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Research Article

Exact Solutions of Coupled Sine-Gordon Equations Using the Simplest Equation Method

Yun-Mei Zhao

Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, China

Correspondence should be addressed to Yun-Mei Zhao; zhaoyunmei2000@126.com

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The simplest equation method has been used for finding the exact solutions of coupled sine-Gordon equations. Such equations have some useful applications in physics and biology, so finding their exact solutions is of great importance.

1. Introduction

Recently, the coupled sine-Gordon equations

$$\begin{gathered} u_{tt} - u_{xx} = -\delta^2 \sin(u-\omega) \\ \omega_{tt} - \alpha^2 \omega_{xx} = \sin(u-\omega) \\ \alpha > 0, \ \delta > 0, \end{gathered} \tag{1}$$

have been introduced by Khusnutdinova and Pelinovsky [1]. The coupled sine-Gordon equations generalize the Frenkel-Kontorova dislocation model [2, 3]. System (1) with $\alpha = 1$ was also proposed to describe the open states in DNA model [4].

Very recently, system (1) was studied by many researchers and various methods. It was studied by Salas, using a special rational exponential ansatz [5]. Zhao et al. obtained some new solutions including Jacobi elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions by the Jacobi elliptic function expansion method [6], the hyperbolic auxiliary function method [7], and the symbolic computation method [8].

In the past four decades, the study of nonlinear partial differential equations (NLEEs) modelling physical phenomena has become an important research topic. Seeking exact solutions of NLEEs has long been one of the central themes of perpetual interest in mathematics and physics. With the development of symbolic computation packages like Maple and Mathematica, many powerful methods for finding exact solutions have been proposed, such as the homogeneous balance method [9, 10], the auxiliary equation method [11, 12], the Exp-function method [13, 14], the Darboux transformation [15, 16], the tanh-function method [17], and the (G'/G)-expansion method [18, 19].

The simplest equation method is a very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations. It has been developed by Kudryashov [20, 21] and used successfully by many authors for finding exact solutions of ODEs in mathematical physics [22, 23].

In this paper, we will apply the simplest equation method [24] to obtain some new and more general explicit exact solutions of the coupled sine-Gordon equations.

2. The Simplest Equation Method

In this section, we will give the detailed description of the simplest equation method.

Step 1. Suppose that we have a nonlinear partial differential equation (PDE) for $u(x,t)$ in the form

$$N(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{2}$$

where N is a polynomial in its arguments.

Step 2. By taking $u(x,t) = u(\xi)$, $\xi = x - ct$, we look for traveling wave solutions of (2) and transform it to the ordinary differential equation (ODE)

$$N\left(u, -cu', u', c^2u'', -cu'', u'', \ldots\right) = 0. \quad (3)$$

Step 3. Suppose that the solution u of (3) can be expressed as a finite series in the form

$$u = \sum_{i=0}^{n} A_i (H(\xi))^i, \quad (4)$$

where $H(\xi)$ satisfies the Bernoulli or Riccati equation, n is a positive integer that can be determined by balancing procedure [21], and A_i $(i = 0, 1, 2, \ldots, n)$ are parameters to be determined.

The Bernoulli equation we consider in this paper is

$$H'(\xi) = aH(\xi) + bH^2(\xi), \quad (5)$$

where a and b are constants. Its solutions can be written as

$$H(\xi) = \frac{-aD_1}{b\left(D_1 + \cosh(a(\xi + C)) - \sinh(a(\xi + C))\right)},$$
$$H(\xi) = \frac{-a\left(\cosh(a(\xi + C)) + \sinh(a(\xi + C))\right)}{b\left(D_2 + \cosh(a(\xi + C)) + \sinh(a(\xi + C))\right)}, \quad (6)$$

where D_1, D_2, and C are constants.

For the Riccati equation

$$H'(\xi) = aH^2(\xi) + bH(\xi) + s, \quad (7)$$

where a, b, and s are constants, we will use the solutions

$$H(\xi) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{\theta}{2}(\xi + C)\right],$$
$$H(\xi) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{\theta}{2}\xi\right) + \frac{\operatorname{sech}(\theta\xi/2)}{C\cosh(\theta\xi/2) - (2a/\theta)\sinh(\theta\xi/2)}, \quad (8)$$

where $\theta^2 = b^2 - 4as$.

