Research Article

Mixed Portmanteau Test for Diagnostic Checking of Time Series Models

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Model criticism is an important stage of model building and thus goodness of fit tests provides a set of tools for diagnostic checking of the fitted model. Several tests are suggested in literature for diagnostic checking. These tests use autocorrelation or partial autocorrelation in the residuals to criticize the adequacy of fitted model. The main idea underlying these portmanteau tests is to identify if there is any dependence structure which is yet unexplained by the fitted model. In this paper, we suggest mixed portmanteau tests based on autocorrelation and partial autocorrelation functions of the residuals. We derived the asymptotic distribution of the mixture test and studied its size and power using Monte Carlo simulations.

1. Introduction

In practice, there are many possible linear and nonlinear models for a problem under study, for example, autoregressive, moving average, mixed ARMA models, threshold autoregressive. Box and Jenkins [1] have described time series model building as a three-stage iterative procedure that consists of identification, estimation, and validation.

Time series models should be able to describe the dependence among the observations; see, for example, Li [2]. It is a well-discussed issue that in time series model criticism the residuals obtained from fitting a potential model to the observed time series play a vital role and can be used to detect departures from the underlying assumptions, [1, 2].

In particular, if the model is a good fit to the observed series then the residuals should behave somewhat like a white noise process. So, taking into account of the effect of estimation, the residuals obtained from a good fit should be approximately uncorrelated. While looking at the significance of residual autocorrelations, one approach is to test the significance of each individual residual autocorrelation which seems to be quite cumbersome. Another approach is to have some portmanteau test capable of testing the significance of the first, say, m, residual autocorrelations [1, 2], an approach we now describe.

Since Box and Pierce [3] paper, the portmanteau test has become the vital part of time series diagnostic checking. Several modifications and versions of Box and Pierce [3] have been suggested in the literature; see, for example, Ljung and Box [4], McLeod and Li [5], Monte [6], Katayama [7], and Katayama [8]. These tests are capable of testing the significance of the autocorrelations (partial autocorrelations) up to a finite number of lags.

The residuals are very commonly used as a diagnostic tool to test the goodness of fit of models. In a time series context, if the fitted model is good then it should be able to explain the dependence pattern among successive observations. In other words, all the dependence in terms of autocorrelations and partial autocorrelations of the data generating process (DGP) should be explained by the fitted model so there should be no significant autocorrelation and partial autocorrelation in successive terms of the residuals.

In practice, the most popular way for diagnostic checking of a time series model is the portmanteau test, which tests whether any of the first m autocorrelations $(\hat{r}_1, \ldots, \hat{r}_m)$ of a time series are significantly different from zero. This type of test was first suggested by Box and Pierce [3], in which they studied the distribution of residual autocorrelations in ARIMA processes.

Based on the autocorrelations of the residuals obtained by fitting an ARMA(p, q) model, Box and Pierce [3] suggested the following portmanteau test:

$$Q_m = n \sum_{k=1}^{m} \widehat{r}_k^2, \qquad (1)$$

where $\widehat{r}_k$ is the residual autocorrelation at lag k. They suggested that $Q_m \sim \chi^2_{m-p-q}$, for moderate values of m, and the fitted model is adequate, under the following conditions:

(1) $\psi_j \leq O(n^{-1/2})$ for $j \geq m - p$,

(2) $m/n = O(n^{-1/2})$,

where ψ_j are the weights in the MA(∞) representation. This approximation requires substitution of residuals, $\widehat{\varepsilon}_t$, for the error term, ε_t, in the model but this sort of substitution can be a serious underestimation of significance level in diagnostic checking [9]. Many other researchers have also questioned the distribution of Q_m (see, e.g., [10] and references therein). The choice of m is an important issue.

