. It follows that

$$\frac{R_{J_j}}{\left(c_2|\ln b|\right)^{3\sigma}} \le R_{I_i} \quad \forall 1 \le i \le k,\ 1 \le j \le m. \tag{63}$$

By using (22) we see that $\cup\{S_{J_j}(U) : S_{J_j} \in \Psi^i\}$ is contained in a ball with center in $S_{I_i}(U)$ and radius $(c_1+c_2)|\ln b|^\sigma b|U|$. Hence it follows from (22) again that

$$\begin{aligned}&\#\Psi^i \mathscr{L}^d(U)\left(\frac{b}{c_2|\ln b|^\sigma}\right)^d \\ &\quad\le \sum_{S_{J_j}\in\Psi^i} \mathscr{L}^d\left(S_{J_j}(U)\right) \\ &\quad\le \left(\left(c_1+c_2\right)|\ln b|^\sigma b|U|\right)^d \mathscr{L}^d\left(B_1(0)\right).\end{aligned} \tag{64}$$

Therefore, there is a constant $c_3 > 0$ such that

$$m \le \sum_{i=1}^k \#\Psi^i \le c_3 |\ln b|^{2d\sigma} k. \tag{65}$$

By interchanging the roles of the two packing families, it can be proved in the same way that there exist constants $c_4 > 0$ and $c_5 > 0$ such that

$$\frac{R_{I_i}}{(c_4 |\ln b|)^{3\sigma}} \le R_{J_j}, \quad \text{for } 1 \le i \le k,\ 1 \le j \le m, \qquad k \le c_5 |\ln b|^{2d\sigma} m. \tag{66}$$

Now, for any two sequences $\Phi_n = \{S_{I_1}, \ldots, S_{I_k}\} \in \mathcal{P}_U(\lambda^n)$ and $\Psi_n = \{S_{J_1}, \ldots, S_{J_m}\} \in \mathcal{P}_U^*(\lambda^n)$, by combining (63)–(66), we have

$$\begin{aligned}
&\sum_{\varphi \in \Phi_n} R_\varphi^s \\
&\le k \max\left\{R_{I_i}^s : 1 \le i \le k\right\} \\
&\le \left(c_5 |\ln \lambda^n|^{2d\sigma}\right) m (c_4 |\ln \lambda^n|)^{3\sigma s} \min\left\{R_{J_j}^s : 1 \le j \le m\right\} \\
&\le \left(c_5 |\ln \lambda^n|^{2d\sigma}\right) (c_4 |\ln \lambda^n|)^{3\sigma s} \sum_{\psi \in \Psi_n} R_\psi^s \\
&\le \left(c_5 |\ln \lambda^n|^{2d\sigma}\right) (c_4 |\ln \lambda^n|)^{3\sigma s} \\
&\quad \times \left(c_3 |\ln \lambda^n|^{2d\sigma}\right) (c_2 |\ln \lambda^n|)^{3\sigma s} \sum_{\varphi \in \Phi_n} R_\phi^s, \quad s \ge 0.
\end{aligned} \tag{67}$$

Similarly,

$$\begin{aligned}
&\sum_{\varphi \in \Phi_n} R_\varphi^s \\
&\le k \max\left\{R_{I_i}^s : 1 \le i \le k\right\} \\
&\le \frac{\left(c_5 |\ln \lambda^n|^{2d\sigma}\right)}{(c_2 |\ln \lambda^n|)^{3\sigma s}} m \min\left\{R_{J_j}^s : 1 \le j \le m\right\} \\
&\le \frac{\left(c_5 |\ln \lambda^n|^{2d\sigma}\right)}{(c_2 |\ln \lambda^n|)^{3\sigma s}} \sum_{\psi \in \Psi_n} R_\psi^s \\
&\le \frac{\left(c_5 |\ln \lambda^n|^{2d\sigma}\right)}{(c_2 |\ln \lambda^n|)^{3\sigma s}} \frac{\left(c_3 |\ln \lambda^n|^{2d\sigma}\right)}{(c_4 |\ln \lambda^n|)^{3\sigma s}} \sum_{\varphi \in \Phi_n} R_\varphi^s, \quad s < 0.
\end{aligned} \tag{68}$$

It now follows from these inequalities and Theorem 10 that (61) holds. The proof is complete. □

4. Examples

In this section we illustrate the applications of our results by some examples.

