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Research Article

Convergence of Variational Iteration Method for Solving Singular Partial Differential Equations of Fractional Order

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We are concerned here with singular partial differential equations of fractional order (FSPDEs). The variational iteration method (VIM) is applied to obtain approximate solutions of this type of equations. Convergence analysis of the VIM is discussed. This analysis is used to estimate the maximum absolute truncated error of the series solution. A comparison between the results of VIM solutions and exact solution is given. The fractional derivatives are described in Caputo sense.

1. Introduction

In recent years, considerable attention has been devoted to the study of the fractional calculus and its numerous applications in many areas such as physics and engineering. The applications of fractional calculus used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, and signal processing can be successfully modeled by linear or nonlinear FDEs [1–7]. Further, fractional partial differential equations appeared in many fields of engineering and science, including fractals theory, statistics, fluid flow, control theory, biology, chemistry, diffusion, probability, and potential theory [8, 9].

The singular partial differential equations of fractional order (FSPDEs), as generalizations of classical singular partial differential equations of integer order (SPDEs), are increasingly used to model problems in physics and engineering. Consequently, considerable attention has been given to the solution of singular partial differential equations of fractional order. Finding approximate or exact solutions of SPDEs is an important task. Except for a limited number of these equations, we have difficulty in finding their analytical solutions. Therefore, there have been attempts to find methods for obtaining approximate solutions. Several such techniques have drawn special attention, such as variational iteration method [10], homotopy analysis method [11], and homotopy iteration method [12].

The variational iteration method (VIM) was proposed by He [13–16] due to its flexibility and convergence and efficiently works with different types of linear and nonlinear partial differential equations of fractional order and gives approximate analytical solution for all these types of equations without linearization or discretization; many author have been studying it; for example, see [17–21]. In this paper, we discuss the VIM for solving FSPDEs and obtain the convergence results of this method. The contribution of this work can be summarized in three points.

(1) Based on the sufficient condition that guarantees the existence of a unique solution to our problem (see Theorem 6) and using the series solution, convergence of VIM is discussed (see Theorem 7).

(2) Using point one, the maximum absolute truncated error of series solution of VIM is estimated (see Theorem 8).

(3) Some numerical examples are given.

Consider fractional singular partial differential equations with variable coefficients

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\mu(x)\frac{\partial^4 u}{\partial x^4}+\lambda(y)\frac{\partial^4 u}{\partial y^4}+h(z)\frac{\partial^4 u}{\partial z^4}=0,\quad a<x,y,z<b,\ t>0, \tag{1}$$

where the variable coefficients subject to initial conditions

$$u(x,y,z,0)=f_0(x,y,z),\qquad \frac{\partial u}{\partial t}(x,y,z,0)=f_1(x,y,z) \tag{2}$$

and boundary conditions

$$\begin{aligned}
&u(a,y,z,t)=g_0(y,z,t), \qquad u(b,y,z,t)=g_1(y,z,t)\\
&u(x,a,z,t)=g_2(x,z,t), \qquad u(x,b,z,t)=g_3(x,z,t)\\
&u(x,y,a,t)=g_4(x,z,t), \qquad u(x,y,b,t)=g_5(x,z,t)\\
&\frac{\partial^2 u}{\partial x^2}(a,y,z,t)=k_0(y,z,t),\\
&\frac{\partial^2 u}{\partial x^2}(b,y,z,t)=k_0(y,z,t),\\
&\frac{\partial^2 u}{\partial y^2}(x,a,z,t)=k_2(x,z,t),\\
&\frac{\partial^2 u}{\partial y^2}(x,a,z,t)=k_3(x,z,t),\\
&\frac{\partial^2 u}{\partial z^2}(x,y,a,t)=k_4(x,y,t),\\
&\frac{\partial^2 u}{\partial z^2}(x,y,b,t)=k_5(y,z,t),
\end{aligned} \tag{3}$$

where $\partial^\alpha/\partial t^\alpha$ is the fractional derivative in the Caputo sense, $a<x,y,z<b$, and g_i and k_i, $i=0,\ldots,5$ are continuous. The $\partial^4 u/\partial x^4$, $\partial^4 u/\partial y^4$, and $\partial^4 u/\partial z^4$ are linear bounded operator; that is, it is possible to find numbers $m_1,m_2,m_3>0$ such that $\|\partial^4 u/\partial x^4\|\le m_1\|u\|$, $\|\partial^4 u/\partial y^4\|\le m_2\|u\|$, $\|\partial^4 u/\partial z^4\|\le m_3\|u\|$. Equation (1) can be written as

$$\begin{aligned}
{}^cD_t^\alpha u(x,y,z,t) &= f\big(t,u(x,y,z,t),D_x^{n_1}u(x,y,z,t),\\
&\quad D_y^{n_2}u(x,y,z,t),D_z^{n_3}u(x,y,z,t)\big),
\end{aligned} \tag{4}$$

where $n_1=n_2=n_3=4$.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory used in this paper.

