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Research Article

Expansions of Functions Based on Rational Orthogonal Basis with Nonnegative Instantaneous Frequencies

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We consider in this paper expansions of functions based on the rational orthogonal basis for the space of square integrable functions. The basis functions have nonnegative instantaneous frequencies so that the expansions make physical sense. We discuss the almost everywhere convergence of the expansions and develop a fast algorithm for computing the coefficients arising in the expansions by combining the characterization of the coefficients with the fast Fourier transform.

1. Introduction

A common approach for understanding a function is to expand it as a sum of basic functions. These basic functions should be chosen for different practical purpose. In the time-frequency analysis, the instantaneous frequency is one of the most important information for understanding a given signal. Then the basic functions used to express a complicated signal are expected to have nonnegative instantaneous frequencies so that the expansion of a signal makes physical sense. In this point of view, recent works have contributed to characterize and construct the basic functions [1–7] and establish numerical algorithms for decomposing a signal based on the basic functions [8–10].

Motivated by the analytic signal approach [11, 12], the authors in [7] constructed rational orthogonal basis for $L^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$, with each element of the basis enjoying physically meaningful instantaneous frequency. Specifically, let $\mathbb{N}$ denote the set of all positive integers. The sequence ψ_n, $|n| \in \mathbb{N}$, forms an orthogonal basis for $L^2(\mathbb{R})$, where for each $n \in \mathbb{N}$

$$\begin{aligned} \psi_{2n-1}(x) &:= -\frac{x}{1+x^2}\left(\frac{x-i}{x+i}\right)^{2n-1}, \quad x \in \mathbb{R}, \\ \psi_{2n}(x) &:= -\frac{1}{1+x^2}\left(\frac{x-i}{x+i}\right)^{2n-1}, \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

and for each $-n \in \mathbb{N}$, $\psi_n := \overline{\psi_{-n}}$. As pointed out in [7], the basis functions all have nonnegative instantaneous frequencies in the sense of the analytic signal. Based upon this basis, we focus in this paper on decomposing a function in $L^2(\mathbb{R})$ into a sum of the basis functions ψ_n, $|n| \in \mathbb{N}$. For simplicity, we first transform the basis into an orthonormal one by normalization. A direct computation shows that there holds, for each $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}} |\psi_{2n-1}(x)|^2 dx &= \int_{\mathbb{R}} \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{2}, \\ \int_{\mathbb{R}} |\psi_{2n}(x)|^2 dx &= \int_{\mathbb{R}} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{2}. \end{aligned} \tag{2}$$

Hence, we obtain the resulting orthonormal basis $\{\phi_n : |n| \in \mathbb{N}\}$ for $L^2(\mathbb{R})$, where

$$\phi_n := \sqrt{\frac{2}{\pi}}\psi_n, \quad |n| \in \mathbb{N}. \tag{3}$$

In fact, due to the completeness of $\{\phi_n, |n| \in \mathbb{N}\}$ in $L^2(\mathbb{R})$, each $f \in L^2(\mathbb{R})$ can be represented as the infinite series

$$\sum_{|n|\in\mathbb{N}} d_n(f)\,\phi_n, \tag{4}$$

where the coefficients are defined by

$$d_n(f) := \int_{\mathbb{R}} f(x)\overline{\phi_n(x)}dx, \quad |n| \in \mathbb{N}. \tag{5}$$

However, for practical purpose, one often needs to decompose a function into a finite sum of basic functions. Thus, we have to approximate a function $f \in L^2(\mathbb{R})$ by the partial sum of the series (4) defined by

$$\mathcal{S}_N(f)(x) := \sum_{|n|=1}^{N} d_n(f)\phi_n(x), \quad x \in \mathbb{R}. \tag{6}$$

From this point of view, we have to deal with two issues. The first one is to study the convergence of the partial sum (6) as N tends to infinity. Since it is known that the sequence of the partial sums converges to f in the sense of norm, we will aim in this paper at discussing the almost everywhere convergence of them. The second one is to develop a fast algorithm for computing the partial sum so that this type of decomposition can be carried out in practical applications.