Example 23. Let A be a $d \times d$ real matrix, $\mathcal{A} = \{A^k : k = 0, 1, 2, \ldots\}$, and let $S_j(x) = \rho_j A_j(x + d_j)$, $j = 1, \ldots, N$, be an IFS with $A_j \in \mathcal{A}$, and $0 < |\rho_j| < 1$. Assume that all eigenvalues of A have moduli 1. Then

(a) LDP is satisfied and thus the conclusions of Theorems 10 and 11 hold;

(b) BDP holds if and only if there is a real invertible matrix B and a real orthogonal matrix Q such that $A = BQB^{-1}$. In this case, the attractor is similar to a self-similar set generated by the IFS with A replaced by Q;

(c) if OSC holds, then $\dim_H K$ is the unique solution of the equation

$$\sum_{j=1}^N \left|\rho_j\right|^s = 1. \tag{69}$$

Proof. Letting $0 < \gamma < \min\{\rho_j^{-1} : 1 \le j \le N\}$ and using the following norm $|\cdot|'$ in [12]:

$$|x|' := \sum_{k=0}^\infty \gamma^k \left|A^{-k} x\right|, \tag{70}$$

we see that $\{S_i\}_{i=1}^N$ is an IFS of essential contractions.

For the matrix A, by the Jordan decomposition theorem, there is an invertible complex matrix P such that

$$P^{-1} A P = \operatorname{diag}(J_1, \ldots, J_\ell), \tag{71}$$

where each J_j is a Jordan block with all diagonal entries being the same and equal to 1 in modulus.

(a) We need only show that the IFS satisfies LDP. Let

$$A_j = A^{k_j}, \qquad \overline{k} = \max\left\{k_j\right\}, \qquad \overline{\rho} = \max\left\{\rho_j\right\}. \tag{72}$$

Since all eigenvalues of A are 1 in modulus, using (71), it is not difficult (see, e.g., [5]) to prove that

$$\begin{gathered}
c^{-1} k^{-d} \le \left\|A^k\right\| \le c k^d, \\
c^{-1} k^{-d} \le \left\|A^{-k}\right\| \le c k^d, \\
k = 1, 2, \ldots
\end{gathered} \tag{73}$$

for some constant $c > 0$.

For any $I = i_1 \cdots i_n \in \mathcal{I}_b$, let $S_{i_1 \cdots i_n}(x) = \rho_{i_1 \cdots i_n} A^k (x + \alpha)$ for some $\alpha \in \mathbb{R}^d$ with $k = k_{i_1} + \cdots + k_{i_n}$. Then the R_I and r_I defined in (4) become

$$r_{i_1 \cdots i_n} = \left|\rho_{i_1 \cdots i_n}\right| \left\|A^{-k}\right\|^{-1}, \qquad R_{i_1 \cdots i_n} = \left|\rho_{i_1 \cdots i_n}\right| \left\|A^k\right\|. \tag{74}$$

By using these, (73), and the inequality $k \le n\overline{k}$, we get

$$\frac{R_{i_1 \cdots i_n}}{r_{i_1 \cdots i_n}} \le c^2 k^{2d} \le c^2 n^{2d} \overline{k}^{2d}. \tag{75}$$

By the definition of $\mathcal{A}_b$, we have $|\rho_{I^-}| \cdot \|A^{k-k_n}\| > b$, and hence (73) implies

$$\begin{aligned}
c\left(n\overline{k}\right)^d (\overline{\rho})^{n-1} &\ge c k^d \left|\rho_{I^-}\right| \\
&\ge c\left(k - k_{i_n}\right)^d \left|\rho_{I^-}\right| \\
&\ge \left|\rho_{I^-}\right| \left|A^{k-k_0}\right| > b
\end{aligned} \tag{76}$$

and thus

$$n < \frac{\ln\left(b\overline{\rho}/c\overline{k}^{d}\right) - d\ln n}{\ln\overline{\rho}}. \tag{77}$$

Therefore, (75) and (77) imply

$$\begin{aligned}\frac{R_{i_1\cdots i_n}}{r_{i_1\cdots i_n}|\ln b|^{3d}} &\le \frac{c^2k^{2d}}{|\ln b|^{3d}} \le \frac{c^2n^{3d}\overline{k}^{2d}}{|\ln b|^{3d}n^d}\\ &\le \frac{\left(\ln\left((b\overline{\rho})/c\overline{k}^{d}\right) - d\ln n\right)^{3d}}{|\ln b|^{3d}n^d}\cdot\frac{c^2\overline{k}^{2d}}{\ln(\overline{\rho})^{3d}}.\end{aligned} \tag{78}$$