Definition 1. A real function $f(x)$, $x>0$ is said to be in space $C\mu$, $\mu\in R$ if there exists a real number $p>\mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if $f^n\in R_\mu$, $n\in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha\ge 0$ of a function $f\in C\mu$, $\mu\ge -1$ is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt,\quad \alpha>0,\ t>0. \tag{5}$$

In particular $J^0 f(x)=f(x)$.

For $\beta\ge 0$ and $\gamma\ge -1$, some properties of the operator J^α are

(1) $J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x)$,

(2) $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma=(\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1))x^{\alpha+\gamma}$.

Definition 3. The Caputo fractional derivative of $f\in C_{-1}^m$, $m\in N$ is defined as

$$D^\alpha f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^m(t)\,dt,\quad m-1<\alpha\le m. \tag{6}$$

Lemma 4. *If $m-1<\alpha\le m$, $m\in N$, $f\in C_\mu^m$, $\mu>-1$, then the following two properties hold:*

(1) $D^\alpha[J^\alpha f(x)]=f(x)$,

(2) $J^\alpha[D^\alpha f(x)]=f(x)-\sum_{k=1}^{m-1}f^k(0)(x^k/k!)$.

Lemma 5. *Suppose that u and their partial derivatives are continuous; then the fractional derivative, ${}^cD_t^\alpha u(x,y,z,t)$, is bonded.*

Proof. We need to prove that it is possible to find number $M>0$ such that $\|{}^cD_t^\alpha u(x,y,z,t)\|\le M\|u\|$. From the definition of Caputo fractional derivative above we have

$$\begin{aligned}
&\|{}^cD_t^\alpha u(x,y,z,t)\|\\
&\quad=\left\|\frac{1}{\Gamma(m-\alpha)}\int_a^b (x-t)^{m-\alpha-1}u^{(m)}(t)\,dt\right\|\\
&\quad\le\frac{|b-a|}{|(m-\alpha)\Gamma(m-\alpha)|}\|u\|=M\|u\|,
\end{aligned} \tag{7}$$

where $M=|b-a|/|(m-\alpha)\Gamma(m-\alpha)|$. □

3. Analysis of the Variational Iteration Method

To solve the fractional singular partial differential equations (4) by using the variational iteration method, with initial and boundary conditions (2) and (3), where $\|{}^{c}D_{t}^{\alpha}u(t)\| = M\|u\|$, we construct the following correction functional:

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n(x,y,z,t) \\ &\quad + J_t^{\alpha}\Big[{}^{c}D_t^{\alpha}u(x,y,z,t) \\ &\qquad - f\big((x,y,z,t),u_n(x,y,z,t), \\ &\qquad D_x^{n_1}u(x,y,z,t),D_y^{n_2}u_n(x,y,z,t), \\ &\qquad D_z^{n_3}u_n(x,y,z,t)\big)\Big] \end{aligned} \tag{8}$$

or

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n(x,y,z,t) + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1}\lambda(s) \\ &\quad \times \Big({}^{c}D_t^{\alpha}u(x,y,z,s) \\ &\qquad - f\big(s,u_n(x,y,z,s),D_x^{n_1}u_n(x,y,z,s), \\ &\qquad D_y^{n_2}u_n(x,y,z,s),D_z^{n_3}u_n(x,y,z,s)\big)\Big)ds. \end{aligned} \tag{9}$$

J_t^{α} is the Riemann-Liouville fractional integral operator of order α, with respect to variable t, and λ is a general Lagrange multiplier which can be identified as optimally variational theory [22], and $\tilde{u}_n(x,t)$ are considered as restricted variation; that is, $\delta\tilde{u}_n(x,t) = 0$.