The organization of this paper is as follows. In Section 2, we establish the relation between the coefficients in (5) and the classical Fourier coefficients and then discuss the almost everywhere convergence of the sequence of the partial sums in (6). We develop in Section 3 an algorithm for computing the coefficients in the expansion (6) by employing the relation between the coefficients and the classical Fourier coefficients and the fast Fourier transform. To show that the proposed algorithm is fast, we also give an estimate of its computational complexity.

2. Almost Everywhere Convergence

We discuss in this section the almost everywhere convergence of the sequence $\mathcal{S}_N(f)$, $N \in \mathbb{N}$. This will be done by making use of some classical results about the Fourier series. To this end, with respect to any $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, a periodic function is introduced as follows.

Let $\sigma : \mathbb{R} \to (0, 2\pi)$ be defined by

$$\sigma(x) = 2\arctan(x) + \pi, \quad x \in \mathbb{R}. \tag{7}$$

Then $\sigma'(x) = 2/(x^2+1) > 0$, $x \in \mathbb{R}$ means σ is a one-to-one mapping from $\mathbb{R}$ onto $(0, 2\pi)$. Then for any $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, we set

$$f_{\mathbf{p}}(t) := \frac{f\left(\sigma^{-1}(t)\right)}{\left(1-e^{it}\right)^2}, \qquad f_{\mathbf{q}}(t) := f\left(\sigma^{-1}(t)\right), \qquad t \in (0, 2\pi), \tag{8}$$

where $\sigma^{-1}(t) = \tan((t-\pi)/2) = -i(e^{it}+1)/(e^{it}-1)$. The following lemma gives a desired properties of the functions $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$.

Lemma 1. *Let $p \in [1, +\infty)$. If $f \in L^p(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} |f(x)|^p (1+x^2)^{p-1} dx < \infty, \tag{9}$$

then $f_{\mathbf{p}}, f_{\mathbf{q}} \in L^p(0, 2\pi)$.

Proof. Taking the change of variables $x = \sigma^{-1}(t)$, we have

$$\begin{aligned} \int_0^{2\pi} \left|f_{\mathbf{p}}(t)\right|^p dt &= \int_0^{2\pi} \left|\frac{f(\sigma^{-1}(t))}{(1-e^{it})^2}\right|^p dt \\ &= \frac{1}{4^{p-1}} \int_{\mathbb{R}} \left|f(x)\right|^p (1+x^2)^{p-1} dx. \end{aligned} \tag{10}$$

Similarly, for the function $f_{\mathbf{q}}$, we get

$$\begin{aligned} \int_0^{2\pi} |f_{\mathbf{q}}(t)|^p dt &= 2\int_{\mathbb{R}} |f(x)|^p \left(1+x^2\right)^{-1} dx. \\ &< 2\int_{\mathbb{R}} |f(x)|^p (1+x^2)^{p-1} dx. \end{aligned} \tag{11}$$

Together with the assumption (9), the above equalities lead to the desired conclusions. □

We need one more lemma to represent the coefficients d_n, $|n| \in \mathbb{N}$, in terms of the classical Fourier coefficients of $f_{\mathbf{p}}$. For any $g \in L^1(0, 2\pi)$, the Fourier coefficients of g are defined by

$$c_n(g) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(t) e^{-int} dt, \quad n \in \mathbb{Z}. \tag{12}$$

