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Research Article

Iterative Algorithms for Systems of Generalized Equilibrium Problems with the Constraints of Variational Inclusion and Fixed Point Problems

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We introduce and analyze a hybrid extragradient-like viscosity iterative algorithm for finding a common solution of a systems of generalized equilibrium problems and a generalized mixed equilibrium problem with the constraints of two problems: a finite family of variational inclusions for maximal monotone and inverse strongly monotone mappings and a fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under some suitable conditions, we prove the strong convergence of the sequence generated by the proposed algorithm to a common solution of these problems.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\overline{\gamma} \in (0, 1]$ such that

$$\langle Vx, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1}$$

A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\left\| Sx - Sy \right\| \leq L \left\| x - y \right\|, \quad \forall x, y \in C. \tag{2}$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then A is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP) [1] which is to find a point $x \in C$ such that

$$\left\langle Ax, y - x \right\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The solution set of VIP (3) is denoted by $\mathrm{VI}(C, A)$.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [2] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \left\langle Ax, y - x \right\rangle \geq 0, \quad \forall y \in C. \tag{4}$$

We denote the set of solutions of GMEP (4) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The system of equilibrium problems or generalized equilibrium problems is a tool to study Nash eequilibrium problems, see for example [3–8]. In fact, the GMEP (4) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP is further considered and studied; see for example, [9–15]. Here we also consider a system of two generalized equilibrium problem that could be usefull to study the Two players game problem, see [16].

Throughout this paper, it is assumed as in [2] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)–(H4) and

$\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (H5), where

(H1) $\Theta(x, x) = 0$ for all $x \in C$;

(H2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for any $x, y \in C$;

(H3) Θ is upper-hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \le \Theta(x, y); \tag{5}$$

(H4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(H5) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{6}$$

Given a positive number $r > 0$. Let $S_r^{(\Theta,\varphi)} : H \to C$ is the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$S_r^{(\Theta,\varphi)}(x) := \Big\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\big\langle K'(y) - K'(x), z - y\big\rangle \ge 0, \forall z \in C \Big\}. \tag{7}$$

In particular, whenever $K(x) = (1/2)\|x\|^2, \forall x \in H$, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions, and $A_1, A_2 : C \to H$ be two nonlinear mappings. Consider the following system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{gathered} \Theta_1(x^*, x) + \langle A_1 y^*, x - x^*\rangle + \frac{1}{\nu_1}\langle x^* - y^*, x - x^*\rangle \ge 0, \quad \forall x \in C, \\ \Theta_2(y^*, y) + \langle A_2 x^*, y - y^*\rangle + \frac{1}{\nu_2}\langle y^* - x^*, y - y^*\rangle \ge 0, \quad \forall y \in C, \end{gathered} \tag{8}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It is introduced and studied in [17]. Whenever $\Theta_1 \equiv \Theta_2 \equiv 0$, the SGEP reduces to a system of variational inequalities, which is considered and studied in [18]. It is worth to mention that the system of variational inequalities is a tool to solve the Nash equilibrium problem for noncooperative games.

In 2010, Ceng and Yao [17] transformed the SGEP into a fixed point problem in the following way.

Proposition CY (see [17]). *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)–(H4) and let $A_k : C \to H$ be ζ_k-inverse-strongly monotone for $k = 1, 2$. Let $\nu_k \in (0, 2\zeta_k)$ for $k = 1, 2$. Then, $(x^*, y^*) \in C \times C$ is a solution of SGEP (8) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $G = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1) T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)$ where $y^* = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)x^*$. Here, we denote the fixed point set of G by SGEP(G).*

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \ge 1$, define a mapping W_n on H as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I, \\ &\cdots \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \\ U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I, \\ &\cdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I, \\ W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I. \end{aligned} \tag{9}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In 2011, for the case where $C = H$, Yao et al. [14] proposed the following hybrid iterative algorithm

$$\begin{gathered} \Theta(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}\big\langle K'(y_n) - K'(x_n), z - y_n\big\rangle \ge 0, \quad z \in H, \\ x_{n+1} = \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n (I + \mu V)) W_n y_n, \quad \forall n \ge 1, \end{gathered} \tag{10}$$

where $f : H \to H$ be a contraction, $K : H \to \mathbf{R}$ is differentiable and strongly convex, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $x_0, u \in H$ are given, for finding a common element of the set MEP(Θ, φ) and the fixed point set $\cap_{n=1}^{\infty} \text{Fix}(T_n)$ of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H. They proved the strong convergence of the sequence generated by the hybrid iterative algorithm (10) to a point $x^* \in \cap_{n=1}^{\infty} \text{Fix}(T_n) \cap \text{MEP}(\Theta, \varphi)$ under some appropriate conditions. This point x^* also solves the following optimization problem:

$$\min_{x \in \cap_{n=1}^{\infty} \text{Fix}(T_n) \cap \text{MEP}(\Theta,\varphi)} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP0}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Let $f : H \to H$ be a contraction and V be a strongly positive bounded linear operator on H. Assume that $\varphi : H \to \mathbf{R}$ is a lower semicontinuous and convex functional, that $\Theta, \Theta_1, \Theta_2 : H \times H \to \mathbf{R}$ satisfy conditions (H1)–(H4), and that $A, A_1, A_2 : H \to H$ are inverse-strongly monotone. Let the mapping G be defined as in Proposition CY. Very recently, Ceng et al. [11] introduced the following hybrid extragradient-like iterative algorithm

$$\begin{aligned} z_n &= S_{r_n}^{(\Theta,\varphi)} (x_n - r_n A x_n), \\ x_{n+1} &= \alpha_n (u + \gamma f(x_n)) + \beta_n x_n \\ &\quad + ((1-\beta_n) I - \alpha_n (I + \mu V)) W_n G z_n, \quad \forall n \geq 0, \end{aligned} \tag{11}$$

for finding a common solution of GMEP (4), SGEP (8) and the fixed point problem of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H, where $\{r_n\} \subset (0, \infty), \{\alpha_n\}, \{\beta_n\} \subset (0,1), \nu_k \in (0, 2\zeta_k), k = 1, 2$, and $x_0, u \in H$ are given. The authors proved the strong convergence of the sequence generated by the hybrid iterative algorithm (11) to a point $x^* \in \cap_{n=1}^{\infty} \operatorname{Fix}(T_n) \cap \operatorname{GMEP}(\Theta, \varphi, A) \cap \operatorname{SGEP}(G)$ under some suitable conditions. This point x^* also solves the following optimization problem:

$$\begin{aligned} \min_{x \in \cap_{n=1}^{\infty} \operatorname{Fix}(T_n) \cap \operatorname{GMEP}(\Theta,\varphi,A) \cap \operatorname{SGEP}(G)} & \frac{\mu}{2} \langle Vx, x \rangle \\ & + \frac{1}{2} \|x - u\|^2 - h(x), \end{aligned} \tag{OP1}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{12}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (12). In particular, if $B = R = 0$, then $I(B, R) = C$. If $B = 0$, then problem (12) becomes the inclusion problem introduced by Rockafellar [19]. It is known that problem (12) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, and so forth. Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H, \tag{13}$$

where λ is a positive number.

In this paper, we will introduce and analyze an iterative algorithm by hybrid extragradient-like viscosity method for finding a common solution of a systems of generalized equilibrium problems and a generalized mixed equilibrium problem with the constraints of two problems: a finite family of variational inclusions for maximal monotone and inverse strongly monotone mappings and a fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under some suitable conditions, we prove the strong convergence of the sequence generated by the proposed algorithm to a common solution of these problems. Such solution also solves an optimization problem. Several special cases are also discussed. The results presented in this paper are the supplement, extension, improvement and generalization of the previously known results in this area.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\begin{aligned} \omega_w(x_n) := \{ x \in H : x_{n_i} \rightharpoonup x \\ \text{for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\} \}. \end{aligned} \tag{14}$$

Definition 1. A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C, \tag{15}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C, \tag{16}$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \zeta \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{17}$$

It is easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2. A differentiable function $K : H \to \mathbf{R}$ is called:

(i) convex, if

$$K(y) - K(x) \geq \langle K'(x), y - x \rangle, \quad \forall x, y \in H, \tag{18}$$

where $K'(x)$ is the Frechet derivative of K at x;

(ii) strongly convex, if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), y - x \rangle \geq \frac{\sigma}{2} \|x - y\|^2, \quad \forall x, y \in H. \tag{19}$$

It is easy to see that if $K : H \to \mathbf{R}$ is a differentiable strongly convex function with constant $\sigma > 0$ then $K' : H \to H$ is strongly monotone with constant $\sigma > 0$.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C). \tag{20}$$

Some important properties of projections are gathered in the following proposition.

Proposition 3. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \le 0, \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2, \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2, \forall y \in H$. *(This implies that P_C is nonexpansive and monotone.)*

By using the technique of [20], we can readily obtain the following elementary result.

Proposition 4 (see [11, Lemma 1 and Proposition 1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying the conditions (H1)–(H4). Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,*

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\left\langle K'(z) - K'(x), y_x - z\right\rangle < 0. \tag{21}$$

Then the following hold:

(a) *for each $x \in H$, $S_r^{(\Theta,\varphi)}(x) \neq \emptyset$;*

(b) *$S_r^{(\Theta,\varphi)}$ is single-valued;*

(c) *$S_r^{(\Theta,\varphi)}$ is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ and*

$$\begin{aligned}&\left\langle K'(x_1) - K'(x_2), u_1 - u_2\right\rangle \\ &\quad \ge \left\langle K'(u_1) - K'(u_2), u_1 - u_2\right\rangle, \quad \forall (x_1, x_2) \in H \times H,\end{aligned} \tag{22}$$

where $u_i = S_r^{(\Theta,\varphi)}(x_i)$ for $i = 1, 2$;

(d) *for all $s, t > 0$ and $x \in H$*

$$\begin{aligned}&\left\langle K'\left(S_s^{(\Theta,\varphi)}x\right) - K'\left(S_t^{(\Theta,\varphi)}x\right), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\rangle \\ &\quad \le \frac{s-t}{s}\left\langle K'\left(S_s^{(\Theta,\varphi)}x\right) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\rangle;\end{aligned} \tag{23}$$

(e) *$\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;*

(f) *$\mathrm{MEP}(\Theta, \varphi)$ is closed and convex.*

In particular, whenever $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying the conditions (H1)–(H4) and $K(x) = (1/2)\|x\|^2, \forall x \in H$, then that is, for any $x, y \in H$,

$$\left\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\right\|^2 \le \left\langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\right\rangle \tag{24}$$

($S_r^{(\Theta,\varphi)}$ is firmly nonexpansive) and

$$\begin{aligned}\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| &\le \frac{|s-t|}{s}\left\|S_s^{(\Theta,\varphi)}x - x\right\|, \\ &\forall s, t > 0, \quad x \in H.\end{aligned} \tag{25}$$

In this case, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$. If, in addition, $\varphi \equiv 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

Remark 5. Suppose $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and $K' : H \to H$ is Lipschitz continuous with constant $\nu > 0$. Then $K' : H \to H$ is σ-strongly monotone and ν-Lipschitz continuous with positive constants $\sigma, \nu > 0$. Utilizing Proposition 4 (d) we can show that for all $s, t > 0$ and $x \in H$,

$$\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\cdot\frac{\nu}{\sigma}\left\|S_s^{(\Theta,\varphi)}x - x\right\|. \tag{26}$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 6. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{27}$$

Lemma 7. *Let H be a real Hilbert space. Then the following hold:*

(a) *$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;*

(b) *$\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*

(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H. \tag{28}$$

We have the following crucial lemmas concerning the W-mappings defined by (9).

Lemma 8 (see [21, Lemma 3.2]). *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H such that $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in H$ and $k \ge 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists, where $U_{n,k}$ is defined by* (9).