Making the above correction functional stationary, the following condition can be obtained:

$$\begin{aligned} \delta u_{k+1}(x,y,z,t) &= \delta u_n(x,y,z,t) + \frac{1}{\Gamma(\alpha)}\delta \\ &\quad \times \int_0^t (t-s)^{\alpha-1}\lambda(s) \\ &\quad \times \Big({}^{c}D_t^{\alpha}u(x,y,z,s) \\ &\qquad - f\big(t,\tilde{u}_n(x,y,z,s),D_x^{n_1}\tilde{u}_n(x,y,z,s), \\ &\qquad D_y^{n_2}\tilde{u}_n(x,y,z,s),D_z^{n_3}\tilde{u}_n(x,y,z,s)\big)\Big)ds \end{aligned} \tag{10}$$

and yields to Lagrange multiplier

$$\lambda(s) = s - t. \tag{11}$$

We obtain the following iteration formula by substitution of (11) in (9)

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n(x,y,z,t) + \frac{1}{\Gamma(\alpha-1)} \\ &\quad \times \int_0^t (t-s)^{\alpha-2}(t-s) \\ &\quad \times \big({}^{c}D_t^{\alpha}u(x,y,z,s) \\ &\qquad - f\big(s,u_k(x,y,z,s),D_x^{n_1}u_n(x,y,z,s)\ , \\ &\qquad D_y^{n_2}u_n(x,y,z,s),D_z^{n_3}u_n(t)\big)\big)ds. \end{aligned} \tag{12}$$

That is,

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n(x,y,z,t) - \frac{(\alpha-1)}{\Gamma(\alpha)} \\ &\quad \times \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \big({}^{c}D_t^{\alpha}u(x,y,z,s) \\ &\qquad - f\big(s,u_n(x,y,z,s), \\ &\qquad D_x^{n_1}u_n(x,y,z,s), \\ &\qquad D_y^{n_2}u_n(x,y,z,s), \\ &\qquad D_z^{n_3}u_n(x,y,z,s)\big)\big)ds. \end{aligned} \tag{13}$$

This yields the following iteration formula:

$$\begin{aligned} u_{n+1}(x,y,z,t) &= u_n(x,y,z,t) - (\alpha-1) \\ &\quad \times J_t^{\alpha}\big[{}^{c}D_t^{\alpha}u(x,y,z,t) \\ &\qquad - f\big(t,u_n(x,y,z,t)\ , \\ &\qquad D_x^{n_1}u(x,y,z,t),D_y^{n_2}u_n(x,y,z,t), \\ &\qquad D_z^{n_3}u_n(t)\big)\big]. \end{aligned} \tag{14}$$

The initial approximation u_0 can be chosen by the following manner which satisfies initial conditions:

$$u_0 = \sum_{j=0}^{1}\gamma_j\frac{t^j}{j!} = \gamma_0 + \gamma_1 t, \tag{15}$$

where $\gamma_0 = f_0(x,y,z)$, $\gamma_1 = f_1(x,y,z)$.

We can obtain the following first-order approximation by substitution of (15) into (14)

$$\begin{aligned} u_1(x,y,z,t) &= u_0(x,y,z,t) - (\alpha-1) J_t^{\alpha} \\ &\times \left[{}^{c}D_t^{\alpha} u_0(x,y,z,t) \right. \\ &\quad - f\left(t, u_0(t), \right. \\ &\qquad D_x^{n_1} u(x,y,z,t), D_y^{n_2} u_0(x,y,z,t), \\ &\qquad \left.\left. D_z^{3_m} u_0(x,y,z,t)\right)\right]. \end{aligned} \tag{16}$$

Finally, by substituting the constant values of γ_0 and γ_1 into (16), we have the results as the first approximate solutions of (4) with (2) and (3).

3.1. Convergence Analysis

3.1.1. Existence and Uniqueness Theorem. Define $F : X \to X$ contentious mapping, and the function $F(t, u_0, u_1, \ldots, u_{n-1})$ exists with continuous and bounded derivatives, where X is the Banach space $(C(J), \|\cdot\|)$, the space of all continuous functions on J with the norm