Step 4. Substituting (4) into (3) with (5) (or (7)), then the left hand side of (3) is converted into a polynomial in $H(\xi)$, and equating each coefficient of the polynomial to zero yields a set of algebraic equations for A_i, a, b $(i = 0, 1, 2, \ldots, n)$. Solving the algebraic equations by symbolic computation, we can determine those parameters explicitly.

Step 5. Assuming that the constants A_i, a, b $(i = 0, 1, 2, \ldots, n)$ can be obtained in Step 4 and substituting the results into (4), then we obtain the exact traveling wave solutions for (2).

Remark 1. In (5), when $a = A$ and $b = -1$ we obtain the Bernoulli equation

$$H'(\xi) = AH(\xi) - H^2(\xi). \quad (9)$$

Equation (9) admits the following exact solutions:

$$H(\xi) = \frac{A}{2}\left(1 + \tanh\left(\frac{A}{2}(\xi + C)\right)\right), \quad (10)$$

when $A > 0$, and

$$H(\xi) = \frac{A}{2}\left(1 - \tanh\left(\frac{A}{2}(\xi + C)\right)\right), \quad (11)$$

when $A < 0$.

3. Exact Solutions of the Coupled Sine-Gordon Equations

In this section, we solve the coupled sine-Gordon equations by the simplest equation method.

In order to solve (1), we introduce a new unknown function $\varphi = \varphi(x,t)$ by the formula

$$\varphi(x,t) = u(x,t) - \omega(x,t), \quad (12)$$

so that $\omega(x,t) = u(x,t) - \varphi(x,t)$. According to (1), we have

$$u_{tt} - u_{xx} = -\delta^2 \sin\varphi,$$
$$u_{tt} - \varphi_{tt} - \alpha^2\left(u_{xx} - \varphi_{xx}\right) = \sin\varphi. \quad (13)$$

Let

$$\xi = \mu(x - ct),$$
$$\varphi = 2\arctan(v(\xi)) = 2\arctan\left(v\left(\mu(x - ct)\right)\right), \quad (14)$$

then

$$\sin\varphi = \sin(2\arctan v(\xi)) = \frac{2v(\xi)}{1 + v^2(\xi)}. \quad (15)$$

Substitution (14)–(15) into (13), we get the following coupled system of nonlinear differential equations:

$$\mu^2\left(c^2 - 1\right)\left(1 + v^2(\xi)\right)u''(\xi) + 2\delta^2 v(\xi) = 0,$$
$$2\mu^2\left(c^2 - \alpha^2\right)v^2(\xi)\left(u''(\xi) - v''(\xi)\right) + \mu^2\left(c^2 - \alpha^2\right)\left(u''(\xi) - 2v''(\xi)\right) + \mu^2\left(c^2 - \alpha^2\right)v^4(\xi)u''(\xi) + v(\xi)\left(4\mu^2\left(c^2 - \alpha^2\right)v'^2(\xi) - 2\right) - 2v^3(\xi) = 0. \quad (16)$$

According to the first equation of (16), we have

$$u''(\xi) = -\frac{2\delta^2 v(\xi)}{\mu^2\left(c^2 - 1\right)\left(1 + v^2(\xi)\right)}. \quad (17)$$

Substituting (17) into the second equation of (16), we obtain a single nonlinear second-order differential equation in the unknown $v = v(\xi)$:

$$\left(c^2 - 1\right)\mu^2\left(c^2 - \alpha^2\right)\left(v^2(\xi) + 1\right)v''(\xi) - 2\mu^2\left(c^2 - 1\right)\left(c^2 - \alpha^2\right)v(\xi)v'^2(\xi) + \left(v^3(\xi) + v(\xi)\right)\left(\delta^2\left(c^2 - \alpha^2\right) + c^2 - 1\right) = 0. \quad (18)$$

As we can see, it suffices to find analytic solutions to (18). Observe that if $v(\xi)$ is a solution of (18), then $-v(\xi)$ is also a solution.