Lemma 2. *Suppose that $f \in L^2(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} |f(x)|^2 \left(1+x^2\right) dx < \infty, \tag{13}$$

and $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$ are defined in (8). *Then the coefficients $d_n(f)$, $|n| \in \mathbb{N}$, defined in* (5) *are determined by*

$$\begin{aligned} &d_n(f) \\ &= \begin{cases} -ic_n\left(f_{\mathbf{p}}\right) + ic_{n-2}\left(f_{\mathbf{p}}\right), & n = \pm(2k-1),\ k \in \mathbb{N}, \\ -c_{n-1}\left(f_{\mathbf{p}}\right) + 2c_{n-2}\left(f_{\mathbf{p}}\right) - c_{n-3}\left(f_{\mathbf{p}}\right), & \\ \qquad n = 2k,\ k \in \mathbb{N}, & \\ -c_{n+1}\left(f_{\mathbf{p}}\right) + 2c_{n}\left(f_{\mathbf{p}}\right) - c_{n-1}\left(f_{\mathbf{p}}\right), & \\ \qquad n = -2k,\ k \in \mathbb{N}, & \end{cases} \end{aligned} \tag{14}$$

or

$$\begin{aligned} &d_n(f) \\ &= \begin{cases} -ic_n\left(f_{\mathbf{p}}\right) + ic_{n-2}\left(f_{\mathbf{p}}\right), & n = \pm(2k-1),\ k \in \mathbb{N}, \\ -c_{n-1}\left(f_{\mathbf{q}}\right), & n = 2k,\ k \in \mathbb{N}, \\ -c_{n+1}\left(f_{\mathbf{q}}\right), & n = -2k,\ k \in \mathbb{N}. \end{cases} \end{aligned} \tag{15}$$

Proof. For simplicity, we only give the proof for the cases of $n = 2k-1$ and $n = 2k$, $k \in \mathbb{N}$. The proof for the other cases can be similarly handled. By definition we have for each $k \in \mathbb{N}$

$$\begin{aligned} d_{2k-1}(f) &= \int_{\mathbb{R}} f(x)\overline{\phi_{2k-1}(x)}dx. \\ &= -\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} f(x) \frac{x}{1+x^2} \left(\frac{x+i}{x-i}\right)^{2k-1} dx. \end{aligned} \tag{16}$$

Taking the change of variables $t = \sigma(x)$, we get

$$d_{2k-1}(f) = -i\sqrt{\frac{1}{2\pi}} \int_0^{2\pi} \frac{f\left(\sigma^{-1}(t)\right)}{\left(1-e^{it}\right)^2} \left(e^{-i(2k-1)t} - e^{-i(2k-3)t}\right) dt, \tag{17}$$

where we used the fact that $(x/(1+x^2))(\sigma^{-1})' = -(i/2)((e^{it}+1)/(e^{it}-1))$. Combining the assumptions with Lemma 1, we point out that $f_{\mathbf{p}} \in L^2(0,2\pi)$. Then the above equation leads to

$$d_{2k-1}(f) = -ic_{2k-1}\left(f_{\mathbf{p}}\right) + ic_{2k-3}\left(f_{\mathbf{p}}\right). \tag{18}$$

In the same method, for the case of $n = 2k$, $k \in \mathbb{N}$, and by the fact that $(1/(1+x^2))(\sigma^{-1}(t))' = 1/2$, we obtain

$$\begin{aligned} d_{2k} &= -\sqrt{\frac{1}{2\pi}} \int_0^{2\pi} f\left(\sigma^{-1}(t)\right) e^{-i(2k-1)t} dt \\ &= -c_{2k-1}\left(f_{\mathbf{q}}\right). \end{aligned} \tag{19}$$

According to the relationship between $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$, we have another form of d_{2k}, which completes the proof. □

It is known that for any $g \in L^1(0,2\pi)$, its Fourier coefficients $c_n(g)$ tend to zero as $|n| \to \infty$. This is the well-known Riemann-Lebesgue lemma [13]. As a direct consequence of Lemma 2, we have the same decay property of the coefficients $d_n(f)$, $|n| \in \mathbb{N}$.