Ljung and Box [4] suggested the use of the modified statistic

$$Q_m^* = n(n+2) \sum_{k=1}^{m} \frac{\widehat{r}_k^2}{n-k}. \qquad (2)$$

They have shown that the modified portmanteau statistic Q_m^* has a finite sample distribution which is much closer to χ^2_{m-p-q}. Their results also show that Q_m^* is insensitive to the normality assumption of ε_t. As pointed out by many researchers, for example, Davies et al. [11], Ansley and Newbold [12], the true significance levels of Q_m tend to be much lower than predicted by the asymptotic theory and though the mean of Q_m^* is much closer to the asymptotic distribution, this corrected version of the portmanteau test has an inflated variance. But Ljung and Box [4] pointed out that approximate expression of variance given by Davies et al. [11] overestimates the variance of Q_m^*.

McLeod [10, Theorem 1] has shown that $\widehat{\mathbf{r}} = (\widehat{r}_1, \ldots, \widehat{r}_m)$ is approximately normal with mean $\mathbf{0}$ and $\text{Var}(\widehat{\mathbf{r}}) = (\mathbf{I} - \mathbf{C})/n$, where $\mathbf{C} = \mathbf{X}\mathbb{I}^{-1}\mathbf{X}^T$, $\mathbf{I}$ is the identity matrix, and $\mathbb{I}$ is the Fisher information matrix. The superscript T stands for transposition of matrix. We noticed that approximation of $\mathbf{C}$ by $\mathbf{D} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$, especially when m is small, is a source of bias in approximating the asymptotic distribution of portmanteau tests.

In practice, the optimal choice of m is difficult as the use of the χ^2_{m-p-q} approximation and diagnostic checking require large values of m which results in less power and unstable size of test, as noticed by Ljung [13] and Katayama [7]. Katayama [8] suggested a multiple portmanteau test to overcome this problem. His suggested test is based on several portmanteau tests for a range of small to medium values of m. He showed using some numerical examples that his suggestion leads to a superior test. He also discussed the linkage between his suggested multiple test and the test due to Peña and Rodríguez [14]. He suggested a method based on

some iterative procedure to approximate joint distribution of the multiple test as the computation of the distribution is very hard due to correlated elements.

The portmanteau tests Q_m and Q_m^* are based on the autocorrelations. Monti [6] suggested a portmanteau test

$$Q_m^*(\widehat{\omega}) = n(n+2) \sum_{k=1}^{m} \frac{\widehat{\omega}_k^2}{n-k}, \qquad (3)$$

where $\widehat{\omega}_k$ is the residual partial autocorrelation at lag k. She showed that $Q_m^*(\widehat{\omega})$, analogously to Q_m^*, has an asymptotic null distribution χ^2_{m-p-q} and that $Q_m^*(\widehat{\omega})$ is more powerful than Q_m^* especially when the order of the moving average component is understated. Chand and Kamal [15] had studied the size of Monti's test.

Portmanteau tests, for example, Box-Pierce test, which are based on residuals autocorrelations, have good power against the understatement of autoregressive component while the tests using residuals partial autocorrelations have shown good power against the understatement of moving average term. A new test using both autocorrelation and partial autocorrelations is expected to perform better in both of these two scenarios and will be capable of picking up understatement in moving average and autoregressive components.

2. Materials and Methods

In this section, we give the algorithms for the Monte Carlo method used to compute the empirical size and power of the diagnostic tests defined in Section 2. For each Monte Carlo run, a sample time series $\{y_t\}_{t=1}^n$ is simulated under the model $\mathcal{M}$. For empirical size, $\mathcal{M}$ is the null model while for the computation of power it is the alternative model. In both of the situations, we estimate the null model for the simulated sample time series and T is calculated from the residuals, $\widehat{\varepsilon}_t$.

Algorithm 1 gives the bootstrap procedure used in our numerical study. From this algorithm, we obtain the bootstrap approximation of the distribution of the test. We will use this algorithm to compute empirical size in the following algorithm of our simulation study consisting of N Monte Carlo runs.

3. Results

Suppose we have an observed time series $\{y_t : t = 1, 2, \ldots, n\}$ generated by an ARMA(p, q) model given by

$$\phi(B) y_t = \theta(B) \varepsilon_t, \qquad (4)$$

where B is the backward shift operator, $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ is the autoregressive polynomial of order p, and $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$ is the moving average polynomial of order q. Let an appropriately identified model be fitted to the observed time series and let residuals $\widehat{\varepsilon}_1, \widehat{\varepsilon}_2, \ldots, \widehat{\varepsilon}_n$ be obtained. These residuals play an important

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Step 1. Generate bootstrap sample y_t^* using resamples of residuals, $\widehat{\varepsilon}_t$, obtained by fitting the model under null hypothesis to the simulated time series. We denoted these resampled residuals as ε_t^*.