$$\|u\| = \max_{\forall t \in J} |u|, \tag{17}$$

and satisfies Lipschitz condition with Lipschitz constant L, such that

$$\begin{aligned} &\left| f\left(t, u_1(x,y,z,t), D^{n_1} u_1(x,y,z,t), \right.\right. \\ &\qquad \left. D^{n_2} u_1(x,y,z,t), D^{n_3} u_1(x,y,z,t)\right) \\ &\quad - f\left(t, u_2(x,y,z,t), D^{n_1} u_2(x,y,z,t), \right. \\ &\qquad \left.\left. D^{n_2} u_2(x,y,z,t), D^{n_m} u_2(x,y,z,t)\right)\right| \\ &\quad \le L \left| \left(u_1(x,y,z,t), D^{n_1} u_1(x,y,z,t), \right.\right. \\ &\qquad \left. D^{n_2} u_1 x, y, z, (t), D^{n_3} u_1(x,y,z,t)\right) \\ &\qquad - \left(u_2(t), D^{n_1} u_2(x,y,z,t), \right. \\ &\qquad \left.\left. D^{n_2} u_2(x,y,z,t), D^{n_3} u_2(x,y,z,t)\right)\right| \\ &\qquad\qquad 0 < L < 1,\ t \ge 0. \end{aligned} \tag{18}$$

Theorem 6. *Let f satisfy the Lipschitz condition* (18) *then the problem* (4) *with* (2) *and* (3) *has unique solution $u(x,t)$, whenever $0 < L < 1$.*

Proof. (1) The existence of the solution. From equation (4) we have

$$u = f\left(t, \sum_{j=0}^{m-1} c_j \frac{t^j}{j!} + J^{\alpha} u, J^{\alpha} D_x^{n_1} u, J^{\alpha} D_y^{n_2} u, J^{\alpha} D_z^{n_3} u\right). \tag{19}$$

The mapping $F : X \to X$ is defined as

$$F(u) = f\left(t, \sum_{j=0}^{m-1} c_j \frac{t^j}{j!} + J^{\alpha} u, J^{\alpha} D_x^{n_1} u, J^{\alpha} D_y^{n_2} u, J^{\alpha} D_z^{n_3} u\right). \tag{20}$$

Let $u, v \in X$; then

$$\begin{aligned} &|F(u) - F(v)| \\ &= \left| f\left(t, \sum_{j=0}^{m-1} c_j \frac{t^j}{j!} + J^{\alpha} u, J^{\alpha} D_x^{n_1} u, J^{\alpha} D_y^{n_2} u, J^{\alpha} D_z^{n_3} u\right) \right. \\ &\qquad \left. - f\left(t, \sum_{j=0}^{m-1} c_j \frac{t^j}{j!} + J^{\alpha} v, J^{\alpha} D_x^{n_1} v, J^{\alpha} D_y^{n_2} v, J^{\alpha} D_z^{n_3} v\right)\right| \\ &\le L \sum_{i=0}^{3} \left| J^{\alpha-n_i} u - J^{\alpha-n_i} v \right| \\ &\le L \sum_{i=0}^{3} \left| \frac{1}{\Gamma(\alpha-n_i)} \int_0^t (t-s)^{\alpha-n_i-1} [u-v]\, ds \right| \\ &\max |F(u) - F(v)| \\ &\le L \sum_{i=0}^{3} \frac{1}{\Gamma(\alpha-n_i)} \max \left| \int_0^t (t-s)^{\alpha-n_i-1} [u-v]\, ds \right| \\ &\le L \sum_{i=0}^{3} \frac{\|u-v\|}{\Gamma(\alpha-n_i)} \left| \int_0^t (t-s)^{\alpha-n_i-1} ds \right| \\ &\le \sum_{i=0}^{3} \frac{\mathrm{LMT}}{\Gamma(\alpha-n_i)} \|u-v\| \\ &\le \gamma \|u-v\|, \end{aligned} \tag{21}$$

where $\gamma = \sum_{i=0}^{3} (\mathrm{LMT}/(\Gamma(\alpha-n_i))) < 1$, then we get

$$\|F(u) - F(v)\| \le \|u-v\|, \tag{22}$$

therefore the mapping F is contraction, and there exists unique solution $u \in C(J)$ to problem (4).

(2) The uniqueness of the solution (see [23]).