Corollary 3. *If $f \in L^2(\mathbb{R})$ satisfies* (13) *then there holds*

$$\lim_{|n|\to\infty} d_n(f) = 0. \tag{20}$$

Proof. It follows from Lemma 1 that $f_{\mathbf{p}} \in L^2(0,2\pi)$. Thus, there holds $c_n(f_{\mathbf{p}}) \to 0$, $|n| \to \infty$. The relation between $d_n(f)$, $|n| \in \mathbb{N}$ and $c_n(f_{\mathbf{p}})$, $n \in \mathbb{Z}$, which is given in Lemma 2, yields the desired result. □

We now turn to the main theorem in this section about the almost everywhere convergence of the sequence $\mathcal{S}_N(f)$, $N \in \mathbb{N}$. The almost everywhere convergence of the Fourier series is one of the most important results in the theory of Fourier series [14, 15]. It is said that for any $g \in L^p(0,2\pi)$, $1 < p < \infty$, the partial sum of its Fourier series

$$\widetilde{\mathcal{S}}_N(g)(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} c_n(g) e^{int}, \quad t \in (0,2\pi) \tag{21}$$

converges to g almost everywhere on $(0,2\pi)$ as $N \to \infty$. We will show that the sequence of $\mathcal{S}_N(f)$, $N \in \mathbb{N}$, enjoys the same convergence property if f satisfies condition (13).

Theorem 4. *If $f \in L^2(\mathbb{R})$ satisfies* (13)*, then the sequence $\mathcal{S}_N(f)$, $N \in \mathbb{N}$, converges to f almost everywhere on $\mathbb{R}$.*

Proof. Associated with any f satisfying (13), due to Lemma 1, we know that $f_{\mathbf{q}} = f \circ \sigma^{-1} \in L^2(0,2\pi)$; it suffices to prove that as $N \to \infty$ there holds almost everywhere on $(0,2\pi)$

$$\mathcal{S}_N(f)\left(\sigma^{-1}\right)(t) \longrightarrow f\left(\sigma^{-1}\right)(t). \tag{22}$$

Consequently, the partial sum of its Fourier series $\widetilde{\mathcal{S}}_N(f_{\mathbf{q}})$ tends to $f_{\mathbf{q}}$ almost everywhere on $(0,2\pi)$ as $N \to \infty$. Upon this observation, (22) can be obtained by proving that as $N \to \infty$,

$$\mathcal{S}_N(f)\left(\sigma^{-1}(t)\right) - \widetilde{\mathcal{S}}_N\left(f_{\mathbf{q}}\right)(t) \longrightarrow 0, \quad t \in (0,2\pi). \tag{23}$$

To get the difference between $\mathcal{S}_N(f)(\sigma^{-1}(t))$ and $\widetilde{\mathcal{S}}_N(f_{\mathbf{q}})(t)$, we need to express the two items, respectively. By definition and change of variables $x = \sigma^{-1}(t)$, we have

$$\mathcal{S}_N(f)\left(\sigma^{-1}(t)\right) = \sum_{|n|=1}^{N} d_n(f)\,\phi_n\left(\sigma^{-1}(t)\right), \quad t \in (0,2\pi). \tag{24}$$

Applying Lemma 2 to the above equation, we can reexpress $\mathcal{S}_{2N}(f)(\sigma^{-1}(t))$ through a direct computation. Specifically, upon introducing four summations

$$\begin{aligned} \mathcal{I}_1 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{2n-1}\left(f_{\mathbf{p}}\right) - 2c_{2n-2}\left(f_{\mathbf{p}}\right) + c_{2n-3}\left(f_{\mathbf{p}}\right)\right) \\ &\quad \times e^{i(2n-1)t}, \\ \mathcal{I}_2 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{-(2n-1)}\left(f_{\mathbf{p}}\right) - 2c_{-2n}\left(f_{\mathbf{p}}\right) + c_{-(2n+1)}\left(f_{\mathbf{p}}\right)\right) \\ &\quad \times e^{-i(2n-1)t}, \\ \mathcal{I}_3 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{2n-2}\left(f_{\mathbf{p}}\right) - c_{2n-3}\left(f_{\mathbf{p}}\right)\right) e^{i(2n-2)t} \\ &\quad - \left(c_{2n-1}\left(f_{\mathbf{p}}\right) - c_{2n-2}\left(f_{\mathbf{p}}\right)\right) e^{i2nt}, \\ \mathcal{I}_4 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{-2n}\left(f_{\mathbf{p}}\right) - c_{-(2n+1)}\left(f_{\mathbf{p}}\right)\right) e^{-i2nt} \\ &\quad - \left(c_{-(2n-1)}\left(f_{\mathbf{p}}\right) - c_{-2n}\left(f_{\mathbf{p}}\right)\right) e^{-i(2n-2)t}, \end{aligned} \tag{25}$$