Remark 9 (see [22, Remark 3.1]). It can be known from Lemma 8 that if D is a nonempty bounded subset of H, then for $\epsilon > 0$ there exists $n_0 \ge k$ such that for all $n > n_0$

$$\sup_{x\in D}\|U_{n,k}x - U_k x\| \le \epsilon. \tag{29}$$

Remark 10 (see [22, Remark 3.2]). Utilizing Lemma 8, we define a mapping $W : H \to H$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in H. \tag{30}$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : H \to H$ is also nonexpansive. Indeed, observe that for each $x, y \in H$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \le \|x - y\|. \tag{31}$$

If $\{x_n\}$ is a bounded sequence in H, then we put $D = \{x_n : n \ge 1\}$. Hence, it is clear from Remark 5 that for an arbitrary $\epsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x\in D} \|U_{n,1}x - U_1 x\| \le \epsilon. \tag{32}$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0. \tag{33}$$

Lemma 11 (see [21, Lemma 3.3]). *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H such that $\cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, $\mathrm{Fix}(W) = \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.*

Lemma 12 (see [23, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 13. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see* Proposition 3 *(i)) implies*

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0. \tag{34}$$

Lemma 14 (see [24]). *Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a real Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) w_n, \quad \forall n \ge 0,$$
$$\limsup_{n\to\infty} (\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{35}$$

Then, $\lim_{n\to\infty}\|w_n - x_n\| = 0$.

Lemma 15 (see [25]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \sigma_n \gamma_n, \quad \forall n \ge 1, \tag{36}$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$ and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\sigma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y\rangle \ge 0. \tag{37}$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Next we provide an example to illustrate the concept of maximal monotone mapping.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \ge 0, \forall u \in C\}. \tag{38}$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{39}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [19].

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Let $\lambda > 0$. In terms of Huang [26] (see also [27]), there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 16. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, that is,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \ge \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H. \tag{40}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 17 (see [28]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0, u \in C$ is a solution of problem* (12) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu). \tag{41}$$

Lemma 18 (see [27]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 19 (see [28]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

Lemma 20 (see [29]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbf{R} \cup +\infty$ be* *a proper lower semicontinuous differentiable convex function. If x^* is a solution the minimization problem*

$$g(x^*) = \inf_{x\in C} g(x), \tag{42}$$

then,

$$\left\langle g'(x), x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{43}$$

In particular, if x^* solves (OP), then

$$\langle u + (\gamma f - (I + \mu V)) x^*, x - x^* \rangle \leq 0. \tag{44}$$

3. Main Results

In this section, we introduce and analyze an iterative algorithm by hybrid extragradient-like viscosity method for finding a common solution of a systems of generalized equilibrium problems and a generalized mixed equilibrium problem with the constraints of two problems: a finite family of variational inclusions for maximal monotone and inverse strongly monotone mappings and a fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under appropriate conditions imposed on the parameter sequences we will prove strong convergence of the proposed algorithm.

Theorem 21. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A, A_k : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2\}$ and $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\overline{\gamma}$. Let W_n be the W-mapping defined by (9). Assume that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \cap_{i=1}^{N} I(B_i, R_i) \neq \emptyset$ where G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$. Assume that:*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x) \rangle$ is weakly upper semicontinuous for each $y \in H$;*

(ii) *for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,*

$$\begin{aligned} &\Theta(y, z_x) + \varphi(z_x) - \varphi(y) \\ &\quad + \frac{1}{r} \left\langle K'(y) - K'(x), z_x - y \right\rangle < 0; \end{aligned} \tag{45}$$

(iii) *$\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;*

(iv) *$\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$, $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies*

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta; \tag{46}$$

(v) *$\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.*

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)} (I - r_n A) x_n, \\ z_n &= J_{R_N,\mu_N} (I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}} \\ &\quad \times (I - \mu_{N-1} B_{N-1}) \cdots J_{R_1,\mu_1} (I - \mu_1 B_1) u_n, \\ y_n &= \delta_n G z_n + (1 - \delta_n) W_n z_n, \\ x_{n+1} &= \alpha_n (u + \gamma f(x_n)) + \beta_n x_n \\ &\quad + ((1 - \beta_n) I - \alpha_n (I + \mu V)) W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{47}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \tag{OP2}$$

where h is the potential function of γf.

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)(1 + \mu\|V\|)^{-1}$. Since V is a $\overline{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup \{\langle Vu, u \rangle : u \in H, \|u\| = 1\}. \tag{48}$$

Observe that

$$\begin{aligned} \langle ((1 - \beta_n) I - \alpha_n (I + \mu V)) u, u \rangle &= 1 - \beta_n - \alpha_n - \alpha_n \mu \langle Vu, u \rangle \\ &\geq 1 - \beta_n - \alpha_n - \alpha_n \mu \|V\| \\ &\geq 0 \end{aligned} \tag{49}$$

that is, $(1 - \beta_n) I - \alpha_n (I + \mu V)$ is positive. It follows that

$$\begin{aligned} &\|(1 - \beta_n) I - \alpha_n (I + \mu V)\| \\ &\quad = \sup \{\langle ((1 - \beta_n) I - \alpha_n (I + \mu V)) u, u \rangle : u \in H, \|u\| = 1\} \\ &\quad = \sup \{1 - \beta_n - \alpha_n - \alpha_n \mu \langle Vu, u \rangle : u \in H, \|u\| = 1\} \\ &\quad \leq 1 - \beta_n - \alpha_n - \alpha_n \mu \overline{\gamma}. \end{aligned} \tag{50}$$

Put

$$\begin{aligned} \Lambda^i &= J_{R_i,\mu_i} (I - \mu_i B_i) J_{R_{i-1},\mu_{i-1}} \\ &\quad \times (I - \mu_{i-1} B_{i-1}) \cdots J_{R_1,\mu_1} (I - \mu_1 B_1) \end{aligned} \tag{51}$$

for all $i \in \{1, 2, \ldots, N\}$, and $\Lambda^0 = I$, where I is the identity mapping on H. Then we have that $z_n = \Lambda^N u_n$.

We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take $p \in \Omega$ arbitrarily. Since $p = S_{r_n}^{(\Theta,\varphi)}(p - r_n Ap)$, A is ζ-inverse strongly monotone and $0 \le r_n \le 2\zeta$, we have, for any $n \ge 1$,

$$\begin{aligned}\|u_n - p\|^2 &= \left\|S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n - S_{r_n}^{(\Theta,\varphi)}(I - r_n A)p\right\|^2\\ &\le \|(I - r_n A)x_n - (I - r_n A)p\|^2\\ &= \|(x_n - p) - r_n(Ax_n - Ap)\|^2\\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle\\ &\quad + r_n^2\|Ax_n - Ap\|^2\\ &\le \|x_n - p\|^2 - 2r_n\zeta\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2\\ &= \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2\\ &\le \|x_n - p\|^2.\end{aligned} \tag{52}$$

Since $p = J_{R_i,\mu_i}(I - \mu_i B_i)p, \Lambda^i p = p$ and B_i is η_i-inverse strongly monotone, where $\mu_i \in (0, 2\eta_i), i \in \{1, 2, \ldots, N\}$, by Lemma 16 we deduce that for each $n \ge 1$

$$\begin{aligned}&\|z_n - p\|^2\\ &\quad = \left\|J_{R_N,\mu_N}(I - \mu_N B_N)\Lambda^{N-1}u_n\right.\\ &\qquad \left.- J_{R_N,\mu_N}(I - \mu_N B_N)\Lambda^{N-1}p\right\|^2\\ &\quad \le \left\|(I - \mu_N B_N)\Lambda^{N-1}u_n - (I - \mu_N B_N)\Lambda^{N-1}p\right\|^2\\ &\quad = \left\|\left(\Lambda^{N-1}u_n - \Lambda^{N-1}p\right) - \mu_N\left(B_N\Lambda^{N-1}u_n - B_N\Lambda^{N-1}p\right)\right\|^2\\ &\quad \le \left\|\Lambda^{N-1}u_n - \Lambda^{N-1}p\right\|^2\\ &\qquad + \mu_N(\mu_N - 2\eta_N)\left\|B_N\Lambda^{N-1}u_n - B_N\Lambda^{N-1}p\right\|^2\\ &\quad \le \left\|\Lambda^{N-1}u_n - \Lambda^{N-1}p\right\|^2\\ &\quad \cdots\\ &\quad \le \left\|\Lambda^0 u_n - \Lambda^0 p\right\|^2 = \|u_n - p\|^2.\end{aligned} \tag{53}$$

Combining (52) and (53), we have

$$\|z_n - p\| \le \|x_n - p\|. \tag{54}$$

Since $p = Gp = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p$, A_k is ζ_k-inverse-strongly monotone for $k = 1, 2$, and $0 \le \nu_k \le 2\zeta_k$ for $k = 1, 2$, we deduce that, for any $n \ge 1$,

$$\begin{aligned}&\|Gz_n - p\|^2\\ &\quad = \left\|T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n\right.\\ &\qquad \left.- T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right\|^2\\ &\quad \le \left\|(I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n\right.\\ &\qquad \left.- (I - \nu_1 A_1)T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right\|^2\\ &\quad = \left\|\left[T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n - T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right]\right.\\ &\qquad \left.- \nu_1\left[A_1 T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n - A_1 T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right]\right\|^2\\ &\quad \le \left\|T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n - T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right\|^2\\ &\qquad + \nu_1(\nu_1 - 2\zeta_1)\\ &\qquad \times \left\|A_1 T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n - A_1 T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right\|^2\\ &\quad \le \left\|T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)z_n - T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p\right\|^2\\ &\quad \le \|(I - \nu_2 A_2)z_n - (I - \nu_2 A_2)p\|^2\\ &\quad = \|(z_n - p) - \nu_2(A_2 z_n - A_2 p)\|^2\\ &\quad \le \|z_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \le \|z_n - p\|^2.\end{aligned} \tag{55}$$

(This shows that G is nonexpansive.) Thus, from (54), we get

$$\begin{aligned}\|y_n - p\| &= \|\delta_n(Gz_n - p) + (1 - \delta_n)(W_n z_n - p)\|\\ &\le \delta_n\|Gz_n - p\| + (1 - \delta_n)\|W_n z_n - p\|\\ &\le \delta_n\|z_n - p\| + (1 - \delta_n)\|z_n - p\|\\ &= \|z_n - p\| \le \|x_n - p\|.\end{aligned} \tag{56}$$

Set $\overline{V} = I + \mu V$. Then from (47) we have

$$\begin{aligned}&\|x_{n+1} - p\|\\ &\quad = \|\alpha_n(u + \gamma f(x_n)) + \beta_n x_n\\ &\qquad + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n y_n - p\|\\ &\quad = \left\|\alpha_n u + \alpha_n\left(\gamma f(x_n) - \overline{V}p\right) + \beta_n(x_n - p)\right.\\ &\qquad \left.+ \left((1 - \beta_n)I - \alpha_n\overline{V}\right)(W_n y_n - p)\right\|\end{aligned}$$

$$\begin{aligned}
&\leq \left\|(1-\beta_n) I-\alpha_n\overline{V}\right\| \left\|W_n y_n - p\right\| \\
&\quad + \beta_n \left\|x_n - p\right\| + \alpha_n \|u\| + \alpha_n \left\|\gamma f(x_n) - \overline{V} p\right\| \\
&\leq (1-\beta_n-\alpha_n-\alpha_n\mu\overline{\gamma}) \left\|y_n - p\right\| \\
&\quad + \beta_n \left\|x_n - p\right\| + \alpha_n \|u\| + \alpha_n \left\|\gamma f(x_n) - \overline{V} p\right\| \\
&\leq (1-\alpha_n-\alpha_n\mu\overline{\gamma}) \left\|x_n - p\right\| + \alpha_n \|u\| \\
&\quad + \alpha_n \left(\gamma \left\|f(x_n) - f(p)\right\| + \left\|\gamma f(p) - \overline{V} p\right\|\right) \\
&\leq (1-\alpha_n-\alpha_n\mu\overline{\gamma}) \left\|x_n - p\right\| + \alpha_n \|u\| \\
&\quad + \alpha_n \left(\gamma l \left\|x_n - p\right\| + \left\|\gamma f(p) - \overline{V} p\right\|\right) \\
&\leq \left[1-((1+\mu)\overline{\gamma}-\gamma l)\alpha_n\right] \left\|x_n - p\right\| \\
&\quad + \alpha_n \left(\left\|\gamma f(p) - \overline{V} p\right\| + \|u\|\right) \\
&= \left[1-((1+\mu)\overline{\gamma}-\gamma l)\alpha_n\right] \left\|x_n - p\right\| + ((1+\mu)\overline{\gamma}-\gamma l)\alpha_n \\
&\quad \times \frac{\left\|\gamma f(p) - \overline{V} p\right\| + \|u\|}{(1+\mu)\overline{\gamma}-\gamma l} \\
&\leq \max\left\{\left\|x_n - p\right\|, \frac{\left\|\gamma f(p) - \overline{V} p\right\| + \|u\|}{(1+\mu)\overline{\gamma}-\gamma l}\right\}.
\end{aligned} \tag{57}$$

By induction, we get

$$\left\|x_n - p\right\| \leq \max\left\{\left\|x_0 - p\right\|, \frac{\left\|\gamma f(p) - \overline{V} p\right\| + \|u\|}{(1+\mu)\overline{\gamma}-\gamma l}\right\}. \tag{58}$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{u_n\}, \{z_n\}, \{y_n\}, \{f(x_n)\}$ and $\{W_n y_n\}$.