Since $n \to \infty$ as $b \to 0^+$,

$$\lim_{b\to 0^+}\frac{|\ln b|^i(\ln n)^j}{|\ln b|^{3d}n^d} = 0, \quad \forall 0 \le i \le 3d,\ 0 \le j \le 3d. \tag{79}$$

Hence (78) implies

$$\lim_{b\to 0^+}\sup_{I\in\mathcal{I}_b}\frac{R_I}{r_I|\ln b|^{3d}} = 0; \tag{80}$$

that is, LDP holds. Part (a) follows.

(b) The sufficiency is obvious. For the necessity, assume BDP holds, and let $A_j = A^{k_j}$. Let $1^n := (1,\dots,1) \in \Sigma^n$. Then $S_{1^n}(x) = \rho_1^n A^{nk_1}(x+\alpha_n)$ for some α_n. Since all eigenvalues of A have moduli 1, it follows that $\mathcal{L}^d(S_{1^n}(V)) = \rho_1^n\mathcal{L}^d(V)$. Hence, by (15), $r_{1^n} \le \rho_1^n \le R_{1^n} = \rho_1^n\|A^{nk_1}\|$. Thus BDP implies $\sup_{n>0}\|A^{nk_1}\| < \infty$. For any ℓ, choosing n so that $nk_1 \le \ell < k_1(n+1)$, we have $\|A^\ell\| \le \|A^{nk_1}\|\cdot\|A^{\ell-nk_1}\|$. Hence

$$\sup_{n>0}\|A^n\| < \infty. \tag{81}$$

Using (71), we have $P^{-1}A^nP = \operatorname{diag}(J_1^n,\dots,J_\ell^n)$. Hence (81) implies

$$\sup_{n>0}\left\|J_j^n\right\| < \infty, \quad j = 1,\dots,\ell. \tag{82}$$

As the operator norm of a matrix is larger than or equal to the maximum of the absolute values of its entries, (82) implies that the entries of J_j^n are uniformly bounded for all $n > 0$ and $j = 1,\dots,\ell$. Hence, similar to the proof of [16, Lemma 2.3], each J_j is of order 1×1, that is, a number with modulus 1. Thus $\ell = d$ and each column of P is an eigenvector of A.

Since the eigenvalues of A must be ±1 or pairs of complex conjugates with moduli 1, by rearranging the columns of P, we may assume, without loss of generality,

$$A = P\operatorname{diag}\left[\lambda_1 I_{n_1},\overline{\lambda}_1 I_{n_1},\dots,\lambda_s I_{n_s},\overline{\lambda}_s I_{n_s}, I_u, -I_v\right]P^{-1}, \tag{83}$$

where $\overline{\lambda}_j$ is the complex conjugate of λ_j, all λ_j are distinct (when $s > 1$), and $2(\sum_{j=1}^s n_j) + u + v = d$.

Decompose P as $P = [P_1, Q_1,\dots,P_s,Q_s,U,V]$ according to (83). Then the columns of P_j, Q_j, U, V consist of a basis for the eigenspace of $\lambda_j, \overline{\lambda}_j, 1, -1$, respectively.

By rechoosing a basis for each eigenspace if necessary, we may assume further that P is of the form $P = [P_1,\overline{P}_1,\dots,P_s,\overline{P}_s,\widetilde{P}_1,\widetilde{P}_{-1}]$, where $\overline{P}_j$ is the conjugate of P_j ($j = 1,\dots,s$), the columns of $P_1,\overline{P}_1,\dots,P_s,\overline{P}_s,\widetilde{P}_1,\widetilde{P}_{-1}$ consist of bases for the eigenspaces of the eigenvalues $\lambda_1,\overline{\lambda}_1,\dots,\lambda_s,\overline{\lambda}_s,1,-1$, respectively, and $\widetilde{P}_1$, $\widetilde{P}_{-1}$ are real matrices.