we rewrite $\mathcal{S}_{2N}(f)(\sigma^{-1}(t))$ as follows:

$$\mathcal{S}_{2N}(f)\left(\sigma^{-1}(t)\right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \tag{26}$$

To be compared with $\mathcal{S}_{2N}(f)(\sigma^{-1}(t))$, the item $\widetilde{\mathcal{S}}_{2N}(f_{\mathbf{q}})(t)$ should be represented in terms of the Fourier coefficients of $f_{\mathbf{p}}$. For this purpose, we rewrite $c_n(f_{\mathbf{q}})$, $n \in \mathbb{Z}$, as follows:

$$\begin{aligned} c_n(g) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{f(\sigma^{-1}(t))}{(1-e^{it})^2} (1-e^{it})^2 e^{-int} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_{\mathbf{p}}(t) \left(e^{-int} - 2e^{-i(n-1)t} + e^{-i(n-2)t}\right) dt \\ &= c_n(f_{\mathbf{p}}) - 2c_{n-1}(f_{\mathbf{p}}) + c_{n-2}(f_{\mathbf{p}}). \end{aligned} \tag{27}$$

Substituting the above formula into the partial sum of the Fourier series of $f_{\mathbf{q}}$, we have

$$\begin{aligned} &\widetilde{\mathcal{S}}_{2N}(f_{\mathbf{q}})(t) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-2N}^{2N} \left(c_n(f_{\mathbf{p}}) - 2c_{n-1}(f_{\mathbf{p}}) + c_{n-2}(f_{\mathbf{p}})\right) e^{int}. \end{aligned} \tag{28}$$

In a similar manner, we split the above equation into five items as follows:

$$\widetilde{\mathcal{S}}_{2N}(f_{\mathbf{q}})(t) = \sum_{k=1}^{5} \mathscr{J}_k, \tag{29}$$

where

$$\begin{aligned} \mathscr{J}_1 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{2n-1}(f_{\mathbf{p}}) - 2c_{2n-2}(f_{\mathbf{p}}) + c_{2n-3}(f_{\mathbf{p}})\right) \\ &\quad \times e^{i(2n-1)t}, \\ \mathscr{J}_2 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{-(2n-1)}(f_{\mathbf{p}}) - 2c_{-2n}(f_{\mathbf{p}}) + c_{-(2n+1)}(f_{\mathbf{p}})\right) \\ &\quad \times e^{-i(2n-1)t}, \\ \mathscr{J}_3 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{2n}(f_{\mathbf{p}}) - 2c_{2n-1}(f_{\mathbf{p}}) + c_{2n-2}(f_{\mathbf{p}})\right) \\ &\quad \times e^{i2nt}, \\ \mathscr{J}_4 &:= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{N} \left(c_{-2n}(f_{\mathbf{p}}) - 2c_{-(2n+1)}(f_{\mathbf{p}}) + c_{-(2n+2)}(f_{\mathbf{p}})\right) \\ &\quad \times e^{-i2nt}, \\ \mathscr{J}_5 &:= \frac{1}{\sqrt{2\pi}} \left(c_0(f_{\mathbf{p}}) - 2c_{-1}(f_{\mathbf{p}}) + c_{-2}(f_{\mathbf{p}})\right). \end{aligned} \tag{30}$$