Step 2. We show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Indeed, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n) w_n, \quad \forall n \geq 1. \tag{59}$$

Then from the definition of w_n, we obtain

$$\begin{aligned}
w_{n+1} - w_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}(u+\gamma f(x_{n+1})) + ((1-\beta_{n+1}) I - \alpha_{n+1}\overline{V}) W_{n+1} y_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n(u+\gamma f(x_n)) + ((1-\beta_n) I - \alpha_n\overline{V}) W_n y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u+\gamma f(x_{n+1})) \\
&\quad - \frac{\alpha_n}{1-\beta_n}(u+\gamma f(x_n)) + W_{n+1} y_{n+1} - W_n y_n \\
&\quad + \frac{\alpha_n}{1-\beta_n}\overline{V} W_n y_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\overline{V} W_{n+1} y_{n+1} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[u+\gamma f(x_{n+1}) - \overline{V} W_{n+1} y_{n+1}\right] \\
&\quad + \frac{\alpha_n}{1-\beta_n}\left[\overline{V} W_n y_n - u - \gamma f(x_n)\right] \\
&\quad + W_{n+1} y_{n+1} - W_{n+1} y_n + W_{n+1} y_n - W_n y_n.
\end{aligned} \tag{60}$$

It follows that

$$\begin{aligned}
&\left\|w_{n+1} - w_n\right\| - \left\|x_{n+1} - x_n\right\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|u\| + \left\|\gamma f(x_{n+1})\right\| + \left\|\overline{V} W_{n+1} y_{n+1}\right\|\right) \\
&\quad\quad + \frac{\alpha_n}{1-\beta_n}\left(\left\|\overline{V} W_n y_n\right\| + \|u\| + \left\|\gamma f(x_n)\right\|\right) \\
&\quad\quad + \left\|W_{n+1} y_{n+1} - W_{n+1} y_n\right\| + \left\|W_{n+1} y_n - W_n y_n\right\| \\
&\quad\quad - \left\|x_{n+1} - x_n\right\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|u\| + \left\|\gamma f(x_{n+1})\right\| + \left\|\overline{V} W_{n+1} y_{n+1}\right\|\right) \\
&\quad\quad + \frac{\alpha_n}{1-\beta_n}\left(\left\|\overline{V} W_n y_n\right\| + \|u\| + \left\|\gamma f(x_n)\right\|\right) \\
&\quad\quad + \left\|W_{n+1} y_n - W_n y_n\right\| + \left\|y_{n+1} - y_n\right\| - \left\|x_{n+1} - x_n\right\|.
\end{aligned} \tag{61}$$

From (9), since W_n, T_n and $U_{n,i}$ are all nonexpansive, we have

$$\begin{aligned}
\left\|W_{n+1} z_n - W_n z_n\right\| &= \left\|\lambda_1 T_1 U_{n+1,2} z_n - \lambda_1 T_1 U_{n,2} z_n\right\| \\
&\leq \lambda_1 \left\|U_{n+1,2} z_n - U_{n,2} z_n\right\| \\
&= \lambda_1 \left\|\lambda_2 T_2 U_{n+1,3} z_n - \lambda_2 T_2 U_{n,3} z_n\right\| \\
&\leq \lambda_1 \lambda_2 \left\|U_{n+1,3} z_n - U_{n,3} z_n\right\| \\
&\cdots \\
&\leq \lambda_1 \lambda_2 \cdots \lambda_n \left\|U_{n+1,n+1} z_n - U_{n,n+1} z_n\right\| \\
&\leq M \prod_{i=1}^{n} \lambda_i,
\end{aligned} \tag{62}$$

$$\begin{aligned}
\left\|W_{n+1} y_n - W_n y_n\right\| &= \left\|\lambda_1 T_1 U_{n+1,2} y_n - \lambda_1 T_1 U_{n,2} y_n\right\| \\
&\leq \lambda_1 \left\|U_{n+1,2} y_n - U_{n,2} y_n\right\| \\
&= \lambda_1 \left\|\lambda_2 T_2 U_{n+1,3} y_n - \lambda_2 T_2 U_{n,3} y_n\right\| \\
&\leq \lambda_1 \lambda_2 \left\|U_{n+1,3} y_n - U_{n,3} y_n\right\|
\end{aligned}$$

$$\begin{aligned}
&\cdots\\
&\leq \lambda_1\lambda_2\cdots\lambda_n \left\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\right\|\\
&\leq M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{62$'$}$$

where M is a constant such that

$$\begin{aligned}
&\sup_{n\geq 1}\left\{\left\|U_{n+1,n+1}z_n\right\| + \left\|U_{n,n+1}z_n\right\|\right\} \leq M,\\
&\sup_{n\geq 1}\left\{\left\|U_{n+1,n+1}y_n\right\| + \left\|U_{n,n+1}y_n\right\|\right\} \leq M.
\end{aligned} \tag{63}$$

On the other hand, we estimate $\|y_{n+1} - y_n\|$. Taking into account that $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1, 0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$ and $0 < \liminf_{n\to\infty}r_n \leq \limsup_{n\to\infty}r_n < 2\zeta$, we may assume, without loss of generality, that $\{r_n\} \subset [c, d] \subset (0, 2\zeta)$ and $\{\beta_n\}, \{\delta_n\} \subset [\widehat{c}, \widehat{d}] \subset (0, 1)$. Utilizing Remark 5 and Lemma 16, we have

$$\begin{aligned}
&\left\|z_{n+1} - z_n\right\|^2\\
&\quad = \Big\|J_{R_N,\mu_N}\left(I - \mu_N B_N\right)\Lambda^{N-1}u_{n+1}\\
&\qquad -J_{R_N,\mu_N}\left(I - \mu_N B_N\right)\Lambda^{N-1}u_n\Big\|^2\\
&\quad \leq \left\|(I - \mu_N B_N)\Lambda^{N-1}u_{n+1} - (I - \mu_N B_N)\Lambda^{N-1}u_n\right\|^2\\
&\quad = \Big\|\left(\Lambda^{N-1}u_{n+1} - \Lambda^{N-1}u_n\right)\\
&\qquad -\mu_N\left(B_N\Lambda^{N-1}u_{n+1} - B_N\Lambda^{N-1}u_n\right)\Big\|^2\\
&\quad \leq \left\|\Lambda^{N-1}u_{n+1} - \Lambda^{N-1}u_n\right\|^2\\
&\qquad + \mu_N\left(\mu_N - 2\eta_N\right)\left\|B_N\Lambda^{N-1}u_{n+1} - B_N\Lambda^{N-1}u_n\right\|^2\\
&\quad \leq \left\|\Lambda^{N-1}u_{n+1} - \Lambda^{N-1}u_n\right\|^2\\
&\quad \cdots\\
&\quad \leq \left\|\Lambda^{0}u_{n+1} - \Lambda^{0}u_n\right\|^2 = \left\|u_{n+1} - u_n\right\|^2,
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\left\|\left(I - r_{n+1}A\right)x_{n+1} - \left(I - r_nA\right)x_n\right\|\\
&\quad = \left\|x_{n+1} - x_n - r_{n+1}\left(Ax_{n+1} - Ax_n\right) + \left(r_n - r_{n+1}\right)Ax_n\right\|\\
&\quad \leq \left\|x_{n+1} - x_n - r_{n+1}\left(Ax_{n+1} - Ax_n\right)\right\| + \left|r_{n+1} - r_n\right|\left\|Ax_n\right\|\\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left|r_{n+1} - r_n\right|\left\|Ax_n\right\|,
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\left\|u_{n+1} - u_n\right\|\\
&\quad = \left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_{n+1}A\right)x_{n+1} - S^{(\Theta,\varphi)}_{r_n}\left(I - r_nA\right)x_n\right\|\\
&\quad = \Big\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_{n+1}A\right)x_{n+1} - S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n\\
&\qquad + S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - S^{(\Theta,\varphi)}_{r_n}\left(I - r_nA\right)x_n\Big\|\\
&\quad \leq \left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_{n+1}A\right)\ x_{n+1} - S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n\right\|\\
&\qquad + \left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - S^{(\Theta,\varphi)}_{r_n}\left(I - r_nA\right)x_n\right\|\\
&\quad \leq \left\|\left(I - r_{n+1}A\right)x_{n+1} - \left(I - r_nA\right)x_n\right\|\\
&\qquad + \left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - S^{(\Theta,\varphi)}_{r_n}\left(I - r_nA\right)x_n\right\|\\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left|r_{n+1} - r_n\right|\left\|Ax_n\right\|\\
&\qquad + \left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - S^{(\Theta,\varphi)}_{r_n}\left(I - r_nA\right)x_n\right\|\\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left|r_{n+1} - r_n\right|\left\|Ax_n\right\| + \frac{\left|r_{n+1} - r_n\right|}{r_{n+1}}\\
&\qquad \cdot\frac{\nu}{\sigma}\left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - \left(I - r_nA\right)x_n\right\|\\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left|r_{n+1} - r_n\right|\\
&\qquad \times\left(\left\|Ax_n\right\| + \frac{\nu}{c\sigma}\left\|S^{(\Theta,\varphi)}_{r_{n+1}}\left(I - r_nA\right)x_n - \left(I - r_nA\right)x_n\right\|\right)\\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left|r_{n+1} - r_n\right|M_1,
\end{aligned} \tag{66}$$

where $\sup_{n\geq 1}\{\|Ax_n\| + (\nu/c\sigma)\|S^{(\Theta,\varphi)}_{r_{n+1}}(I - r_nA)x_n - (I - r_nA)x_n\|\} \leq M_1$ for some $M_1 > 0$.