Step 2. Fit the null model to the bootstrap sample y_t^* and obtain residuals as $\widehat{\varepsilon}_t^* = y_t^* - \widehat{y}_t^*$, where $\widehat{y}_t^*$ is the fitted series.

Step 3. Using the residuals, $\widehat{\varepsilon}_t^*$, obtained in Step 2 above, calculate test-statistic T. We denote it as T^*.

Step 4. Repeat Step 1–3 for each of the B bootstrap samples.

ALGORITHM 1: Bootstrap sampling procedure.

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role in the diagnostic checking of the fitted model. The residuals autocorrelation at lag k can be defined as

$$\widehat{r}_k = \frac{\sum_{t=k+1}^{n} \widehat{\varepsilon}_t \widehat{\varepsilon}_{t-k}}{\sum_{t=1}^{n} \widehat{\varepsilon}_t^2} \quad k = 1, 2, \ldots, \tag{5}$$

where $\{\widehat{\varepsilon}_{t-k}\}$ is the series at lag k of $\{\widehat{\varepsilon}_t\}$. Let $\boldsymbol{\beta} = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$ be the vector of model parameters and let $\widehat{\boldsymbol{\beta}} = (\widehat{\phi}_1, \ldots, \widehat{\phi}_p, \widehat{\theta}_1, \ldots, \widehat{\theta}_q)$ be the corresponding estimator of $\boldsymbol{\beta}$.

Let us define autocorrelation of error term, ε_t, that is, white noise at lag k, as

$$r_k = \frac{\sum_{t=k+1}^{n} \varepsilon_t \varepsilon_{t-k}}{\sum_{t=1}^{n} \varepsilon_t^2} \quad k = 1, 2, \ldots. \tag{6}$$

The root mean square error of $\widehat{\boldsymbol{\beta}}$ can be defined as

$$\sqrt{E\left(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}\right)^2}. \tag{7}$$

As the above term is of order $1/\sqrt{n}$, thus we can use Taylor series of first order to approximate $\widehat{r}_k$ around r_k. We can write

$$\widehat{r}_k = r_k + \sum_{j=1}^{p+q} \left(\beta_j - \widehat{\beta}_j\right)^2 \lambda_{jk} + O_p\left(\frac{1}{n}\right), \tag{8}$$

where

$$\lambda_{jk} = -\left.\frac{\partial r_k}{\partial \beta_j}\right|_{\beta = \widehat{\beta}},$$

$$\widehat{\lambda}_{jk} = -\left.\frac{\partial \widehat{r}_k}{\partial \beta_j}\right|_{\beta = \widehat{\beta}}. \tag{9}$$

Take

$$\varepsilon_t = y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} - \theta_1 \varepsilon_{t-1} - \cdots - \theta_q \varepsilon_{t-q}. \tag{10}$$

For an AR(p) process

$$\varepsilon_t = y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p}, \tag{11}$$

at $\boldsymbol{\beta} = \dot{\boldsymbol{\beta}}$, that is, for an AR($p$), $\boldsymbol{\phi} = \dot{\boldsymbol{\phi}}$, we can write

$$\dot{\varepsilon}_t = y_t - \dot{\phi}_1 y_{t-1} - \cdots - \dot{\phi}_p y_{t-p},$$

$$\frac{\partial \dot{\varepsilon}_t}{\partial \dot{\phi}_j} = -y_{t-j}. \tag{12}$$

From (10) and summing over observed sample, we get

$$\sum \varepsilon_t^2 = \sum \left(y_t - \phi_1 y_{t-1} \cdots - \phi_p y_{t-p}\right)^2. \tag{13}$$