It is clear that $\mathscr{I}_k = \mathscr{J}_k$, $k = 1, 2$. For $k = 3, 4$, there holds

$$\begin{aligned} \mathscr{I}_3 - \mathscr{J}_3 &= \frac{1}{\sqrt{2\pi}} \left(c_0(f_{\mathbf{p}}) - c_{-1}(f_{\mathbf{p}})\right) \\ &\quad + \frac{1}{\sqrt{2\pi}} \left(c_{2N-1}(f_{\mathbf{p}}) - c_{2N}(f_{\mathbf{p}})\right) e^{i2Nt}, \\ \mathscr{I}_4 - \mathscr{J}_4 &= \frac{1}{\sqrt{2\pi}} \left(c_{-2}(f_{\mathbf{p}}) - c_{-1}(f_{\mathbf{p}})\right) \\ &\quad + \frac{1}{\sqrt{2\pi}} \left(c_{-(2N+1)}(f_{\mathbf{p}}) - c_{-(2N+2)}(f_{\mathbf{p}})\right) e^{-i2Nt}. \end{aligned} \tag{31}$$

Then by subtracting (29) from (26), we obtain

$$\begin{aligned} &\mathcal{S}_{2N}(f)(\sigma^{-1}(t)) - \widetilde{\mathcal{S}}_{2N}(f_{\mathbf{q}})(t) \\ &\quad = \frac{1}{\sqrt{2\pi}} \left(c_{2N-1}(f_{\mathbf{p}}) - c_{2N}(f_{\mathbf{p}})\right) e^{i2Nt} \\ &\qquad + \frac{1}{\sqrt{2\pi}} \left(c_{-(2N+1)}(f_{\mathbf{p}}) - c_{-(2N+2)}(f_{\mathbf{p}})\right) e^{-i2Nt}. \end{aligned} \tag{32}$$

Together with the fact that $f_{\mathbf{p}} \in L^2(0, 2\pi)$, Riemann-Lebesgue lemma yields for any $t \in (0, 2\pi)$

$$\lim_{N \to \infty} \left(\mathcal{S}_{2N}(f)(\sigma^{-1}(t)) - \widetilde{\mathcal{S}}_{2N}(f_{\mathbf{q}})(t)\right) = 0. \tag{33}$$

□

3. Numerical Algorithm

Since the expansion (6) can serve as a representation of f with both mathematical and physical meanings, developing a fast approach for computing the coefficients $d_n(f)$, $|n| \in \mathbb{N}$, is practically important. Lemma 2 states that evaluating the expansion coefficients of f can be carried by calculating the Fourier coefficients of $f_{\mathbf{p}}$, and then the well-known fast Fourier transform [16, 17] can be applied to compute the latter ones.

We first recall the fast Fourier transform. The algorithm is developed to reduce the computational cost of the discrete Fourier transform which is defined as follows:

$$Y_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \omega_N^{-nk}, \quad 0 \le n \le N-1, \tag{34}$$

where $\omega_N := e^{2i\pi(1/N)}$. It requires $(N-1)^2$ number of multiplications to compute Y_n, $0 \le n \le N-1$, directly by formula (34). When N is large, the computational cost is huge. The key idea of the fast Fourier transform is to compute Y_n by splitting the sum into two parts according to the even and odd indices. More precisely, assume that N is an positive even integer, that is, $N = 2m$. We rewrite formula (34) as follows:

$$Y_n = \frac{1}{2} \left(P_n + \omega_N^{-n} I_n\right), \tag{35}$$

where

$$\begin{aligned} P_n &= \frac{1}{m} \left(y_0 + y_2 \omega_N^{-2n} + \cdots + y_{N-2} \omega_N^{-(N-2)n}\right), \\ I_n &= \frac{1}{m} \left(y_1 + y_3 \omega_N^{-2n} + \cdots + y_{N-1} \omega_N^{-(N-2)n}\right). \end{aligned} \tag{36}$$

Note that there holds the relations $P_{n+m} = P_n$ and $I_{n+m} = I_n$, for $n = 0, 1, \ldots, m-1$. Thus, one can obtain the same result for half of the cost by the following three steps.