Note that

$$\begin{aligned}
&y_{n+1} - y_n\\
&\quad = \delta_n\left(Gz_{n+1} - Gz_n\right) + \left(\delta_{n+1} - \delta_n\right)\left(Gz_{n+1} - W_{n+1}z_{n+1}\right)\\
&\qquad + \left(1 - \delta_n\right)\left(W_{n+1}z_{n+1} - W_nz_n\right).
\end{aligned} \tag{67}$$

Since G is nonexpansive, from (62), (64) and (66) it follows that

$$\begin{aligned}
&\left\|y_{n+1} - y_n\right\|\\
&\quad \leq \delta_n\left\|Gz_{n+1} - Gz_n\right\| + \left|\delta_{n+1} - \delta_n\right|\left\|Gz_{n+1} - W_{n+1}z_{n+1}\right\|\\
&\qquad + \left(1 - \delta_n\right)\left\|W_{n+1}z_{n+1} - W_nz_n\right\|\\
&\quad \leq \delta_n\left\|z_{n+1} - z_n\right\| + \left|\delta_{n+1} - \delta_n\right|\left\|Gz_{n+1} - W_{n+1}z_{n+1}\right\|\\
&\qquad + \left(1 - \delta_n\right)\left(\left\|W_{n+1}z_{n+1} - W_{n+1}z_n\right\|\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| W_{n+1} z_n - W_n z_n \right\|) \\
&\le \delta_n \left\| z_{n+1} - z_n \right\| + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| \\
&\quad + (1-\delta_n) \left(\left\| z_{n+1} - z_n \right\| + M \prod_{i=1}^{n} \lambda_i \right) \\
&\le \left\| z_{n+1} - z_n \right\| + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| \\
&\quad + M \prod_{i=1}^{n} \lambda_i \\
&\le \left\| u_{n+1} - u_n \right\| + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| \\
&\quad + M \prod_{i=1}^{n} \lambda_i \\
&\le \left\| x_{n+1} - x_n \right\| + \left| r_{n+1} - r_n \right| M_1 \\
&\quad + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| + M \prod_{i=1}^{n} \lambda_i .
\end{aligned} \tag{68}
$$

Utilizing (61), $(62)'$ and (68), we have

$$
\begin{aligned}
&\left\| w_{n+1} - w_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\|u\| + \left\| \gamma f(x_{n+1}) \right\| + \left\| \overline{V} W_{n+1} y_{n+1} \right\| \right) \\
&\quad + \frac{\alpha_n}{1-\beta_n} \left(\left\| \overline{V} W_n y_n \right\| + \|u\| + \left\| \gamma f(x_n) \right\| \right) \\
&\quad + \left\| W_{n+1} y_n - W_n y_n \right\| + \left\| y_{n+1} - y_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\|u\| + \left\| \gamma f(x_{n+1}) \right\| + \left\| \overline{V} W_{n+1} y_{n+1} \right\| \right) \\
&\quad + \frac{\alpha_n}{1-\beta_n} \left(\left\| \overline{V} W_n y_n \right\| + \|u\| + \left\| \gamma f(x_n) \right\| \right) \\
&\quad + M \prod_{i=1}^{n} \lambda_i + \left\| x_{n+1} - x_n \right\| + \left| r_{n+1} - r_n \right| M_1 \\
&\quad + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| \\
&\quad + M \prod_{i=1}^{n} \lambda_i - \left\| x_{n+1} - x_n \right\| \\
&\le \frac{\alpha_{n+1}}{1-\widehat{d}} \left(\|u\| + \left\| \gamma f(x_{n+1}) \right\| + \left\| \overline{V} W_{n+1} y_{n+1} \right\| \right) \\
&\quad + \frac{\alpha_n}{1-\widehat{d}} \left(\left\| \overline{V} W_n y_n \right\| + \|u\| + \left\| \gamma f(x_n) \right\| \right) \\
&\quad + 2M \prod_{i=1}^{n} \lambda_i + \left| r_{n+1} - r_n \right| M_1 \\
&\quad + \left| \delta_{n+1} - \delta_n \right| \left\| G z_{n+1} - W_{n+1} z_{n+1} \right\| \\
&\le M_2 \left(\alpha_{n+1} + \alpha_n + \prod_{i=1}^{n} \lambda_i + \left| r_{n+1} - r_n \right| + \left| \delta_{n+1} - \delta_n \right| \right) \\
&\le M_2 \left(\alpha_{n+1} + \alpha_n + b^n + \left| r_{n+1} - r_n \right| + \left| \delta_{n+1} - \delta_n \right| \right),
\end{aligned} \tag{69}
$$

where $\sup_{n\ge 1}\{(1/(1-\widehat{d}))(\|u\| + \|\gamma f(x_n)\| + \|\overline{V} W_n y_n\|) + \|G z_n - W_n z_n\| + M_1 + 2M\} \le M_2$ for some $M_2 > 0$. Since $b \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty}(|\delta_{n+1}-\delta_n| + |r_{n+1}-r_n|) = 0$, we deduce from (69) that

$$
\limsup_{n\to\infty} \left(\left\| w_{n+1} - w_n \right\| - \left\| x_{n+1} - x_n \right\| \right) = 0. \tag{70}
$$

Since $x_{n+1} = \beta_n x_n + (1-\beta_n) w_n$ for all $n \ge 1$, by Lemma 14 we obtain from $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ that

$$
\lim_{n\to\infty} \left\| w_n - x_n \right\| = 0, \tag{71}
$$

which immediately yields

$$
\lim_{n\to\infty} \left\| x_{n+1} - x_n \right\| = \lim_{n\to\infty} (1-\beta_n) \left\| w_n - x_n \right\| = 0. \tag{72}
$$

Step 3. $\|y_n - G z_n\| \to 0$ as $n \to \infty$.

Indeed, utilizing Lemmas 6 and 7(b) we obtain from (47) and (54) that

$$
\begin{aligned}
&\left\| x_{n+1} - p \right\|^2 \\
&= \left\| \alpha_n \left((u + \gamma f(x_n)) - \overline{V} W_n y_n \right) + \beta_n (x_n - p) + (1-\beta_n)(W_n y_n - p) \right\|^2 \\
&\le \left\| \beta_n (x_n - p) + (1-\beta_n)(W_n y_n - p) \right\|^2 \\
&\quad + 2\alpha_n \left\langle (u + \gamma f(x_n)) - \overline{V} W_n y_n, x_{n+1} - p \right\rangle \\
&= \beta_n \left\| x_n - p \right\|^2 + (1-\beta_n) \left\| W_n y_n - p \right\|^2 \\
&\quad - \beta_n (1-\beta_n) \left\| x_n - W_n y_n \right\|^2 \\
&\quad + 2\alpha_n \left\| (u + \gamma f(x_n)) - \overline{V} W_n y_n \right\| \left\| x_{n+1} - p \right\| \\
&\le \beta_n \left\| x_n - p \right\|^2 + (1-\beta_n) \left\| y_n - p \right\|^2 \\
&\quad - \beta_n (1-\beta_n) \left\| x_n - W_n y_n \right\|^2 \\
&\quad + 2\alpha_n \left\| (u + \gamma f(x_n)) - \overline{V} W_n y_n \right\| \left\| x_{n+1} - p \right\| \\
&\le \beta_n \left\| x_n - p \right\|^2 + (1-\beta_n) \left\| x_n - p \right\|^2 \\
&\quad - \beta_n (1-\beta_n) \left\| x_n - W_n y_n \right\|^2 \\
&\quad + 2\alpha_n \left\| (u + \gamma f(x_n)) - \overline{V} W_n y_n \right\| \left\| x_{n+1} - p \right\| \\
&= \left\| x_n - p \right\|^2 - \beta_n (1-\beta_n) \left\| x_n - W_n y_n \right\|^2 \\
&\quad + 2\alpha_n \left\| (u + \gamma f(x_n)) - \overline{V} W_n y_n \right\| \left\| x_{n+1} - p \right\| ,
\end{aligned} \tag{73}
$$

which leads to

$$\begin{aligned}
&\hat{c}\left(1-\hat{d}\right)\left\|x_n-W_ny_n\right\|^2\\
&\quad\le \beta_n\left(1-\beta_n\right)\left\|x_n-W_ny_n\right\|^2\\
&\quad\le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|\\
&\quad\le \left\|x_n-x_{n+1}\right\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|.
\end{aligned}\tag{74}$$

Since $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ and $\lim_{n\to\infty}\alpha_n=0$, we deduce from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$ and $\{W_ny_n\}$ that

$$\lim_{n\to\infty}\left\|x_n-W_ny_n\right\|=0.\tag{75}$$

Also, by Lemma 7(b) we deduce from (47) and (54) that

$$\begin{aligned}
\left\|y_n-p\right\|^2&=\delta_n\left\|Gz_n-p\right\|^2+\left(1-\delta_n\right)\left\|W_nz_n-p\right\|^2\\
&\quad-\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\\
&\le\delta_n\left\|z_n-p\right\|^2+\left(1-\delta_n\right)\left\|z_n-p\right\|^2\\
&\quad-\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\\
&=\left\|z_n-p\right\|^2-\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\\
&\le\left\|x_n-p\right\|^2-\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2.
\end{aligned}\tag{76}$$

From (73) and (76) we get

$$\begin{aligned}
&\left\|x_{n+1}-p\right\|^2\\
&\quad\le\beta_n\left\|x_n-p\right\|^2+\left(1-\beta_n\right)\left\|y_n-p\right\|^2\\
&\qquad-\beta_n\left(1-\beta_n\right)\left\|x_n-W_ny_n\right\|^2\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|\\
&\quad\le\beta_n\left\|x_n-p\right\|^2+\left(1-\beta_n\right)\left\|y_n-p\right\|^2\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|\\
&\quad\le\beta_n\left\|x_n-p\right\|^2+\left(1-\beta_n\right)\\
&\qquad\times\left[\left\|x_n-p\right\|^2-\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\right]\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|\\
&\quad=\left\|x_n-p\right\|^2-\left(1-\beta_n\right)\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|,
\end{aligned}\tag{77}$$

which immediately implies that

$$\begin{aligned}
&\hat{c}\left(1-\hat{d}\right)^2\left\|Gz_n-W_nz_n\right\|^2\\
&\quad\le\left(1-\beta_n\right)\delta_n\left(1-\delta_n\right)\left\|Gz_n-W_nz_n\right\|^2\\
&\quad\le\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|\\
&\quad\le\left\|x_n-x_{n+1}\right\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)\\
&\qquad+2\alpha_n\left\|\left(u+\gamma f\left(x_n\right)\right)-\overline{V}W_ny_n\right\|\left\|x_{n+1}-p\right\|.
\end{aligned}\tag{78}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we deduce from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$ and $\{W_ny_n\}$ that

$$\lim_{n\to\infty}\left\|Gz_n-W_nz_n\right\|=0.\tag{79}$$

So, it follows that

$$\lim_{n\to\infty}\left\|y_n-Gz_n\right\|=\lim_{n\to\infty}\left(1-\delta_n\right)\left\|W_nz_n-Gz_n\right\|=0.\tag{80}$$

Step 4. $\|x_n-u_n\|\to 0$, $\|u_n-z_n\|\to 0$, $\|z_n-Gz_n\|\to 0$ and $\|z_n-Wz_n\|\to 0$ as $n\to\infty$.

Indeed, for $p\in\Omega$, we find that

$$\begin{aligned}
&\left\|u_n-p\right\|^2\\
&\quad=\left\|S_{r_n}^{(\Theta,\varphi)}\left(I-r_nA\right)x_n-S_{r_n}^{(\Theta,\varphi)}\left(I-r_nA\right)p\right\|^2\\
&\quad\le\left\|(I-r_nA)x_n-\left(I-r_nA\right)p\right\|^2\\
&\quad=\left\|x_n-p-r_n\left(Ax_n-Ap\right)\right\|^2\\
&\quad\le\left\|x_n-p\right\|^2+r_n\left(r_n-2\zeta\right)\left\|Ax_n-Ap\right\|^2.
\end{aligned}\tag{81}$$

From (47), (53) and (81), we obtain

$$\begin{aligned}
\left\|y_n-p\right\|^2&\le\delta_n\left\|Gz_n-p\right\|^2+\left(1-\delta_n\right)\left\|W_nz_n-p\right\|^2\\
&\le\left\|z_n-p\right\|^2\le\left\|u_n-p\right\|^2\\
&\le\left\|x_n-p\right\|^2+r_n\left(r_n-2\zeta\right)\left\|Ax_n-Ap\right\|^2,
\end{aligned}\tag{82}$$

which together with (73), implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\qquad-\beta_n(1-\beta_n)\|x_n-W_ny_n\|^2 \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left[\|x_n-p\|^2+r_n(r_n-2\zeta)\|Ax_n-Ap\|^2\right] \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad=\|x_n-p\|^2+(1-\beta_n)r_n(r_n-2\zeta)\|Ax_n-Ap\|^2 \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.
\end{aligned}\tag{83}$$

So, it follows that

$$\begin{aligned}
&\left(1-\widehat{d}\right)c(2\zeta-d)\|Ax_n-Ap\|^2 \\
&\quad\le(1-\beta_n)r_n(2\zeta-r_n)\|Ax_n-Ap\|^2 \\
&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad\le\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.
\end{aligned}\tag{84}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{W_ny_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0.\tag{85}$$