Partially differentiating with respect to ϕ_j

$$\frac{\partial \sum \varepsilon_t^2}{\partial \phi_j} = 2 \sum \left(y_t - \phi_1 y_{t-1} \cdots - \phi_p y_{t-p}\right)\left(-y_{t-j}\right). \tag{14}$$

At $\boldsymbol{\phi} = \dot{\boldsymbol{\phi}}$

$$\frac{\partial \sum \varepsilon_t^2}{\partial \phi_j} = 0. \tag{15}$$

Now

$$\widehat{\lambda}_{jk} = -\frac{\partial \widehat{r}_k}{\partial \dot{\phi}_j} = -\frac{\partial \left(c_k/c_0\right)}{\partial \phi_j}$$

$$= -\frac{\partial}{\partial \phi_j}\left(\frac{\sum \varepsilon_t \varepsilon_{t-k}}{\sum \varepsilon_t^2}\right) = -\frac{1}{\sum \varepsilon_t^2}\frac{\partial \left(\sum \varepsilon_t \varepsilon_{t-k}\right)}{\partial \phi_j}. \tag{16}$$

As

$$\sum \varepsilon_t \varepsilon_{t-k} = \sum \left(\phi\left(B\right) y_t\right)\left(\phi\left(B\right) y_{t-k}\right)$$

$$= \sum_t \left(\sum_{\substack{i \ j \\ i = k+j}} \phi_i \phi_j y_{t-i} y_{t-k-j} + \sum_{\substack{i \ j \\ i \neq k+j}} \phi_i \phi_j y_{t-i} y_{t-k-j}\right), \tag{17}$$

thus

$$\widehat{\lambda}_{jk} = -\frac{\sum_t y_t^2}{\sum_t \widehat{\varepsilon}_t^2} \sum_i \left(r_{k-i+j}\left(y\right) + r_{k+i-j}\left(y\right)\right)$$

$$= -\frac{\sum_i \widehat{phi}_i \left(r_{k-i+j}\left(y\right) + r_{k+i-j}\left(y\right)\right)}{\sum_i \sum_j \widehat{\phi}_i \widehat{\phi}_j r_{i-j}\left(y\right)}, \tag{18}$$

where

$$r_k\left(y\right) = \frac{\sum y_t y_{t-k}}{\sum y_t^2}. \tag{19}$$

Using Bartlett's formula, we can write

$$r_k(y) = \rho_k + O_p\left(n^{-1/2}\right). \tag{20}$$

Using $\widehat{\phi}_j = \phi_j + O_p(n^{-1/2})$ and the above results

$$\widehat{\lambda}_{jk} = \lambda_{jk} + O_p\left(n^{-1/2}\right). \tag{21}$$

Consider the AR(p) process

$$\phi(B)\,y_t = \varepsilon_t,$$

$$\left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)y_t = \varepsilon_t, \tag{22}$$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p} = \varepsilon_t.$$

Multiplying both sides by y_{t-k} and on taking expectation, we get $E(y_t y_{t-k}) - \phi_1 E(y_{t-1} y_{t-k}) - \phi_2 E(y_{t-2} y_{t-k}) - \cdots - \phi_p E(y_{t-p} y_{t-k}) = E(\varepsilon_t y_{t-k})$. Without loss of generality, we can assume $E(y_t) = 0$; moreover it is considered a stationary process; therefore $E(y_t^2) = \text{var}(y_t) = c_0$ and $E(y_t y_{t-k}) = \text{cov}(y_t, yt - k) = c_k$. Moreover, $E(\varepsilon_t y_{t-k}) = 0$ for $k \geq 1$. Thus

$$c_k - \phi_1 c_{k-1} - \phi_2 c_{k-2} - \cdots - \phi_p c_{k-p} = 0. \tag{23}$$

Divide both sides by c_0 and using $r_k = c_k/c_0$, we get

$$r_k - \phi_1 r_{k-1} - \phi_2 r_{k-2} - \cdots - \phi_p r_{k-p} = 0,$$

$$\phi(B)\,r_k = 0. \tag{24}$$

Using this result and the structure of AR(p) process, it implies

$$\lambda_{jk} = \frac{\sum_i \phi_i \phi_{k-j+i}}{\sum_i \phi_i \rho_i}. \tag{25}$$