3.1. Solutions of (18) *Using the Bernoulli Equation as the Simplest Equation.* The balancing procedure yields $n = 1$. Thus, the solution of (18) is of the form

$$v(\xi) = A_0 + A_1 H(\xi). \quad (19)$$

Substituting (19) into (18) and making use of the Bernoulli equation (5) and then equating the coefficients of the functions $H^i(\xi)$ to zero, we obtain an algebraic system of equations in terms of A_i $(i = 0, 1)$, a, and b. Solving this system of algebraic equations, with the aid of Maple, one possible set of values of A_i $(i = 0, 1)$, a, and b is

$$A_0 = \pm i, \qquad A_1 = \pm\frac{2bi}{a}, \qquad \delta = \pm\sqrt{\frac{1-c^2}{c^2-\alpha^2}}. \quad (20)$$

Therefore, using solutions (6) of (5), ansatz (19), we obtain the following exact solution of (18):

$$v_1(\xi) = \pm i \mp \frac{2iD_1}{D_1 + \cosh(a(\xi + C)) - \sinh(a(\xi + C))}, \quad (21)$$

$$v_2(\xi) = \pm i \mp \frac{2i(\cosh(a(\xi + C)) + \sinh(a(\xi + C)))}{D_2 + \cosh(a(\xi + C)) + \sinh(a(\xi + C))}. \quad (22)$$

Substituting (21) into (17) with (12), the exact traveling wave solution to (1) can be written as

$$\begin{aligned}
u_1(\xi) &= \mp\left(i\left(\left(D_1^2-1\right)\cosh(a(\xi+C))+\left(D_1^2+1\right)\sinh(a(\xi+C))\right)\right) \\
&\quad \times\left(2a^2\left(c^2-\alpha^2\right)\mu^2 D_1\right)^{-1} + C_1\xi + C_2, \\
\omega_1(\xi) &= \mp\left(i\left(\left(D_1^2-1\right)\cosh(a(\xi+C)) + \left(D_1^2+1\right)\sinh(a(\xi+C))\right)\right) \\
&\quad \times\left(2a^2\left(c^2-\alpha^2\right)\mu^2 D_1\right)^{-1} + C_1\xi + C_2 - \varphi_1, \\
\varphi_1 &= 2\arctan\left(\pm i \mp \frac{2iD_1}{D_1 + \cosh(a(\xi + C)) - \sinh(a(\xi + C))}\right),
\end{aligned} \quad (23)$$

where $\xi = \mu(x - ct)$, $\delta = \pm\sqrt{(1-c^2)/(c^2-\alpha^2)}$, and D_1, a, C, C_1, C_2 are arbitrary parameters.

Now, to obtain some special cases of the above solutions, we set $D_1 = i$, $a = 1$, $C = 0$, "±" take "+"; we have

$$\begin{aligned}
u_2(\xi) &= \frac{\delta^2\cosh(\xi)}{\mu^2(c^2-1)} + C_1\xi + C_2, \\
\omega_2(\xi) &= \frac{\delta^2\cosh(\xi)}{\mu^2(c^2-1)} + C_1\xi + C_2 - \varphi_2, \\
\varphi_2 &= 2\arctan\left(-i - \frac{2}{i + \cosh(\xi) - \sinh(\xi)}\right) \\
&= 2\arctan\left(\frac{-1 + i\sinh(\xi)}{\cosh(\xi)}\right),
\end{aligned} \quad (24)$$

where $\xi = \mu(x - ct)$, $c = \pm\sqrt{(1+\delta^2\alpha^2)/(1+\delta^2)}$.

The equations in (24) are the same as those in (36) of [7].

If we set $D_1 = 1$, $a = 1$, $C = 0$, we have

$$\begin{aligned}
u_3(\xi) &= \mp\frac{i\delta^2\sinh(\xi)}{\mu^2(c^2-1)} + C_1\xi + C_2, \\
\omega_3(\xi) &= \mp\frac{i\delta^2\sinh(\xi)}{\mu^2(c^2-1)} + C_1\xi + C_2 - \varphi_3, \\
\varphi_3 &= \mp 2\arctan\left(i - \frac{2i}{1 + \cosh(\xi) - \sinh(\xi)}\right) \\
&= \pm 2i\,\mathrm{arctanh}\left(\frac{\sinh(\xi)}{1 + \cosh(\xi)}\right),
\end{aligned} \quad (25)$$

where $\xi = \mu(x - ct)$, $c = \pm\sqrt{(1+\delta^2\alpha^2)/(1+\delta^2)}$.