Furthermore, from the firm nonexpansivity of $S_{r_n}^{(\Theta,\varphi)}$, we have

$$\begin{aligned}
&\|u_n-p\|^2 \\
&\quad=\left\|S_{r_n}^{(\Theta,\varphi)}(I-r_nA)x_n-S_{r_n}^{(\Theta,\varphi)}(I-r_nA)p\right\|^2 \\
&\quad\le\langle(I-r_nA)x_n-(I-r_nA)p,u_n-p\rangle \\
&\quad=\frac{1}{2}\left[\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|u_n-p\|^2\right. \\
&\qquad\left.-\|(I-r_nA)x_n-(I-r_nA)p-(u_n-p)\|^2\right] \\
&\quad\le\frac{1}{2}\left[\|x_n-p\|^2+\|u_n-p\|^2\right. \\
&\qquad\left.-\|x_n-u_n-r_n(Ax_n-Ap)\|^2\right] \\
&\quad=\frac{1}{2}\left[\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\right. \\
&\qquad\left.+2r_n\langle Ax_n-Ap,x_n-u_n\rangle-r_n^2\|Ax_n-Ap\|^2\right],
\end{aligned}\tag{86}$$

which implies that

$$\begin{aligned}
\|u_n-p\|^2\le{}&\|x_n-p\|^2-\|x_n-u_n\|^2 \\
&+2r_n\|Ax_n-Ap\|\|x_n-u_n\|.
\end{aligned}\tag{87}$$

From (47) and (87), we have

$$\begin{aligned}
\|y_n-p\|^2&\le\delta_n\|Gz_n-p\|^2+(1-\delta_n)\|W_nz_n-p\|^2 \\
&\le\|z_n-p\|^2\le\|u_n-p\|^2 \\
&\le\|x_n-p\|^2-\|x_n-u_n\|^2 \\
&\quad+2r_n\|Ax_n-Ap\|\|x_n-u_n\|,
\end{aligned}\tag{88}$$

which together with (73), implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad\le\beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left[\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|Ax_n-Ap\|\|x_n-u_n\|\right] \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad\le\|x_n-p\|^2-(1-\beta_n)\|x_n-u_n\|^2 \\
&\qquad+2r_n\|Ax_n-Ap\|\|x_n-u_n\| \\
&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.
\end{aligned}\tag{89}$$

So, it follows that

$$\begin{aligned}
&\left(1-\widehat{d}\right)\|x_n-u_n\|^2 \\
&\quad\le(1-\beta_n)\|x_n-u_n\|^2
\end{aligned}$$

$$\begin{aligned}&\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2r_n\|Ax_n-Ap\|\|x_n-u_n\|\\&\quad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|\\&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\&\quad+2r_n\|Ax_n-Ap\|\|x_n-u_n\|\\&\quad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.\end{aligned}\tag{90}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, from (85) and the boundedness of $\{x_n\},\{y_n\},\{W_ny_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{91}$$

Next we show that $\lim_{n\to\infty}\|A_i\Lambda^iu_n-A_ip\|=0$, $i=1,2,\ldots,N$. Observe that

$$\begin{aligned}&\left\|\Lambda^iu_n-p\right\|^2\\&\quad=\left\|J_{R_i,\mu_i}(I-\mu_iB_i)\Lambda^{i-1}u_n-J_{R_i,\mu_i}(I-\mu_iB_i)p\right\|^2\\&\quad\leq\left\|(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p\right\|^2\\&\quad\leq\left\|\Lambda^{i-1}u_n-p\right\|^2+\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\\&\quad\leq\|u_n-p\|^2+\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\\&\quad\leq\|x_n-p\|^2+\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2.\end{aligned}\tag{92}$$

From (47) and (92), we have

$$\begin{aligned}\|y_n-p\|^2&\leq\delta_n\|Gz_n-p\|^2+(1-\delta_n)\|W_nz_n-p\|^2\\&\leq\|z_n-p\|^2\leq\left\|\Lambda^iu_n-p\right\|^2\\&\leq\|x_n-p\|^2+\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2,\end{aligned}\tag{93}$$

which together with (73), implies that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|\\&\quad\leq\beta_n\|x_n-p\|^2+(1-\beta_n)\\&\qquad\times\left[\|x_n-p\|^2+\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\right]\\&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|\\&\quad=\|x_n-p\|^2+(1-\beta_n)\mu_i(\mu_i-2\eta_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\\&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.\end{aligned}\tag{94}$$

So, it follows that

$$\begin{aligned}&\left(1-\hat{d}\right)\mu_i(2\eta_i-\mu_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\\&\quad\leq(1-\beta_n)\mu_i(2\eta_i-\mu_i)\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|^2\\&\quad\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|\\&\quad\leq\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\&\qquad+2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.\end{aligned}\tag{95}$$

Since $\mu_i\in(0,2\eta_i)$, $i=1,2,\ldots,N$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, from the boundedness of $\{x_n\},\{y_n\},\{W_ny_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty}\left\|B_i\Lambda^{i-1}u_n-B_ip\right\|=0,\quad\forall i\in\{1,2,\ldots,N\}.\tag{96}$$

By Lemmas 7 (a) and 16, we obtain

$$\begin{aligned}&\left\|\Lambda^iu_n-p\right\|^2\\&\quad=\left\|J_{R_i,\mu_i}(I-\mu_iB_i)\Lambda^{i-1}u_n-J_{R_i,\mu_i}(I-\mu_iB_i)p\right\|^2\\&\quad\leq\left\langle(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p,\Lambda^iu_n-p\right\rangle\\&\quad=\frac{1}{2}\left(\left\|(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p\right\|^2+\left\|\Lambda^iu_n-p\right\|^2\right.\\&\qquad\left.-\left\|(I-\mu_iB_i)\Lambda^{i-1}u_n-(I-\mu_iB_i)p-\left(\Lambda^iu_n-p\right)\right\|^2\right)\\&\quad\leq\frac{1}{2}\left(\left\|\Lambda^{i-1}u_n-p\right\|^2+\left\|\Lambda^iu_n-p\right\|^2\right.\\&\qquad\left.-\left\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\right\|^2\right)\\&\quad\leq\frac{1}{2}\left(\|u_n-p\|^2+\left\|\Lambda^iu_n-p\right\|^2\right.\\&\qquad\left.-\left\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\right\|^2\right)\\&\quad\leq\frac{1}{2}\left(\|x_n-p\|^2+\left\|\Lambda^iu_n-p\right\|^2\right.\\&\qquad\left.-\left\|\Lambda^{i-1}u_n-\Lambda^iu_n-\mu_i(B_i\Lambda^{i-1}u_n-B_ip)\right\|^2\right),\end{aligned}\tag{97}$$

which implies that

$$\begin{aligned}\left\|\Lambda^i u_n - p\right\|^2 &\le \left\|x_n - p\right\|^2 \\ &\quad - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n - \mu_i \left(B_i \Lambda^{i-1} u_n - B_i p\right)\right\|^2 \\ &= \left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad - \mu_i^2 \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|^2 \\ &\quad + 2\mu_i \left\langle \Lambda^{i-1} u_n - \Lambda^i u_n, B_i \Lambda^{i-1} u_n - B_i p\right\rangle \\ &\le \left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|.\end{aligned} \tag{98}$$

From (47) and (98), we have

$$\begin{aligned}\left\|y_n - p\right\|^2 &\le \delta_n \left\|G z_n - p\right\|^2 + \left(1 - \delta_n\right) \left\|W_n z_n - p\right\|^2 \\ &\le \left\|z_n - p\right\|^2 \le \left\|\Lambda^i u_n - p\right\|^2 \\ &\le \left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|,\end{aligned} \tag{99}$$

which together with (73), implies that

$$\begin{aligned}&\left\|x_{n+1} - p\right\|^2 \\ &\quad \le \beta_n \left\|x_n - p\right\|^2 + \left(1 - \beta_n\right) \left\|y_n - p\right\|^2 \\ &\qquad + 2\alpha_n \left\|\left(u + \gamma f\left(x_n\right)\right) - \overline{V} W_n y_n\right\| \left\|x_{n+1} - p\right\| \\ &\quad \le \beta_n \left\|x_n - p\right\|^2 + \left(1 - \beta_n\right) \\ &\qquad \times \left[\left\|x_n - p\right\|^2 - \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \right. \\ &\qquad\qquad \left. + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\|\right] \\ &\qquad + 2\alpha_n \left\|\left(u + \gamma f\left(x_n\right)\right) - \overline{V} W_n y_n\right\| \left\|x_{n+1} - p\right\| \\ &\quad \le \left\|x_n - p\right\|^2 - \left(1 - \beta_n\right) \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\qquad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\qquad + 2\alpha_n \left\|\left(u + \gamma f\left(x_n\right)\right) - \overline{V} W_n y_n\right\| \left\|x_{n+1} - p\right\|.\end{aligned} \tag{100}$$

So, it follows that

$$\begin{aligned}&\left(1 - \widehat{d}\right) \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad \le \left(1 - \beta_n\right) \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\|^2 \\ &\quad \le \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\ &\qquad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\qquad + 2\alpha_n \left\|\left(u + \gamma f\left(x_n\right)\right) - \overline{V} W_n y_n\right\| \left\|x_{n+1} - p\right\| \\ &\quad \le \left\|x_n - x_{n+1}\right\| \left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right) \\ &\qquad + 2\mu_i \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| \left\|B_i \Lambda^{i-1} u_n - B_i p\right\| \\ &\qquad + 2\alpha_n \left\|\left(u + \gamma f\left(x_n\right)\right) - \overline{V} W_n y_n\right\| \left\|x_{n+1} - p\right\|.\end{aligned} \tag{101}$$

Since $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$, from (96) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{W_n y_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n \to \infty} \left\|\Lambda^{i-1} u_n - \Lambda^i u_n\right\| = 0, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{102}$$

From (102) we get

$$\begin{aligned}\left\|u_n - z_n\right\| &= \left\|\Lambda^0 u_n - \Lambda^N u_n\right\| \\ &\le \left\|\Lambda^0 u_n - \Lambda^1 u_n\right\| + \left\|\Lambda^1 u_n - \Lambda^2 u_n\right\| \\ &\quad + \cdots + \left\|\Lambda^{N-1} u_n - \Lambda^N u_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{103}$$

By (91) and (103), we have

$$\begin{aligned}\left\|x_n - z_n\right\| &\le \left\|x_n - u_n\right\| + \left\|u_n - z_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{104}$$

On the other hand, for simplicity, we write $\widetilde{p} = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)p$, $v_n = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2) z_n$ and $\widetilde{v}_n = G z_n = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1) v_n$ for all $n \ge 1$. Then

$$\begin{aligned}p = Gp &= T_{\nu_1}^{\Theta_1}\left(I - \nu_1 A_1\right) \widetilde{p} \\ &= T_{\nu_1}^{\Theta_1}\left(I - \nu_1 A_1\right) T_{\nu_2}^{\Theta_2}\left(I - \nu_2 A_2\right) p.\end{aligned} \tag{105}$$

We now show that $\lim_{n\to\infty}\|Gz_n - z_n\| = 0$, that is, $\lim_{n\to\infty}\|\tilde{v}_n - z_n\| = 0$. As a matter of fact, for $p \in \Omega$, it follows from (47), (54) and (55) that

$$\begin{aligned}
\|y_n - p\|^2 &\le \delta_n \|Gz_n - p\|^2 + (1-\delta_n)\|W_n z_n - p\|^2 \\
&\le \delta_n \|\tilde{v}_n - p\|^2 + (1-\delta_n)\|z_n - p\|^2 \\
&\le \delta_n \left[\|v_n - \tilde{p}\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right] \\
&\quad + (1-\delta_n)\|z_n - p\|^2 \\
&\le \delta_n \left[\|z_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad \left. + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right] \\
&\quad + (1-\delta_n)\|z_n - p\|^2 \\
&= \|z_n - p\|^2 + \delta_n \left(\nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad \left. + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right) \\
&\le \|x_n - p\|^2 \\
&\quad + \delta_n \left(\nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad \left. + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right),
\end{aligned} \tag{106}$$

which together with (73), implies that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\qquad + 2\alpha_n \left\|(u + \gamma f(x_n)) - \overline{V} W_n y_n\right\| \|x_{n+1} - p\| \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n) \\
&\qquad \times \left[\|x_n - p\|^2 \right. \\
&\qquad\quad + \delta_n \left(\nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad\qquad \left.\left. + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right)\right] \\
&\qquad + 2\alpha_n \left\|(u + \gamma f(x_n)) - \overline{V} W_n y_n\right\| \|x_{n+1} - p\| \\
&\quad = \|x_n - p\|^2 \\
&\qquad + (1-\beta_n)\delta_n \left(\nu_2(\nu_2 - 2\zeta_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad\qquad \left. + \nu_1(\nu_1 - 2\zeta_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right) \\
&\qquad + 2\alpha_n \left\|(u + \gamma f(x_n)) - \overline{V} W_n y_n\right\| \|x_{n+1} - p\|.
\end{aligned} \tag{107}$$