The AR(p) process can be written in an equivalent moving average representation, that is, MA(∞), given as

$$\phi(B)\,y_t = \varepsilon_t,$$

$$y_t = \psi(B)\,\varepsilon_t, \tag{26}$$

where

$$\psi(B) = \frac{1}{\phi(B)} = \sum_{i=0}^{\infty} \psi_i B^i. \tag{27}$$

Using Box and Pierce [3, page. 1514] and the above results, we can write

$$\lambda_{jk} = \psi_{k-j}, \tag{28}$$

where $k = 1, 2, \ldots, m$, and thus using (8), we get

$$\widehat{r}_k = r_k + \sum_i \left(\phi_i - \widehat{\phi}_i\right)\psi_{k-i} + O_p\left(\frac{1}{n}\right). \tag{29}$$

Using McLeod [10, Theorem 1], the asymptotic distribution of $\widehat{\mathbf{r}} = (\widehat{r}_1, \ldots, \widehat{r}_m)$, where $m \geq p + q$ is normal with mean $\mathbf{0}$ and covariance matrix $(1 - \mathbf{X}\mathbb{I}^{-1}\mathbf{X}^T)/n$; that is,

$$\widehat{\mathbf{r}} \sim N\left(\mathbf{0}, \frac{\left(\mathbf{I} - \mathbf{X}\mathbb{I}^{-1}\mathbf{X}^T\right)}{n}\right), \tag{30}$$

where for an ARMA(p, q) process

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \psi_1 & 1 & \cdots & 0 & \pi_1 & 1 & \cdots & 0 \\ \psi_2 & \psi_1 & \cdots & 0 & \pi_2 & \pi_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \psi_{m-1} & \psi_{m-2} & \cdots & \psi_{m-p} & \pi_{m-1} & \pi_{m-2} & \cdots & \pi_{m-p} \end{bmatrix}. \tag{31}$$

The elements ψ_j and π_j are the coefficients of B^j in the infinite expansion of inverse of AR and MA polynomials. For an AR(p) process, the above matrix reduces to

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \psi_1 & 1 & \cdots & 0 \\ \psi_2 & \psi_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \psi_{m-1} & \psi_{m-2} & \cdots & \psi_{m-p} \end{bmatrix}. \tag{32}$$

Let ω_k be the partial autocorrelation at lag k; then using Monti [6, page 777, equation (4)], we have

$$\widehat{\omega} = \widehat{\mathbf{r}} + O_p\left(\frac{1}{n}\right). \tag{33}$$

According to Monti [6], the $\widehat{\mathbf{r}}$ can be obtained using a singular linear transformation of the noise autocorrelations, that is, $\mathbf{r}$, given as

$$\widehat{\mathbf{r}} = \left(\mathbf{I} - \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}\right)\mathbf{r} + O_p\left(\frac{1}{n}\right). \tag{34}$$

Using (30), we can write

$$\widehat{\omega} \sim N\left(\mathbf{0}, \frac{\left(\mathbf{I} - \mathbf{X}\mathbb{I}^{-1}\mathbf{X}^T\right)}{n}\right). \tag{35}$$

Ljung and Box [4], using (30), suggested a portmanteau test for diagnostic testing of the fitted ARMA(p, q) model given as

$$Q_m^*(\widehat{r}) = n(n+2)\sum_{k=1}^{m} \frac{\widehat{r}_k^2}{n-k} \sim \chi_{m-p-q}^2. \tag{36}$$

Following the same lines and using the result in (35), Monti [6] suggested the following portmanteau test:

$$Q_m^*(\widehat{\omega}) = n(n+2)\sum_{k=1}^{m} \frac{\widehat{\omega}_k^2}{n-k} \sim \chi_{m-p-q}^2. \tag{37}$$

4. New Test

We suggest a new test $Q_m^\dagger(\widehat{r}, \widehat{\omega}) = Q_m^*(\widehat{r}) + Q_m^*(\widehat{\omega})$, that is, the sum of Ljung and Box [4] and Monti [6], given as

$$Q_m^*(\widehat{r}, \widehat{\omega}) = n(n+2)\sum_{k=1}^{m} \frac{\left(\widehat{r}_k^2 + \widehat{\omega}_k^2\right)}{n-k}. \tag{38}$$