The equations in (25) are the same as those in (32) of [7].

Substituting (22) into (17) with (12), the exact traveling wave solution to (1) can be written as

$$\begin{aligned}
u_4(\xi) &= \pm\left(i\left(\left(D_2^2-1\right)\cosh(a(\xi+C)) - \left(D_2^2+1\right)\sinh(a(\xi+C))\right)\right) \\
&\quad \times\left(2a^2\left(c^2-\alpha^2\right)\mu^2 D_2\right)^{-1} + C_1\xi + C_2, \\
\omega_4(\xi) &= \pm\left(i\left(\left(D_2^2-1\right)\cosh(a(\xi+C)) - \left(D_2^2+1\right)\sinh(a(\xi+C))\right)\right) \\
&\quad \times\left(2a^2\left(c^2-\alpha^2\right)\mu^2 D_2\right)^{-1} + C_1\xi + C_2 - \varphi_4, \\
\varphi_4 &= 2\arctan\left(\pm i \mp \frac{2i(\cosh(a(\xi + C)) + \sinh(a(\xi + C)))}{D_2 + \cosh(a(\xi + C)) + \sinh(a(\xi + C))}\right),
\end{aligned} \quad (26)$$

where $\xi = \mu(x - ct)$, $\delta = \pm\sqrt{(1-c^2)/(c^2-\alpha^2)}$.

Substituting (19) along with (9) into (18) and setting all the coefficients of powers $H^i(\xi)$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \pm i, \qquad A_1 = \mp\frac{2i}{A}, \qquad \delta = \pm\sqrt{\frac{1-c^2}{c^2-\alpha^2}}. \tag{27}$$

Therefore, using solutions (10) and (11) of (9), ansatz (19), we obtain the following exact solution of (18):

$$v_3(\xi) = \mp i \tanh\left(\frac{A}{2}(\xi + C)\right). \tag{28}$$

Then the exact solution to (1) can be written as

$$\begin{aligned}
u_5(\xi) &= \mp\frac{i\sinh(A(\xi + C))}{A^2\mu^2(c^2-\alpha^2)} + C_1\xi + C_2,\\
\omega_5(\xi) &= \mp\frac{i\sinh(A(\xi + C))}{A^2\mu^2(c^2-\alpha^2)} + C_1\xi + C_2 - \varphi_5,\\
\varphi_5 &= 2\arctan\left(\mp i\tanh\left(\frac{A}{2}(\xi + C)\right)\right),
\end{aligned} \tag{29}$$

where $\xi = \mu(x - ct)$, $\delta = \pm\sqrt{(1-c^2)/(c^2-\alpha^2)}$, and A, C, C_1, C_2 are arbitrary parameters.

Now, to obtain some special cases of the above solutions, we set $A = 2$, $C = 0$; we have

$$\begin{aligned}
u_6(\xi) &= \mp\frac{i\delta^2\sinh(2\xi)}{4\mu^2(1-c^2)} + C_1\xi + C_2,\\
\omega_6(\xi) &= \mp\frac{i\delta^2\sinh(2\xi)}{4\mu^2(1-c^2)} + C_1\xi + C_2 - \varphi_6,\\
\varphi_6 &= 2\arctan(\mp i\tanh(\xi)) = \pm 2i\,\mathrm{arctanh}(\tanh(\xi)),
\end{aligned} \tag{30}$$

where $\xi = \mu(x - ct)$, $c = \pm\sqrt{(1+\delta^2\alpha^2)/(1+\delta^2)}$.

The equations in (30) are the same as those in (31) of [7].