So, it follows that

$$\begin{aligned}
&(1-\hat{d})\,\hat{c}\left(\nu_2(2\zeta_2 - \nu_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad \left. + \nu_1(2\zeta_1 - \nu_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right) \\
&\quad \le (1-\beta_n)\delta_n \left(\nu_2(2\zeta_2 - \nu_2)\|A_2 z_n - A_2 p\|^2 \right. \\
&\qquad \left. + \nu_1(2\zeta_1 - \nu_1)\|A_1 v_n - A_1 \tilde{p}\|^2\right) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + 2\alpha_n \left\|(u + \gamma f(x_n)) - \overline{V} W_n y_n\right\| \|x_{n+1} - p\| \\
&\quad \le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad + 2\alpha_n \left\|(u + \gamma f(x_n)) - \overline{V} W_n y_n\right\| \|x_{n+1} - p\|.
\end{aligned} \tag{108}$$

Since $\nu_k \in (0, 2\zeta_k)$, $k = 1, 2$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, from the boundedness of $\{x_n\}$, $\{y_n\}$, $\{W_n y_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty}\|A_2 z_n - A_2 p\| = 0, \qquad \lim_{n\to\infty}\|A_1 v_n - A_1 \tilde{p}\| = 0. \tag{109}$$

Also, in terms of the firm nonexpansivity of $T^{\Theta_k}_{\nu_k}$ and the ζ_k-inverse strong monotonicity of A_k for $k = 1, 2$, we obtain from $\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$ and (54)-(55) that

$$\begin{aligned}
&\|v_n - \tilde{p}\|^2 \\
&\quad = \left\|T^{\Theta_2}_{\nu_2}(I - \nu_2 A_2) z_n - T^{\Theta_2}_{\nu_2}(I - \nu_2 A_2) p\right\|^2 \\
&\quad \le \langle (I - \nu_2 A_2) z_n - (I - \nu_2 A_2) p, v_n - \tilde{p}\rangle \\
&\quad = \frac{1}{2}\left[\|(I - \nu_2 A_2) z_n - (I - \nu_2 A_2) p\|^2 + \|v_n - \tilde{p}\|^2 \right. \\
&\qquad \left. - \|(I - \nu_2 A_2) z_n - (I - \nu_2 A_2) p - (v_n - \tilde{p})\|^2\right] \\
&\quad \le \frac{1}{2}\left[\|z_n - p\|^2 + \|v_n - \tilde{p}\|^2 \right. \\
&\qquad \left. - \|(z_n - v_n) - \nu_2(A_2 z_n - A_2 p) - (p - \tilde{p})\|^2\right] \\
&\quad \le \frac{1}{2}\left[\|x_n - p\|^2 + \|v_n - \tilde{p}\|^2 - \|(z_n - v_n) - (p - \tilde{p})\|^2 \right. \\
&\qquad + 2\nu_2 \langle (z_n - v_n) - (p - \tilde{p}), A_2 z_n - A_2 p\rangle \\
&\qquad \left. - \nu_2^2 \|A_2 z_n - A_2 p\|^2\right], \\
&\|\tilde{v}_n - p\|^2 \\
&\quad = \left\|T^{\Theta_1}_{\nu_1}(I - \nu_1 A_1) v_n - T^{\Theta_1}_{\nu_1}(I - \nu_1 A_1)\tilde{p}\right\|^2 \\
&\quad \le \langle (I - \nu_1 A_1) v_n - (I - \nu_1 A_1)\tilde{p}, \tilde{v}_n - p\rangle \\
&\quad = \frac{1}{2}\left[\|(I - \nu_1 A_1) v_n - (I - \nu_1 A_1)\tilde{p}\|^2 + \|\tilde{v}_n - p\|^2 \right. \\
&\qquad \left. - \|(I - \nu_1 A_1) v_n - (I - \nu_1 A_1)\tilde{p} - (\tilde{v}_n - p)\|^2\right]
\end{aligned}$$

$$\leq \frac{1}{2}\left[\|v_n - \widetilde{p}\|^2 + \|\widetilde{v}_n - p\|^2 - \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|^2 + 2\nu_1 \langle A_1 v_n - A_1 \widetilde{p}, (v_n - \widetilde{v}_n) + (p - \widetilde{p})\rangle - \nu_1^2 \|A_1 v_n - A_1 \widetilde{p}\|^2\right]$$
$$\leq \frac{1}{2}\left[\|x_n - p\|^2 + \|\widetilde{v}_n - p\|^2 - \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|^2 + 2\nu_1 \langle A_1 v_n - A_1 \widetilde{p}, (v_n - \widetilde{v}_n) + (p - \widetilde{p})\rangle\right]. \tag{110}$$

Thus, we have

$$\begin{aligned}\|v_n - \widetilde{p}\|^2 \leq{}& \|x_n - p\|^2 - \|(z_n - v_n) - (p - \widetilde{p})\|^2 \\ &+ 2\nu_2 \langle (z_n - v_n) - (p - \widetilde{p}), A_2 z_n - A_2 p\rangle \\ &- \nu_2^2 \|A_2 z_n - A_2 p\|^2,\end{aligned} \tag{111}$$

$$\|\widetilde{v}_n - p\|^2 \leq \|x_n - p\|^2 - \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|^2 + 2\nu_1 \|A_1 v_n - A_1 \widetilde{p}\| \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|. \tag{112}$$

Consequently, from (47), (54), (55) and (111) it follows that

$$\begin{aligned}\|y_n - p\|^2 &\leq \delta_n \|G z_n - p\|^2 + (1 - \delta_n) \|W_n z_n - p\|^2 \\ &\leq \delta_n \|\widetilde{v}_n - p\|^2 + (1 - \delta_n) \|z_n - p\|^2 \\ &\leq \delta_n \|v_n - \widetilde{p}\|^2 + (1 - \delta_n) \|z_n - p\|^2 \\ &\leq \delta_n \left[\|x_n - p\|^2 - \|(z_n - v_n) - (p - \widetilde{p})\|^2 + 2\nu_2 \langle (z_n - v_n) - (p - \widetilde{p}), A_2 z_n - A_2 p\rangle\right] \\ &\quad + (1 - \delta_n) \|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \delta_n \|(z_n - v_n) - (p - \widetilde{p})\|^2 + 2\nu_2 \|(z_n - v_n) - (p - \widetilde{p})\| \|A_2 z_n - A_2 p\|,\end{aligned} \tag{113}$$

which together with (73), implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 + 2\alpha_n \|(u + \gamma f(x_n)) - \overline{V} W_n y_n\| \|x_{n+1} - p\| \\ &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \times \left[\|x_n - p\|^2 - \delta_n \|(z_n - v_n) - (p - \widetilde{p})\|^2 + 2\nu_2 \|(z_n - v_n) - (p - \widetilde{p})\| \|A_2 z_n - A_2 p\|\right] \\ &\quad + 2\alpha_n \|(u + \gamma f(x_n)) - \overline{V} W_n y_n\| \|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 - (1 - \beta_n) \delta_n \|(z_n - v_n) - (p - \widetilde{p})\|^2 + 2\nu_2 \|(z_n - v_n) - (p - \widetilde{p})\| \|A_2 z_n - A_2 p\| \\ &\quad + 2\alpha_n \|(u + \gamma f(x_n)) - \overline{V} W_n y_n\| \|x_{n+1} - p\|.\end{aligned} \tag{114}$$

So, it follows that

$$\begin{aligned}\left(1 - \widehat{d}\right) \widehat{c} \|(z_n - v_n) - (p - \widetilde{p})\|^2 &\leq (1 - \beta_n) \delta_n \|(z_n - v_n) - (p - \widetilde{p})\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\nu_2 \|(z_n - v_n) - (p - \widetilde{p})\| \|A_2 z_n - A_2 p\| \\ &\quad + 2\alpha_n \|(u + \gamma f(x_n)) - \overline{V} W_n y_n\| \|x_{n+1} - p\| \\ &\leq \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) + 2\nu_2 \|(z_n - v_n) - (p - \widetilde{p})\| \|A_2 z_n - A_2 p\| \\ &\quad + 2\alpha_n \|(u + \gamma f(x_n)) - \overline{V} W_n y_n\| \|x_{n+1} - p\|.\end{aligned} \tag{115}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, from (109) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{W_n y_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty} \|(z_n - v_n) - (p - \widetilde{p})\| = 0. \tag{116}$$

Furthermore, from (47), (54) and (112) it follows that

$$\begin{aligned}\|y_n - p\|^2 &\leq \delta_n \|G z_n - p\|^2 + (1 - \delta_n) \|W_n z_n - p\|^2 \\ &\leq \delta_n \|\widetilde{v}_n - p\|^2 + (1 - \delta_n) \|z_n - p\|^2 \\ &\leq \delta_n \left[\|x_n - p\|^2 - \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|^2 + 2\nu_1 \|A_1 v_n - A_1 \widetilde{p}\| \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|\right] \\ &\quad + (1 - \delta_n) \|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \delta_n \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|^2 + 2\nu_1 \|A_1 v_n - A_1 \widetilde{p}\| \|(v_n - \widetilde{v}_n) + (p - \widetilde{p})\|,\end{aligned} \tag{117}$$

which together with (73), implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\qquad +2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad \le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad \times\left[\|x_n-p\|^2-\delta_n\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2\right. \\
&\qquad\quad \left.+2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|\right] \\
&\qquad +2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad \le \|x_n-p\|^2-(1-\beta_n)\delta_n\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2 \\
&\qquad +2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\| \\
&\qquad +2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.
\end{aligned} \tag{118}$$

So, it follows that

$$\begin{aligned}
&(1-\tilde{d})\tilde{c}\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2 \\
&\quad \le (1-\beta_n)\delta_n\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|^2 \\
&\quad \le \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad +2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\| \\
&\qquad +2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\| \\
&\quad \le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\qquad +2\nu_1\|A_1v_n-A_1\tilde{p}\|\|(v_n-\tilde{v}_n)+(p-\tilde{p})\| \\
&\qquad +2\alpha_n\left\|(u+\gamma f(x_n))-\overline{V}W_ny_n\right\|\|x_{n+1}-p\|.
\end{aligned} \tag{119}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$, from (109) and the boundedness of $\{x_n\}, \{y_n\}, \{W_ny_n\}$ and $\{f(x_n)\}$ we get

$$\lim_{n\to\infty}\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|=0. \tag{120}$$

Note that

$$\|z_n-\tilde{v}_n\|\le\|(z_n-v_n)-(p-\tilde{p})\|+\|(v_n-\tilde{v}_n)+(p-\tilde{p})\|. \tag{121}$$

Hence from (116) and (120) we get

$$\lim_{n\to\infty}\|z_n-\tilde{v}_n\|=\lim_{n\to\infty}\|z_n-Gz_n\|=0, \tag{122}$$

which together with (79), implies that

$$\begin{aligned}
\|z_n-W_nz_n\| &\le \|z_n-Gz_n\|+\|Gz_n-W_nz_n\| \\
&\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{123}$$

Also, observe that

$$\|z_n-Wz_n\|\le\|z_n-W_nz_n\|+\|W_nz_n-Wz_n\|. \tag{124}$$

From (122), Remark 10 and the boundedness of $\{z_n\}$ we immediately obtain

$$\lim_{n\to\infty}\|z_n-Wz_n\|=0. \tag{125}$$

Step 5. We show that

$$\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{V}\right)x^*, x_n-x^*\right\rangle\le 0, \tag{126}$$

where x^* is a solution of (OP2).