TABLE 1: Monte Carlo estimates of ρ_m.

n	m							
	2	3	5	10	15	20	25	30
8	0.799	—	—	—	—	—	—	—
12	0.882	0.872	—	—	—	—	—	—
20	0.932	0.909	0.851	—	—	—	—	—
40	0.962	0.947	0.910	0.829	—	—	—	—
60	0.974	0.968	0.934	0.876	0.828	—	—	—
80	0.982	0.974	0.951	0.898	0.859	0.825	—	—
100	0.986	0.979	0.958	0.923	0.881	0.851	0.826	—
125	0.989	0.983	0.968	0.936	0.902	0.873	0.851	0.829
150	0.990	0.983	0.972	0.941	0.920	0.887	0.875	0.854
200	0.993	0.988	0.979	0.956	0.934	0.915	0.900	0.874

TABLE 2: Empirical size at 1% nominal size for an AR(1) process: $y_t = \phi y_{t-1} + \varepsilon_t$.

m	T	ϕ				
		0.1	0.4	0.7	0.8	0.9
5	$Q_m^*(\hat{r})$	0.021	0.020	0.027	0.024	0.027
	$Q_m^*(\hat{\omega})$	0.012	0.011	0.010	0.005	0.015
	$Q_m^*(\hat{r}, \hat{\omega})$	0.014	0.015	0.016	0.007	0.020
10	$Q_m^*(\hat{r})$	0.008	0.015	0.016	0.017	0.022
	$Q_m^*(\hat{\omega})$	0.007	0.012	0.006	0.009	0.018
	$Q_m^*(\hat{r}, \hat{\omega})$	0.006	0.011	0.008	0.010	0.017
15	$Q_m^*(\hat{r})$	0.013	0.025	0.020	0.018	0.025
	$Q_m^*(\hat{\omega})$	0.006	0.010	0.007	0.010	0.015
	$Q_m^*(\hat{r}, \hat{\omega})$	0.007	0.013	0.010	0.010	0.015
20	$Q_m^*(\hat{r})$	0.024	0.022	0.022	0.022	0.034
	$Q_m^*(\hat{\omega})$	0.011	0.011	0.011	0.012	0.009
	$Q_m^*(\hat{r}, \hat{\omega})$	0.011	0.015	0.018	0.014	0.019
25	$Q_m^*(\hat{r})$	0.021	0.020	0.027	0.024	0.027
	$Q_m^*(\hat{\omega})$	0.012	0.011	0.010	0.005	0.015
	$Q_m^*(\hat{r}, \hat{\omega})$	0.014	0.015	0.016	0.007	0.018

TABLE 3: Empirical size at 5% nominal size for an AR(1) process: $y_t = \phi y_{t-1} + \varepsilon_t$.

m	T	ϕ				
		0.1	0.4	0.7	0.8	0.9
5	$Q_m^*(\hat{r})$	0.077	0.063	0.078	0.072	0.085
	$Q_m^*(\hat{\omega})$	0.050	0.051	0.053	0.047	0.055
	$Q_m^*(\hat{r}, \hat{\omega})$	0.056	0.049	0.055	0.049	0.052
10	$Q_m^*(\hat{r})$	0.041	0.053	0.057	0.059	0.077
	$Q_m^*(\hat{\omega})$	0.044	0.053	0.059	0.056	0.07
	$Q_m^*(\hat{r}, \hat{\omega})$	0.036	0.046	0.045	0.049	0.061
15	$Q_m^*(\hat{r})$	0.06	0.069	0.072	0.074	0.089
	$Q_m^*(\hat{\omega})$	0.052	0.065	0.055	0.057	0.068
	$Q_m^*(\hat{r}, \hat{\omega})$	0.053	0.061	0.051	0.049	0.055
20	$Q_m^*(\hat{r})$	0.065	0.07	0.059	0.087	0.085
	$Q_m^*(\hat{\omega})$	0.051	0.054	0.056	0.064	0.061
	$Q_m^*(\hat{r}, \hat{\omega})$	0.053	0.051	0.049	0.065	0.072
25	$Q_m^*(\hat{r})$	0.077	0.063	0.078	0.072	0.085
	$Q_m^*(\hat{\omega})$	0.050	0.051	0.053	0.047	0.055
	$Q_m^*(\hat{r}, \hat{\omega})$	0.056	0.049	0.055	0.049	0.062