Step 1. For $n = 0, 1, 2, \ldots, m-1$, compute P_n and $\omega_N^{-n} I_n$.

Step 2. Compute Y_n, $n = 0, 1, 2, \ldots, m-1$, by $Y_n = (1/2)(P_n + \omega_N^{-n} I_n)$.

Step 3. Compute Y_n, $n = m, m+1, 2, \ldots, 2m-1$, by $Y_n = (1/2)(P_{n-m} - \omega_N^{-(n-m)} I_{n-m})$.

Assume that N is a power of 2. It is clear that P_n and I_n are still two discrete Fourier transform of order m. Then the above process can be iterated until we arrive at the discrete Fourier transform of order 2. Consequently, we get the celebrated fast Fourier transform, which brings the computational cost for computing Y_n, $0 \le n \le N-1$, down to $\mathcal{O}(N\log_2 N)$.

Applying the fast Fourier transform to compute the Fourier coefficients which are used to represent the coefficients d_n, we have the following fast algorithm for computing d_n, $|n| = 1, 2, \ldots, N$.

Algorithm 5. Let $N > 0$ be a power of 2.

Step 1. Compute $\widehat{f}_{\mathbf{p},n}$, $n = 0, 1, \ldots, 4N-1$, by applying the fast Fourier transform to $f_{\mathbf{p}}(k\pi/2N)$, $k = 0, 1, \ldots, 4N-1$. Set $\widetilde{c}_n := \widehat{f}_{\mathbf{p},n}$, $n = 0, 1, \ldots, 2N-1$, and $\widetilde{c}_n := \widehat{f}_{\mathbf{p},n+4N}$, $n = -1, -2 \ldots, -2N$.

Step 2. Compute $\widetilde{d}_n$, $|n| = 1, 3, \ldots, N-1$, according to formula $\widetilde{d}_n = i(\widetilde{c}_{n-2} - \widetilde{c}_n)$.

Step 3. Compute $\widetilde{d}_n$, $n = 2, 4, \ldots, N$, according to formula $\widetilde{d}_n = -\widetilde{c}_{n-1} + 2\widetilde{c}_{n-2} - \widetilde{c}_{n-3}$.

Step 4. Compute $\widetilde{d}_n$, $n = -2, -4, \ldots, -N$, according to formula $\widetilde{d}_n = -\widetilde{c}_{n+1} + 2\widetilde{c}_n - \widetilde{c}_{n-1}$.

The output $\widetilde{d}_n$, $|n| = 1, 2, \ldots, N$, of Algorithm 5 is regarded as an approximation of the coefficients d_n, $|n| = 1, 2, \ldots, N$. The number of multiplications used in Algorithm 5 is estimated in the next theorem.

Theorem 6. *The total number of multiplications needed for Algorithm 5 is $\mathcal{O}(N\log_2 N)$.*

Proof. The number of multiplications of Step 1, used to compute the discrete Fourier transform of order $4N$ by the fast Fourier transform, is $2N(\log_2 4N - 2) + 1$. In the other steps, the number of multiplications for evaluating a family of $\widetilde{d}_n$ can be obtained as $N/2$. We use $\mathcal{M}_N$ to denote the number of multiplications used in Algorithm 5. Then we have

$$\mathcal{M}_N = 2N\left(\log_2 4N - 2\right) + 1 + 2N = 2N\log_2 N + 2N + 1, \tag{37}$$

which gives the desired result. □

It is noticed that the change of variable could impact the regularity of the related function $f_{\mathbf{p}}$ in (8), thus impacting the accuracy of the numerical scheme. Therefore we require the conditions of Lemma 2 to keep the boundedness of the coefficients $d_n(f)$, $n \in \mathbb{N}$. Other kinds of change of variable should be tried in the future study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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