Indeed, we note that V is a $\overline{\gamma}$-strongly positive bounded linear operator and $f: H\to H$ is an l-Lipschitzian mapping with $\gamma l<(1+\mu)\overline{\gamma}$. It is clear that

$$\begin{aligned}
&\left\langle\left(\overline{V}x-(u+\gamma f(x))\right)-\left(\overline{V}y-(u+\gamma f(y))\right), x-y\right\rangle \\
&\quad \ge ((1+\mu)\overline{\gamma}-\gamma l)\|x-y\|^2, \quad \forall x,y\in H.
\end{aligned} \tag{127}$$

Hence we deduce that $\overline{V}x-(u+\gamma f(x))$ is $((1+\mu)\overline{\gamma}-\gamma l)$-strongly monotone. In the meantime, it is easy to see that $\overline{V}x-(u+\gamma f(x))$ is $(\|\overline{V}\|+\gamma l)$-Lipschitzian with constant $\|\overline{V}\|+\gamma l>0$. Thus, there exists a unique solution x^* in Ω to the VIP

$$\left\langle u+\left(\gamma f-\overline{V}\right)x^*, u-x^*\right\rangle\le 0, \quad \forall u\in\Omega. \tag{128}$$

Equivalently, $x^*\in\Omega$ solves (OP2) (due to Lemma 20).

First, we observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{V}\right)x^*, x_n-x^*\right\rangle \\
&\quad =\lim_{i\to\infty}\left\langle u+\left(\gamma f-\overline{V}\right)x^*, x_{n_i}-x^*\right\rangle.
\end{aligned} \tag{129}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some w. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup w$. From (91) and (102)–(104), we have that $u_{n_i}\rightharpoonup w, \Lambda^m u_{n_i}\rightharpoonup w$ and $z_{n_i}\rightharpoonup w$, where $m\in\{1,2,\ldots,N\}$. By (122) and (125) we have that $\|Gz_n-z_n\|\to 0$ and $\|Wz_n-z_n\|\to 0$ as $n\to\infty$. Utilizing the similar arguments to those of (55), we know that G is nonexpansive. Hence, by Lemma 12 we obtain $w\in\mathrm{Fix}(G)=\mathrm{SGEP}(G)$ and $w\in\mathrm{Fix}(W)=\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ (due to Lemma 11). Next, we prove that $w\in\cap_{m=1}^{N}I(B_m,R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 19 that R_m+B_m is maximal monotone. Let $(v,g)\in G(R_m+B_m)$, that is, $g-B_mv\in R_mv$. Again, since $\Lambda^m u_n=J_{R_m,\mu_m}(I-\mu_mB_m)\Lambda^{m-1}u_n, n\ge 1, m\in\{1,2,\ldots,N\}$, we have

$$\Lambda^{m-1}u_n-\mu_mB_m\Lambda^{m-1}u_n\in(I+\mu_mR_m)\Lambda^m u_n. \tag{130}$$

that is,

$$\frac{1}{\mu_m}\left(\Lambda^{m-1}u_n-\Lambda^m u_n-\mu_mB_m\Lambda^{m-1}u_n\right)\in R_m\Lambda^m u_n. \tag{131}$$

In terms of the monotonicity of R_m, we get

$$\left\langle v-\Lambda^m u_n, g-B_m v - \frac{1}{\mu_m}\left(\Lambda^{m-1}u_n-\Lambda^m u_n-\mu_m B_m \Lambda^{m-1}u_n\right)\right\rangle \geq 0, \tag{132}$$

and hence

$$\begin{aligned}\langle v-\Lambda^m u_n, g\rangle &\geq \left\langle v-\Lambda^m u_n, B_m v + \frac{1}{\mu_m}\left(\Lambda^{m-1}u_n-\Lambda^m u_n-\mu_m B_m \Lambda^{m-1}u_n\right)\right\rangle \\ &= \left\langle v-\Lambda^m u_n, B_m v - B_m\Lambda^m u_n + B_m\Lambda^m u_n - B_m\Lambda^{m-1}u_n + \frac{1}{\mu_m}\left(\Lambda^{m-1}u_n-\Lambda^m u_n\right)\right\rangle \\ &\geq \left\langle v-\Lambda^m u_n, B_m\Lambda^m u_n - B_m\Lambda^{m-1}u_n\right\rangle + \left\langle v-\Lambda^m u_n, \frac{1}{\mu_m}\left(\Lambda^{m-1}u_n-\Lambda^m u_n\right)\right\rangle.\end{aligned} \tag{133}$$

In particular,

$$\begin{aligned}\left\langle v-\Lambda^m u_{n_i}, g\right\rangle \geq{}& \left\langle v-\Lambda^m u_{n_i}, B_m\Lambda^m u_{n_i} - B_m\Lambda^{m-1}u_{n_i}\right\rangle \\ &+ \left\langle v-\Lambda^m u_{n_i}, \frac{1}{\mu_m}\left(\Lambda^{m-1}u_{n_i}-\Lambda^m u_{n_i}\right)\right\rangle.\end{aligned} \tag{134}$$

Since $\|\Lambda^m u_n - \Lambda^{m-1}u_n\| \to 0$ (due to (102)) and $\|B_m\Lambda^m u_n - B_m\Lambda^{m-1}u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda^m u_{n_i} \rightharpoonup w$ and $\mu_m \in (0,2\eta_m), m \in \{1,2,\ldots,N\}$ that

$$\lim_{i\to\infty}\left\langle v-\Lambda^m u_{n_i}, g\right\rangle = \langle v-w, g\rangle \geq 0. \tag{135}$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$, that is, $w \in I(B_m, R_m)$. Therefore, $w \in \cap_{m=1}^N I(B_m, R_m)$.

Next, we show that $w \in \mathrm{GMEP}(\Theta,\varphi,A)$. In fact, from $u_n = S_{r_n}^{(\Theta,\varphi)}(I-r_nA)x_n$, we know that

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\left\langle K'(u_n)-K'(x_n), y-u_n\right\rangle \geq 0, \quad \forall y \in C. \tag{136}$$

From (H2) it follows that

$$\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\left\langle K'(u_n)-K'(x_n), y-u_n\right\rangle \geq \Theta(y,u_n), \quad \forall y \in C. \tag{137}$$

Replacing n by n_i, we have

$$\varphi(y)-\varphi(u_{n_i})+\langle Ax_{n_i}, y-u_{n_i}\rangle + \left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}}, y-u_{n_i}\right\rangle \geq \Theta(y,u_{n_i}), \quad \forall y \in C. \tag{138}$$

Put $u_t = ty+(1-t)w$ for all $t \in (0,1]$ and $y \in C$. Then, from (138) we have

$$\begin{aligned}\langle u_t-u_{n_i}, Au_t\rangle \geq{}& \langle u_t-u_{n_i}, Au_t\rangle - \varphi(u_t) + \varphi(u_{n_i}) - \langle u_t-u_{n_i}, Ax_{n_i}\rangle \\ &- \left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}}, u_t-u_{n_i}\right\rangle + \Theta(u_t,u_{n_i}) \\ \geq{}& \langle u_t-u_{n_i}, Au_t-Au_{n_i}\rangle + \langle u_t-u_{n_i}, Au_{n_i}-Ax_{n_i}\rangle - \varphi(u_t) + \varphi(u_{n_i}) \\ &- \left\langle \frac{K'(u_{n_i})-K'(x_{n_i})}{r_{n_i}}, u_t-u_{n_i}\right\rangle + \Theta(u_t,u_{n_i}).\end{aligned} \tag{139}$$

Since $\|u_{n_i}-x_{n_i}\| \to 0$ as $i \to \infty$, we deduce from the Lipschitz continuity of A and K' that $\|Au_{n_i}-Ax_{n_i}\| \to 0$ and $\|K'(u_{n_i})-K'(x_{n_i})\| \to 0$ as $i \to \infty$. Further, from the monotonicity of A, we have $\langle u_t-u_{n_i}, Au_t-Au_{n_i}\rangle \geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $(K'(u_{n_i})-K'(x_{n_i}))/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle u_t-w, Au_t\rangle \geq -\varphi(u_t)+\varphi(w)+\Theta(u_t,w), \quad \text{as } i \longrightarrow \infty. \tag{140}$$

From (H1), (H4) and (140) we also have

$$\begin{aligned}0 &= \Theta(u_t,u_t)+\varphi(u_t)-\varphi(u_t)\\ &\le t\Theta(u_t,y)+(1-t)\Theta(u_t,w)+t\varphi(y)\\ &\quad+(1-t)\varphi(w)-\varphi(u_t)\\ &= t\left[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)\right]\\ &\quad+(1-t)\left[\Theta(u_t,w)+\varphi(w)-\varphi(w)-\varphi(u_t)\right]\\ &\le t\left[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)\right]+(1-t)\langle u_t-w,Au_t\rangle\\ &= t\left[\Theta(u_t,y)+\varphi(y)-\varphi(u_t)\right]+(1-t)t\langle y-w,Au_t\rangle,\end{aligned} \tag{141}$$

and hence

$$0\le\Theta(u_t,y)+\varphi(y)-\varphi(u_t)+(1-t)\langle y-w,Au_t\rangle. \tag{142}$$

Letting $t\to 0$, we have, for each $y\in C$,

$$0\le\Theta(w,y)+\varphi(y)-\varphi(w)+\langle Aw,y-w\rangle. \tag{143}$$

This implies that $w\in\mathrm{GMEP}(\Theta,\varphi,A)$. Therefore, $w\in\cap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{GMEP}(\Theta,\varphi,A)\cap\mathrm{SGEP}(G)\cap\cap_{i=1}^{N}I(B_i,R_i):=\Omega$. This shows that $\omega_w(x_n)\subset\Omega$. Consequently, from (128) and (129) we have

$$\begin{aligned}&\limsup_{n\to\infty}\left\langle u+\left(\gamma f-\overline{V}\right)x^*,x_n-x^*\right\rangle\\ &\quad=\left\langle u+\left(\gamma f-\overline{V}\right)x^*,w-x^*\right\rangle\le 0.\end{aligned} \tag{144}$$

Step 6. Finally, we show that $x_n\to x^*\in\Omega$ as $n\to\infty$.

Indeed, from (47) and (54), we have

$$\begin{aligned}\|y_n-x^*\|^2&\le\delta_n\|Gz_n-x^*\|^2+(1-\delta_n)\|W_nz_n-x^*\|^2\\ &\le\|z_n-x^*\|^2\le\|x_n-x^*\|^2.\end{aligned} \tag{145}$$

In terms of Lemma 6 we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\quad=\left\|\alpha_n\left(u+\gamma f(x_n)-\overline{V}x^*\right)+\beta_n(x_n-x^*)\right.\\ &\qquad\left.+\left((1-\beta_n)I-\alpha_n\overline{V}\right)(W_ny_n-x^*)\right\|^2\\ &\quad\le\left\|\beta_n(x_n-x^*)+\left((1-\beta_n)I-\alpha_n\overline{V}\right)(W_ny_n-x^*)\right\|^2\\ &\qquad+2\alpha_n\left\langle u+\gamma f(x_n)-\overline{V}x^*,x_{n+1}-x^*\right\rangle\\ &\quad\le\left[\left\|\left((1-\beta_n)I-\alpha_n\overline{V}\right)(W_ny_n-x^*)\right\|+\beta_n\|x_n-x^*\|\right]^22\\ &\qquad+2\alpha_n\gamma\langle f(x_n)-f(x^*),x_{n+1}-x^*\rangle\\ &\qquad+2\alpha_n\left\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\right\rangle\\ &\quad\le\left[(1-\beta_n-\alpha_n(1+\mu)\overline{\gamma})\|y_n-x^*\|+\beta_n\|x_n-x^*\|\right]^2\\ &\qquad+2\alpha_n\gamma l\|x_n-x^*\|\|x_{n+1}-x^*\|\\ &\qquad+2\alpha_n\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\rangle\\ &\quad\le(1-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-x^*\|^2\\ &\qquad+\alpha_n\gamma l\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right)\\ &\qquad+2\alpha_n\left\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\right\rangle,\end{aligned} \tag{146}$$