Step 1. Obtain T^* using Algorithm 1, for each of the B bootstrap samples, reject null model if $T^* \geq T$, otherwise accept it.
Step 2. Determine the proportion of B bootstrap samples, say $\widehat{p}_c$, for which the null hypothesis is rejected.
Step 3. Repeat Step 1-2 for each of the N Monte Carlo runs.
Step 4. Empirical size, $\widehat{\alpha}$, is determined as the proportion of Monte Carlo runs for which the $\widehat{p}_c \leq \alpha$, where α is the level of significance,
$$\widehat{\alpha} = \frac{\#(\widehat{p}_c \leq \alpha)}{N}.$$

ALGORITHM 2: Computation of empirical size.

Step 1. Calculate $100(1 - \alpha)$th percentile, say $T^*_{1-\alpha}$, of the bootstrap distribution of T^* obtained using Algorithm 1.
Step 2. Reject null model if $T \geq T^*_{1-\alpha}$ otherwise accept it.
Step 3. Repeat Step 1-2 for each of the N Monte Carlo runs.
Step 4. Empirical power, $1 - \widehat{\beta}$, is determined as below,
$$1 - \widehat{\beta} = \frac{\#(T \geq T_{1-\alpha})}{N}.$$

ALGORITHM 3: Computation of empirical power.

It is obvious that this new statistic Q^*_m is the sum of two chi-square variables as discussed above in (36) and (37). Moreover, it can be noticed that both of these random variables will be positively correlated due to their structure of using residual autocorrelation and partial autocorrelation. We have discussed their correlational structure later in this report.

Using the result of Joarder and Omar [16, Theorem 3.1], we can write the probability density function of $Z = Q^*_m = Q_m(\widehat{r}) + Q_m(\widehat{\omega})$ as

$$f_Z(z) = \frac{\left(1 - \rho_m^2\right)^{-m/2}}{2^m \Gamma(m)} z^{m-1} \exp\left(\frac{-z}{2 - 2\rho_m^2}\right)$$
$$\times {}_0F_1\left(\frac{m+1}{2}; \frac{\rho_m^2 z^2}{\left(4 - 4\rho_m^2\right)^2}\right); \qquad (39)$$
$$0 \leq z < \infty,$$

where ρ_m is the correlation between $Q^*_m(\widehat{r})$ and $Q^*_m(\widehat{\omega})$:

$$_pF_q\left(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z\right) = \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{p} \left(a_j\right)_{\{i\}} z^i}{\prod_{j=1}^{q} \left(b_j\right)_{\{i\}} i!},$$
$$a_{\{i\}} = a(a+1)\cdots(a+i-1). \qquad (40)$$

Now the important issue is to estimate ρ_m which is discussed in Section 4.1.

4.1. Estimation of ρ_m. We have used Monte Carlo simulation to estimate ρ_m. We have considered various choices of sample size and m and obtained the following results. Empirical results suggest that, for any n, we have

$$0.8 \leq \rho_m \leq 1, \qquad (41)$$

where $m \leq n/4$. Results are given in Table 1.

4.2. Empirical Size. The size of a test is helpful in assessing how reasonable is our assumption of the null distribution. We can compute the size when the sample is simulated under the null model. This is the probability of rejecting the null model when the null model is true model. The model under null hypothesis is considered an AR(1) process $y_t = \phi y_{t-1} + \varepsilon_t$. To see the effect of stationarity and near to unit-root problem, we have considered various choices of ϕ. The value of ϕ near to 1, for example, $\phi = 0.9$, is an example of near nonstationary process.