□

3.1.2. Proof of Convergence

Theorem 7. *Suppose that X is Banach space and $F : X \to X$ satisfies condition* (18)*. Then, the sequence* (14) *converges to the solution of* (4) *with* (2) *and* (3)*.*

Proof. Defined $(C(J), \|\cdot\|)$ is the Banach space, the space of all continuous functions on J with the norm

$$\|u(x,y,z,t)\| = \max_{\forall t \in J} |u(x,y,z,t)|. \tag{23}$$

We need to show that $\{u_n\}$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
&\|u_n - u_m\| \\
&\quad = \max |u_n - u_m| \\
&\quad = \max \Bigg| u_{n-1} - \frac{(\alpha-1)}{\Gamma(\alpha)} \\
&\qquad \times \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[D^{\alpha} u_{n-1}(x,y,z,s) \right. \\
&\qquad - F\left(s, u_{n-1}(s), D_x^{n_1} u_{n-1}(s), \right. \\
&\qquad \left.\left. D_y^{n_2} u_{n-1}(s), D_z^{n_3} u_{n-1}(s)\right)\right] ds \\
&\qquad - u_{m-1} + \frac{(\alpha-1)}{\Gamma(\alpha)} \\
&\qquad \times \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[D^{\alpha} u_{m-1} \right. \\
&\qquad - F\left(s, u_{m-1}(x,y,z,s) D_x^{n_1} u_{m-1}, \right. \\
&\qquad \left.\left. D_y^{n_2} u_{m-1}, D_z^{n_3} u_{m-1}\right)\right] ds \Bigg| \\
&\quad \leq \max \Bigg[|u_{n-1} - u_{m-1}| \\
&\qquad - \Bigg| \frac{(\alpha-1)}{\Gamma(\alpha)} \\
&\qquad \times \int_0^t (t-s)^{\alpha-1} \left[D^{\alpha} u_{n-1} - D^{\alpha} u_{m-1} \right] \\
&\qquad - F\left(s, u_{n-1}, D_x^{n_1} u_{n-1}, D_y^{n_2} u_{n-1}, D_z^{n_3} u_{n-1}\right) ds \\
&\qquad - F\left(s, u_{m-1}, D_x^{n_1} u_{m-1}, D_y^{n_2} u_{m-1}, D_z^{n_3} u_{n-1}\right) ds \Bigg| \Bigg] \\
&\quad \leq \max \Bigg[|u_{n-1} - u_{m-1}| \\
&\qquad - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha)} \\
&\qquad \times \int_0^t \left|(t-s)^{\alpha-1}\right| |u_{n-1} - u_{m-1}| \, ds \Bigg] \\
&\quad \leq \max |u_{n-1} - u_{m-1}| \\
&\qquad \times \left(1 - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha-1)} \right. \\
&\qquad \left. \times \int_0^t \left|(t-s)^{\alpha-1}\right| ds \right),
\end{aligned} \tag{24}$$

where

$$R = \max_{0 \leq s \leq t, 0 \leq t \leq T} \left|(t-s)^{\alpha-1}\right|. \tag{25}$$

Finally, we have

$$\begin{aligned}
&\|u_n - u_m\| \\
&\quad \leq \left(1 - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha-1)} \right) \\
&\qquad \times \|u_{n-1} - u_{m-1}\| \\
&\quad \leq \gamma \|u_{n-1} - u_{m-1}\|,
\end{aligned} \tag{26}$$

where $M, R, T, \Gamma(\alpha)$ are constants and

$$\gamma = \left(1 - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha-1)} \right). \tag{27}$$

Let $n = m + 1$. Then

$$\begin{aligned}
&\|u_{m+1} - u_m\| \\
&\quad \leq \gamma \|u_m - u_{m-1}\| \leq \gamma^2 \|u_{m-1} - u_{m-2}\| \\
&\quad \leq \cdots \leq \gamma^m \|u_1 - u_0\|.
\end{aligned} \tag{28}$$

From the triangle inequality, we have

$$\begin{aligned}
&\|u_n - u_m\| \\
&\quad \leq \|u_{m+1} - u_m\| + \|u_{m+2} - u_{m+1}\| \\
&\quad \leq \cdots \leq \|u_n - u_{n-1}\| \\
&\quad \leq \gamma^m \|u_1 - u_0\| + \gamma^{m+1} \|u_1 - u_0\| \\
&\qquad + \cdots + \gamma^{n-1} \|u_1 - u_0\| \\
&\quad \leq \left[\gamma^m + \gamma^{m+1} + \gamma^{m+2} + \cdots + \gamma^{n-1}\right] \|u_1 - u_0\| \\
&\quad \leq \gamma^m \left[1 + \gamma + \gamma^2 + \cdots + \gamma^{n-m-1}\right] \|u_1 - u_0\| \\
&\quad \leq \gamma^m \left(\frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \|u_1 - u_0\|.
\end{aligned} \tag{29}$$