Let $\{\alpha_{j,k} + i\beta_{j,k}\}_{k=1}^{n_j}$ be the columns of P_j. Then $\{\alpha_{j,k} - i\beta_{j,k}\}_{k=1}^{n_j}$ are the columns of $\overline{P}_j$. Since P is invertible, column operations show that so is

$$\begin{aligned}B = \big[&\alpha_{1,1},\beta_{1,1},\dots,\alpha_{1,n_1},\beta_{1,n_1},\dots,\\ &\alpha_{s,1},\beta_{s,1},\dots,\alpha_{s,n_s},\beta_{s,n_s},U,V\big].\end{aligned} \tag{84}$$

Moreover, as each column of P_j is an eigenvector of A, by letting $\lambda_j = a_j + ib_j$, we have

$$\begin{aligned}a_j^2 + b_j^2 = 1, \qquad A\alpha_{j,k} &= a_j\alpha_{j,k} - b_j\beta_{j,k},\\ A\beta_{j,k} &= b_j\alpha_{j,k} + a_j\beta_{j,k}.\end{aligned} \tag{85}$$

Therefore,

$$\begin{aligned}A\left[\alpha_{j,k},\beta_{j,k}\right] &= \left[\alpha_{j,k},\beta_{j,k}\right]\begin{bmatrix} a_j & b_j\\ -b_j & a_j\end{bmatrix},\\ j &= 1,\dots,s, \quad k = 1,\dots,n_j,\end{aligned} \tag{86}$$

with $\begin{bmatrix} a_j & b_j\\ -b_j & a_j\end{bmatrix}$ being real orthogonal. Hence $AB = BQ$ for a real orthogonal matrix Q, and our conclusion follows.

(c) Since $\mathcal{A}_{\lambda^n}^*(U) = \{S_{i_1\cdots i_k} : \rho_{i_1\cdots i_k} \le \lambda^n < \rho_{i_1\cdots i_{k-1}}\}$ is the unique packing family of itself for every $n > 0$, a simple calculation shows that the solution s of (69) satisfies

$$\sum_{\varphi\in\mathcal{A}_{\lambda^n}^*(U)}\left[\mathcal{L}^d(\varphi(U))\right]^{s/d} = \left[\mathcal{L}^d(U)\right]^{s/d}. \tag{87}$$

Hence $Q_\lambda(s) = 0$ and so the conclusion follows by using Theorem 11. □

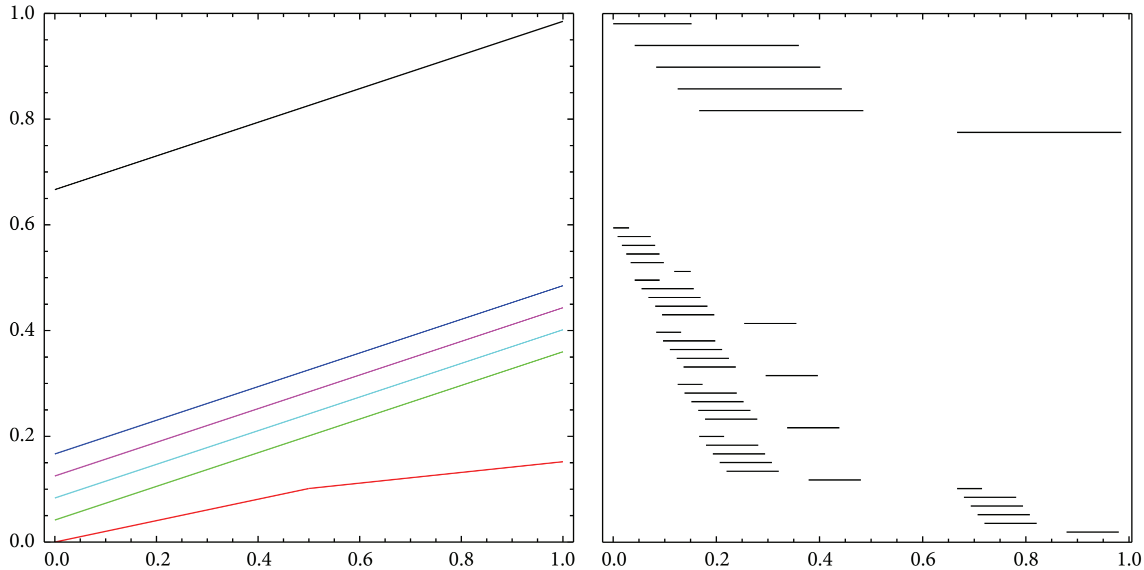
Example 12, proved below, is an illustration of Example 23(c).