which leads to

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &\quad\le\frac{1-2\alpha_n(1+\mu)\overline{\gamma}+\alpha_n^2(1+\mu)^2\overline{\gamma}^2+\alpha_n\gamma l}{1-\alpha_n\gamma l}\\ &\qquad\times\|x_n-x^*\|^2\\ &\qquad+\frac{2\alpha_n}{1-\alpha_n\gamma l}\left\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\right\rangle\\ &\quad=\left[1-\frac{2\left((1+\mu)\overline{\gamma}-\gamma l\right)\alpha_n}{1-\alpha_n\gamma l}\right]\|x_n-x^*\|^2\\ &\qquad+\frac{\left((1+\mu)\alpha_n\overline{\gamma}\right)^2}{1-\alpha_n\gamma l}\|x_n-x^*\|^2\\ &\qquad+\frac{2\alpha_n}{1-\alpha_n\gamma l}\left\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\right\rangle\\ &\quad\le\left[1-\frac{2\left((1+\mu)\overline{\gamma}-\gamma l\right)\alpha_n}{1-\alpha_n\gamma l}\right]\|x_n-x^*\|^2\\ &\qquad+\frac{2\left((1+\mu)\overline{\gamma}-\gamma l\right)\alpha_n}{1-\alpha_n\gamma l}\\ &\qquad\times\left\{\frac{\left(\alpha_n(1+\mu)^2\overline{\gamma}^2\right)M_0}{2\left((1+\mu)\overline{\gamma}-\gamma l\right)}+\frac{1}{(1+\mu)\overline{\gamma}-\gamma l}\right.\\ &\qquad\qquad\left.\times\left\langle u+\gamma f(x^*)-\overline{V}x^*,x_{n+1}-x^*\right\rangle\right\}\\ &\quad=(1-\gamma_n)\|x_n-x^*\|^2+\sigma_n\gamma_n,\end{aligned} \tag{147}$$

where $M_0 = \sup\{\|x_n - x^*\|^2 : n \geq 1\}$, $\gamma_n = 2((1+\mu)\overline{\gamma} - \gamma l)\alpha_n/(1-\alpha_n\gamma l)$ and $\sigma_n = ((\alpha_n(1+\mu)^2\overline{\gamma}^2)M_0/2((1+\mu)\overline{\gamma} - \gamma l)) + (1/((1+\mu)\overline{\gamma} - \gamma l))\langle u + \gamma f(x^*) - \overline{V}x^*, x_{n+1} - x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. Hence by Lemma 15, we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1+\mu)\overline{\gamma}$. Let W_n be the W-mapping defined by (9). Assume that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \cap_{i=1}^{N} I(B_i, R_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0,1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \tag{148}$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$;

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n &= J_{R_N,\mu_N}(I - \mu_N B_N)J_{R_{N-1},\mu_{N-1}} \\ &\quad \times (I - \mu_{N-1}B_{N-1})\cdots J_{R_1,\mu_1}(I - \mu_1 B_1)u_n, \\ y_n &= \delta_n z_n + (1-\delta_n)W_n z_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{149}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP3}$$

where h is the potential function of γf.

Proof. In Theorem 21, putting $\Theta_1 \equiv \Theta_2 \equiv 0$ and $A_1 \equiv A_2 \equiv 0$, we get $Gz_n = z_n$ and $\mathrm{SGEP}(G) = C$. Utilizing Theorem 21 we derive the desired result. □

Corollary 23. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A, A_k : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $i = 1, 2$ and $k = 1, 2$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1+\mu)\overline{\gamma}$. Let W_n be the W-mapping defined by (9). Assume that $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap I(B_2, R_2) \cap I(B_1, R_1) \neq \emptyset$ where G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0,1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \tag{150}$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k = 1, 2$, $\mu_i \in (0, 2\eta_i)$, $i = 1, 2$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$;

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n &= J_{R_2,\mu_2}(I - \mu_2 B_2)J_{R_1,\mu_1}(I - \mu_1 B_1)u_n, \\ y_n &= \delta_n G z_n + (1-\delta_n)W_n z_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{151}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP4}$$

where h is the potential function of γf.

Corollary 24. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R : C \to 2^H$ be a maximal monotone mapping and let $A, A_k : H \to H$ and $B : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η-inverse strongly monotone, respectively, for $k = 1, 2$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\overline{\gamma}$. Let W_n be the W-mapping defined by (9). Assume that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap I(B, R) \neq \emptyset$ where G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \tag{152}$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k = 1, 2$, $\lambda \in (0, 2\eta)$, and $\{r_n\} \subset [0, 2\zeta]$ satisfies $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$;

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ y_n &= \delta_n G J_{R,\lambda}(I - \lambda B)u_n + (1 - \delta_n) W_n J_{R,\lambda}(I - \lambda B)u_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{153}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP5}$$

where h is the potential function of γf.

Corollary 25. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2\}$ and $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1+\mu)\overline{\gamma}$. Let W_n be the W-mapping defined by (9). Assume that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{MEP}(\Theta, \varphi) \cap SGEP(G) \cap \cap_{i=1}^{N} I(B_i, R_i) \neq \emptyset$ where G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \tag{154}$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$, $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, and $\{r_n\}$ is a bounded sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$;

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\left\langle k'(u_n) - k'(x_n), y - u_n\right\rangle \geq 0, \quad \forall y \in C, \\ &z_n = J_{R_N,\mu_N}(I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}} \times (I - \mu_{N-1}B_{N-1}) \cdots J_{R_1,\mu_1}(I - \mu_1 B_1)u_n, \\ &y_n = \delta_n G z_n + (1 - \delta_n) W_n z_n, \\ &x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n(I + \mu V))W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{155}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP6}$$

where h is the potential function of γf.

Proof. In Theorem 21, for all $n \geq 1$, $u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n$ is equivalent to

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\left\langle k'(u_n) - k'(x_n), y - u_n\right\rangle \geq 0, \quad \forall y \in C. \tag{156}$$

Put $A \equiv 0$. Then it follows that

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle k'(u_n) - k'(x_n), y - u_n\rangle \geq 0, \quad \forall y \in C. \tag{157}$$

Observe that for all $\zeta \in (0, \infty)$

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in H. \tag{158}$$

So, whenever $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\zeta$ for some $\zeta \in (0, \infty)$, we obtain the desired result by using Theorem 21. □

Let $T : H \to H$ be a κ-strictly pseudocontractive mapping. For recent convergence result for strictly pseudocontractive mappings, we refer to [16]. Putting $A = I - T$, we know that for all $x, y \in H$

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \kappa\|Ax - Ay\|^2. \tag{159}$$

Note that

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle Ax - Ay, x - y\rangle. \tag{160}$$

Hence we have for all $x, y \in H$

$$\langle Ax - Ay, x - y\rangle \geq \frac{1-\kappa}{2}\|Ax - Ay\|^2. \tag{161}$$

Consequently, if $T : H \to H$ is a κ-strictly pseudocontractive mapping, then the mapping $A = I - T$ is $(1-\kappa)/2$-inverse strongly monotone.

Corollary 26. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $T, A_k : H \to H$ and $B_i : C \to H$ be κ-strictly pseudocontractive, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2\}$ and $i \in \{1, 2, \ldots, N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\overline{\gamma}$. Let W_n be the W-mapping defined by* (9). *Assume that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \cap_{i=1}^{N} I(B_i, R_i) \neq \emptyset$ where $A = I - T$ and G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0; \tag{162}$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$, $\mu_i \in (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, and $\{r_n\} \subset [0, 1 - \kappa]$ satisfies

$$0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 1 - \kappa; \tag{163}$$

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}((1 - r_n)x_n + r_n T x_n), \\ z_n &= J_{R_N,\mu_N}(I - \mu_N B_N) J_{R_{N-1},\mu_{N-1}} \times (I - \mu_{N-1}B_{N-1}) \cdots J_{R_1,\mu_1}(I - \mu_1 B_1) u_n, \\ y_n &= \delta_n G z_n + (1 - \delta_n) W_n z_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n(I + \mu V)) W_n y_n, \quad \forall n \geq 1, \end{aligned} \tag{164}$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP7}$$

where h is the potential function of γf.

Proof. Since T is a κ-strictly pseudocontractive mapping, the mapping $A = I - T$ is $(1 - \kappa)/2$-inverse strongly monotone. In this case, put $\zeta = (1 - \kappa)/2$. Moreover, we obtain that

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_n A) x_n \\ &= S_{r_n}^{(\Theta,\varphi)}(x_n - r_n(I - T) x_n) \\ &= S_{r_n}^{(\Theta,\varphi)}((1 - r_n) x_n + r_n T x_n). \end{aligned} \tag{165}$$

So, from Theorem 21, we obtain the desired result. □

Corollary 27. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let $\Theta, \Theta_1, \Theta_2$ be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A, A_k : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2\}$ and $i \in \{1, 2, \ldots, N\}$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\overline{\gamma}$. Assume that $\Omega := \mathrm{GMEP}(\Theta, \varphi, A) \cap SGEP(G) \cap \cap_{i=1}^{N} I(B_i, R_i) \neq \emptyset$ where G is defined as in Proposition CY. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be three sequences in $[0, 1]$. Assume that:*

(i) $K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$;

(ii) for each $x \in H$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\left\langle K'(y) - K'(x), z_x - y\right\rangle < 0; \quad (166)$$

(iii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $0 < \liminf_{n\to\infty}\delta_n \le \limsup_{n\to\infty}\delta_n < 1$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k \in \{1,2\}$, $\mu_i \in (0, 2\eta_i)$, $i \in \{1,2,\ldots,N\}$, *and* $\{r_n\} \subset [0, 2\zeta]$ *satisfies*

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta; \quad (167)$$

(v) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in H$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} u_n &= S_{r_n}^{(\Theta,\varphi)}(I - r_nA)x_n, \\ z_n &= J_{R_N,\mu_N}(I - \mu_NB_N)J_{R_{N-1},\mu_{N-1}} \\ &\quad\times(I - \mu_{N-1}B_{N-1})\cdots J_{R_1,\mu_1}(I - \mu_1B_1)u_n, \\ y_n &= \delta_nGz_n + (1 - \delta_n)z_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_nx_n \\ &\quad+ ((1 - \beta_n)I - \alpha_n(I + \mu V))y_n, \quad \forall n \ge 1, \end{aligned} \quad (168)$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \quad \text{(OP8)}$$

where h is the potential function of γf.

Proof. Put $T_nx = x$ for all integers $n \ge 1$ and all $x \in H$. Then, the desired result follows from Theorem 21. □

Corollary 28. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ_1 and Θ_2 be two bifunctions from $C\times C$ to* $\mathbf{R}$ *satisfying (H1)–(H4). Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1,2\}$ and $i \in \{1,2,\ldots,N\}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Let V be a $\overline{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1+\mu)\overline{\gamma}$. Let W_n be the W-mapping defined by* (9). *Assume that* $\Omega := \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{SGEP}(G) \cap \cap_{i=1}^{N} I(B_i, R_i) \ne \emptyset$ *where G is defined as in Proposition CY. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be three sequences in* $[0,1]$. *Assume that:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $0 < \liminf_{n\to\infty}\delta_n \le \limsup_{n\to\infty}\delta_n < 1$;

(ii) $\nu_k \in (0, 2\zeta_k)$, $k \in \{1,2\}$, $\mu_i \in (0, 2\eta_i)$, $i \in \{1,2,\ldots,N\}$, *and* $\{r_n\} \subset [0, 2\zeta]$ *satisfies*

$$0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta; \quad (169)$$

(iii) $\lim_{n\to\infty}(|\delta_{n+1} - \delta_n| + |r_{n+1} - r_n|) = 0$.

Given $x_1 \in C$ arbitrarily, then the sequence $\{x_n\}$ generated iteratively by

$$\begin{aligned} z_n &= J_{R_N,\mu_N}(I - \mu_NB_N)J_{R_{N-1},\mu_{N-1}} \\ &\quad\times(I - \mu_{N-1}B_{N-1})\cdots J_{R_1,\mu_1}(I - \mu_1B_1)x_n, \\ y_n &= \delta_nGz_n + (1 - \delta_n)W_nz_n, \\ x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_nx_n \\ &\quad+ ((1 - \beta_n)I - \alpha_n(I + \mu V))W_ny_n, \quad \forall n \ge 1, \end{aligned} \quad (170)$$

converges strongly to $x^* \in \Omega$ which solves the following optimization problem provided $S_r^{(\Theta,\varphi)}$ is firmly nonexpansive:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \quad \text{(OP9)}$$

where h is the potential function of γf.

Proof. Put $\Theta(x,y) = 0$, $\varphi(x) = 0$ for all $x, y \in C$, $Ax = 0$ for all $x \in H$ and $r_n = 1$. Take $K(x) = (1/2)\|x\|^2$ for all $x \in H$. Then we get $u_n = x_n$ in Theorem 21 and the conclusion follows. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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