3.2. Solutions of (18) *Using Riccati Equation as the Simplest Equation.* The balancing procedure yields $n = 1$. Thus, the solution of (18) is of the form

$$v(\xi) = B_0 + B_1 H(\xi). \tag{31}$$

Substituting (31) into (18) and making use of the Riccati Equation (7) and then equating the coefficients of the functions $H^i(\xi)$ to zero, we obtain an algebraic system of equations in terms of B_i $(i = 0, 1)$, a, b, and s. Solving this system of algebraic equations, with the aid of Maple, one possible set of values of B_i $(i = 0, 1)$, a, b, and s is

$$B_1 = \frac{2aB_0}{b}, \qquad s = \frac{b^2(1+B_0^2)}{4aB_0^2}, \qquad \delta = \pm\sqrt{\frac{1-c^2}{c^2-\alpha^2}}. \tag{32}$$

Therefore, using solutions (8) of (7), ansatz (31), we obtain the following exact solution of (18):

$$v_1(\xi) = \tan\left(\frac{b(\xi + C)}{2B_0}\right), \tag{33}$$

$$v_2(\xi) = -\frac{Cb\sin((b/2B_0)\xi) + 2B_0 a\cos((b/2B_0)\xi)}{-Cb\cos((b/2B_0)\xi) + 2B_0 a\sin((b/2B_0)\xi)}. \tag{34}$$

Substituting (33) into (17) with (12), the exact traveling wave solution to (1) can be written as

$$\begin{aligned}
u_1(\xi) &= -\frac{B_0^2\sin((b/B_0)(\xi + C))}{(c^2-\alpha^2)\mu^2b^2} + C_1\xi + C_2,\\
\omega_1(\xi) &= -\frac{B_0^2\sin((b/B_0)(\xi + C))}{(c^2-\alpha^2)\mu^2b^2} + C_1\xi + C_2 - \varphi_1,\\
\varphi_1 &= 2\arctan\left(\tan\left(\frac{b}{2B_0}(\xi + C)\right)\right),
\end{aligned} \tag{35}$$

where $\xi = \mu(x - ct)$, $\delta = \pm\sqrt{(1-c^2)/(c^2-\alpha^2)}$.

Substituting (34) into (17) with (12), the exact traveling wave solution to (1) can be written as

$$\begin{aligned}
u_2(\xi) &= -\left(C^2B_0^2\sin\left(\frac{b}{B_0}\xi\right) + \frac{4aB_0^3}{b^2}\left(bC\cos\left(\frac{b}{B_0}\xi\right) - B_0 a\sin\left(\frac{b}{B_0}\xi\right)\right)\right)\\
&\quad\times\left((c^2-\alpha^2)\mu^2\left(C^2b^2 + 4B_0^2a^2\right)\right)^{-1} + C_1\xi + C_2,\\
\omega_2(\xi) &= -\left(C^2B_0^2\sin\left(\frac{b}{B_0}\xi\right) + \frac{4aB_0^3}{b^2}\left(bC\cos\left(\frac{b}{B_0}\xi\right) - B_0 a\sin\left(\frac{b}{B_0}\xi\right)\right)\right)\\
&\quad\times\left((c^2-\alpha^2)\mu^2\left(C^2b^2 + 4B_0^2a^2\right)\right)^{-1} + C_1\xi + C_2 - \varphi_2,\\
\varphi_2 &= 2\arctan\left(\frac{Cb\sin((b/2B_0)\xi) + 2B_0 a\cos((b/2B_0)\xi)}{-Cb\cos((b/2B_0)\xi) + 2B_0 a\sin((b/2B_0)\xi)}\right),
\end{aligned} \tag{36}$$

where $\xi = \mu(x - ct)$, $\delta = \pm\sqrt{(1-c^2)/(c^2-\alpha^2)}$.

Remark 2. Compared with [7], the exact solutions of this paper are more general, such that when $D_1 = i$, $a = 1$, and $C = 0$ in (23), the solutions become as those in (36) of [7]. When $D_1 = 1$, $a = 1$, and $C = 0$ in (23), the solutions become as those in (32) of [7]. When $A = 2$ and $C = 0$ of (29), the solutions become as those in (31) of [7]. There are many such examples; thus, it is easy to see that the study of [7] is a special case in this paper. So the exact solutions of this paper are more general, and all the solutions are new solutions which are not reported in the relevant literature reported.

4. Conclusions

In this paper, we obtained some exact solutions of the coupled sine-Gordon equations by using the simplest equation method. The Bernoulli equation and Riccati equation have been used as the simplest equation. The solutions obtained may be significant and important for the explanation of some practical physical problems. The method may also be applied to other nonlinear partial differential equations. Also, we have verified that the solutions that we have found are indeed solutions to the original nonlinear evolution equations.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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