Since $0 < \gamma < 1$, so $1 - \gamma^{n-m} < 1$, and then

$$\|u_n - u_m\| \leq \frac{\gamma^m}{1 - \gamma} \|u_1 - u_0\|. \tag{30}$$

But $\|u_1 - u_0\| < \infty$; then $\|u_n - u_m\| \to 0$ as $m \to \infty$. We conclude that u_n is a Cauchy sequence in $C[J]$, so the sequence converges and the proof is complete. □

3.1.3. Error Analysis

Theorem 8. *The maximum absolute error of the approximate solution u_m to problem* (4)-(3) *is estimated to be*

$$\max_{t \in J} |u_{exact} - u_n| \leq k, \tag{31}$$

where

$$k = \left(\frac{\gamma^m \left(M + (m_1 + m_2 + m_3 RT)\,\beta \right)}{(1-\gamma)} \right) \|u_0\|,$$
$$\beta = \left| \left(\frac{\alpha - 1}{\Gamma(\alpha)} \right) \right|. \tag{32}$$

Proof. From Theorem (9) and inequality (30) we have

$$\|u_n - u_m\| \le \left(\frac{\gamma^m}{1-\gamma} \right) \|u_1 - u_0\| \tag{33}$$

as $n \to \infty$; then $u_n \to u_{\text{exact}}$ and

$$\begin{aligned} &\|u_1 - u_0\| \\ &= \max_{t\in J} \left| -\frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \right. \\ &\qquad \times \left[D^\alpha u_0 \right. \\ &\qquad \left. \left. - F\left(s, u_0, D_x^{n_1} u_0, D_y^{n_2}, D_z^{n_3} u_0\right)\right] ds \right| \\ &= \left[\left(M + (m_1 + m_2 + m_3 RT)\,\beta \right) \right] \|u_0\|, \end{aligned} \tag{34}$$

where $\beta = |((\alpha-1)/\Gamma(\alpha))|$, and thus, the maximum absolute error in the interval J is

$$\|u_{\text{exact}} - u_n\| \le \max_{t\in J} |u_{\text{exact}} - u_n| \le k. \tag{35}$$

This completes the proof. □

4. Numerical Examples

Example 1. Consider the following fourth-order fractional singular partial differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(\frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4} = 0,$$
$$\frac{1}{2} < x < 1, \quad t > 0, \quad 1 < \alpha \le 2. \tag{36}$$

With initial conditions

$$u(x,0) = 0, \quad \frac{1}{2} < x < 1,$$
$$\frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \quad 0 < x < 1 \tag{37}$$

and boundary conditions

$$u\left(\frac{1}{2}, t\right) = \left(1 + \frac{(1/2)^5}{120} \right) \sin t,$$
$$u(1,t) = \frac{121}{120} \sin t, \quad t > 0,$$
$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin t,$$
$$\frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6} \sin t, \quad t > 0, \tag{38}$$

the exact solution in special case $\alpha = 2$ is

$$u(x,t) = \left(1 + \frac{x^5}{120} \right) \sin t \tag{39}$$

and we solve the problem (36) by variational iteration method. According to variational iteration method, formula (14) for (36) can be expressed in the following form:

$$\begin{aligned} &u_{n+1}(x,t) \\ &= u_n(x,t) - (\alpha-1) J_t^\alpha \\ &\quad \times \left(\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} + \left(\frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4} \right). \end{aligned} \tag{40}$$

Suppose that an initial approximation has the following form which satisfies the initial conditions:

$$u_0(x,t) = \left(1 + \frac{x^5}{120} \right) t. \tag{41}$$

Now by iteration formula (16), we obtain the following approximations:

$$\begin{aligned} &u_1(x,t) \\ &= u_0(x,t) - (\alpha-1) J_t^\alpha \\ &\quad \times \left(\frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} + \left(\frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u_0}{\partial x^4} \right) \\ &= \left(\left(1 + \frac{x^5}{120} \right) \right) t - (\alpha-1) \left(1 + \frac{x^5}{120} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \end{aligned} \tag{42}$$