Proof of Example 12. The IFS satisfies LDP by the conclusion of Example 23(a). By a result in [10], OSC holds. Note that $\rho_1 = \rho_2 = 1/2$. The conclusion $\dim_H K = \dim_P K = \dim_B K = 1$ follows from Example 23(c). □

The following IFS consists of a nondifferentiable map. It satisfies LDP but not the natural extension of WSC.

Example 24. Let $S_i : [0,1] \to [0,1]$, $i = 1,\dots,N$, be an IFS of contractions defined as follows:

$$\begin{aligned}S_1(x) &= \begin{cases}\rho_0(x+a_1), & x \le \dfrac{1}{2},\\[2ex] \rho_0\left(\dfrac{1}{2}+a_1\right) + \rho_1\left(x-\dfrac{1}{2}\right), & x > \dfrac{1}{2},\end{cases}\\ S_i(x) &= \rho_i(x+a_i), \quad i = 2,\dots,N.\end{aligned} \tag{88}$$



(a) (b)

FIGURE 2: (a) Graphs of the maps in the IFS in (90). (b) First two levels of iterations of the interval $[0,1]$ under the IFS.

Assume $\rho_0\rho_1 > 0$, $0 < |\rho_i| \le 1/2$ for $i = 0, 1, \ldots, N$, and, for each i, either $S_i((0,1)) \subseteq (0,(1/2))$ or $S_i((0,1)) \subseteq (1/2, 1)$. Then the IFS $\{S_i\}_{i=1}^N$ satisfies LDP.

Proof. It follows from the assumptions that $0 < r_i \le R_i \le 1/2$ for $0 \le i \le N$. Let

$$c = \frac{\max\{|\rho_0|, |\rho_1|\}}{\min\{|\rho_0|, |\rho_1|\}}. \tag{89}$$

We need only prove $R_I \le c r_I$ for all $I \in \Sigma^*$. We use induction on the length of I. It is easy to see that this is true when $|I| = 1$. Assume it is true for the case $|I| \le n$. Consider the case $|I| = n+1$. Let $I = i_1 \cdots i_{n+1}$.

If $i_1 > 1$, then $S_I(x) = \rho_{i_1}(S_{i_2\cdots i_{n+1}}(x) + a_{i_1})$ for all $x \in [0,1]$. Hence $R_I = |\rho_{i_1}| R_{i_2\cdots i_{n+1}}$ and $r_I = |\rho_{i_1}| r_{i_2\cdots i_{n+1}}$, and so $R_I \le c r_I$ by induction hypothesis.

Now assume $i_1 = 1$. If $S_{i_2}[0,1] \subseteq [0, 1/2]$, then $S_I(x) = \rho_0(S_{i_2\cdots i_{n+1}}(x) + a_1)$ for all $x \in [0,1]$. Hence $R_I = |\rho_0| R_{i_2\cdots i_{n+1}}$ and $r_I = |\rho_0| r_{i_2\cdots i_{n+1}}$. If $i_1 = 1$ and $S_{i_2}[0,1] \subseteq [1/2, 1]$, then $S_I(x) = \rho_0(1/2 + a_1) + \rho_1(S_{i_2\cdots i_{n+1}}(x) - 1/2)$. Hence $R_I = |\rho_1| R_{i_2\cdots i_{n+1}}$ and $r_I = |\rho_1| r_{i_2\cdots i_{n+1}}$. In both cases it follows by induction hypothesis that $R_I \le c r_I$. Thus LDP holds. □

The following is an explicit IFS from Example 24 (see Figure 2):

$$\begin{aligned}
S_1(x) &= \begin{cases} \dfrac{2x}{\pi^2}, & x \le \dfrac{1}{2}, \\[2mm] \dfrac{x}{\pi^2} + \dfrac{1}{2\pi^2}, & x > \dfrac{1}{2}, \end{cases} \\
S_2(x) &= \frac{x}{\pi} + \frac{1}{24}, \\
S_3(x) &= \frac{x}{\pi} + \frac{1}{12}, \\
S_4(x) &= \frac{x}{\pi} + \frac{1}{8}, \\
S_5(x) &= \frac{x}{\pi} + \frac{1}{6}, \\
S_6(x) &= \frac{x}{\pi} + \frac{2}{3}.
\end{aligned} \tag{90}$$

The IFS in (90) does not satisfy the natural extension of WSC to the IFSs we consider in this paper, since the sub-IFS $\{S_2, S_3, S_4, S_5\}$ does not.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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