It can be seen from the results shown in Tables 2 and 3 that our new suggested mixture test, that is, $Q^*_m(\widehat{r}, \widehat{\omega})$, have close approximation of corresponding nominal sizes. These results show the good approximation of asymptotic distribution of our new suggested test.

4.3. Empirical Power. The power of a test is the probability of rejecting a false null hypothesis. For empirical power, as mentioned earlier, the sample is generated under the alternative model. Algorithms 2 and 3 state the Monte Carlo procedure we use to determine the empirical size and power of test.

For power computation, we have simulated under an ARMA$(1, 1)$ process $y_t = \phi y_{t-1} + \varepsilon_t + 0.8\varepsilon_{t-1}$ and $y_t = 0.8 y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$ as alternative models, while AR(1) process $y_t = \phi y_{t-1} + \varepsilon_t$ is the model specified in null hypothesis. Results are provided in Tables 4 and 5. These results show that our suggested mixed portmanteau tests have empirical power

Table 4: Empirical power by fitting an AR(1) process against an ARMA(1, 1) process: $y_t = \phi y_{t-1} + \varepsilon_t + 0.8\varepsilon_{t-1}$ using 1000 Monte Carlo runs.

m	T	ϕ				
		0.1	0.4	0.7	0.8	0.9
5	$Q_m^*(\hat{r})$	0.955	0.991	1	1	1
	$Q_m^*(\hat{\omega})$	0.992	0.997	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.997	0.999	1	1	1
10	$Q_m^*(\hat{r})$	0.819	0.941	0.983	0.986	0.995
	$Q_m^*(\hat{\omega})$	0.980	1	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.985	1	1	1	1
15	$Q_m^*(\hat{r})$	0.748	0.871	0.940	0.957	0.976
	$Q_m^*(\hat{\omega})$	0.958	0.990	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.953	0.970	0.995	0.997	1
20	$Q_m^*(\hat{r})$	0.680	0.828	0.900	0.940	0.948
	$Q_m^*(\hat{\omega})$	0.926	0.989	0.995	0.999	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.943	0.957	0.987	0.998	0.999
25	$Q_m^*(\hat{r})$	0.614	0.810	0.872	0.910	0.934
	$Q_m^*(\hat{\omega})$	0.864	0.974	0.990	0.999	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.927	0.940	0.975	0.995	0.998

Table 5: Empirical power by fitting an AR(1) process against an ARMA(1, 1) process: $y_t = 0.8 y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$ using 1000 Monte Carlo runs.

m	T	θ				
		0.1	0.4	0.7	0.8	0.9
5	$Q_m^*(\hat{r})$	0.103	0.703	1	1	1
	$Q_m^*(\hat{\omega})$	0.081	0.771	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.208	0.846	1	1	1
10	$Q_m^*(\hat{r})$	0.092	0.520	0.917	1	0.978
	$Q_m^*(\hat{\omega})$	0.115	0.601	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.228	0.784	1	1	1
15	$Q_m^*(\hat{r})$	0.063	0.554	0.941	0.963	0.955
	$Q_m^*(\hat{\omega})$	0.051	0.552	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.196	0.747	1	1	1
20	$Q_m^*(\hat{r})$	0.094	0.441	0.832	0.964	0.907
	$Q_m^*(\hat{\omega})$	0.063	0.491	1	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.187	0.698	1	1	1
25	$Q_m^*(\hat{r})$	0.020	0.475	0.801	0.896	0.858
	$Q_m^*(\hat{\omega})$	0.053	0.571	0.982	1	1
	$Q_m^*(\hat{r}, \hat{\omega})$	0.176	0.669	0.994	1	1

comparable with other tests. It can be noticed that our new test has shown power better than other considered tests when moving average part is understated and the moving average parameter is relatively small.

5. Conclusion

Portmanteau tests based on residuals autocorrelations and partial autocorrelations have their individual good properties. Our suggested mixture portmanteau tests have a good approximation of their asymptotic distribution. As it is important to study both the residuals autocorrelation and the partial autocorrelations for model diagnostic checking, so

this new test is capable of doing it simultaneously instead of running separate diagnostic tests.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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