The second approximation takes the following form:

$$\begin{aligned} &u_2(x,t) \\ &= u_1(x,t) - (\alpha-1) J_t^\alpha \\ &\quad \times \left(\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + \left(\frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u_1}{\partial x^4} \right) \\ &= \left(1 + \frac{x^5}{120} \right) t - (\alpha-1) \left(1 + \frac{x^5}{120} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad + (\alpha-1)^2 \left(1 + \frac{x^5}{120} \right) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)} \\ &= \left(1 + \frac{x^5}{120} \right) \\ &\quad \times \left(t - (\alpha-1)^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\alpha-1)^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right), \end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= u_2(x,t) - (\alpha-1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_2(x,t)}{\partial t^{\alpha}} + \left(1 + \frac{x^4}{120}\right) \frac{\partial^4 u_1}{\partial x^4} \right) \\
&= \left(1 + \frac{x^5}{120}\right) \\
&\quad \times \left(t - (\alpha-1)^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\alpha-1)^2 \right. \\
&\qquad \left. \times \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (\alpha-1)^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \right) \\
&\ \ \vdots
\end{aligned} \tag{43}
$$

Table 1 shows the absolute error of VIM solution of example (36) (when $\alpha = 1.999$, $x = 0.1$, and $n = 2$), while Table 2 shows the maximum absolute truncated error of VIM solution (using Theorem 8) at different values of n (when $t = 2$).

Example 2. Consider the following fourth-order fractional singular partial differential equation:

$$
\begin{gathered}
\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4} = 0, \\
0 < x < 1, \quad t > 0, \quad 1 < \alpha \leq 2.
\end{gathered} \tag{44}
$$

With initial conditions

$$
\begin{aligned}
u(x,0) &= x - \sin x, \quad 0 < x < 1 \\
\frac{\partial u}{\partial t}(x,0) &= -(x - \sin x), \quad 0 < x < 1
\end{aligned} \tag{45}
$$

and boundary conditions

$$
\begin{aligned}
&u(0,t) = 0, \quad u(1,t) = e^{-t}(1 - \sin 1), \quad t > 0, \\
&\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = e^{-t} \sin 1, \quad t > 1,
\end{aligned} \tag{46}
$$

the exact solution in special case $\alpha = 2$ is

$$
u(x,t) = (x - \sin x) e^{-t}. \tag{47}
$$

According to variational iteration method, formula (14) for (44) can be expressed in the following form:

$$
\begin{aligned}
u_{k+1}(x,t) &= u_k(x,t) - (\alpha-1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_k(x,t)}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u}{\partial x^4} \right).
\end{aligned} \tag{48}
$$

Table 1: Absolute error.

t	Error of VIM ($n = 2$)
0.2	1.9635×10^{-7}
0.4	9.44308×10^{-6}
0.6	5.15266×10^{-5}
0.8	1.7613×10^{-4}
1	4.98115×10^{-4}
1.2	0.00127331
1.4	0.00301987
1.6	0.00669301
1.8	0.0139188
2	0.02729

Table 2: Maximum absolute error.

n	Maximum error VIM
2	0.0272901
3	0.00186871
4	0.00328421

Suppose that an initial approximation has the following form which satisfies the initial condition:

$$
u_0(x,t) = (x - \sin x) - (x - \sin x) t. \tag{49}
$$

Now by iteration formula (48), we obtain the first approximation

$$
\begin{aligned}
u_1(x,t) &= u_0(x,t) - (\alpha-1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_0(x,t)}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u_0}{\partial x^4} \right) \\
&= (x - \sin x) - (x - \sin x) t \\
&\quad + (\alpha-1)(x - \sin x) \frac{t^{\alpha}}{\Gamma(\alpha+1)} - (\alpha-1) \\
&\quad \times (x - \sin x) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}
\end{aligned} \tag{50}
$$

and second approximation

$$
\begin{aligned}
u_2(x,t) &= u_1(x,t) - (\alpha-1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_1(x,t)}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1 \right) \frac{\partial^4 u_1}{\partial x^4} \right) \\
&= (x - \sin x) - (x - \sin x) t \\
&\quad - (\alpha-3)(\alpha-1)(x - \sin x) \frac{t^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

$$+ (\alpha - 3)(\alpha - 1)(x - \sin x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$
$$+ (\alpha - 1)^2 (x - \sin x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- (\alpha - 1)^2 (x - \sin x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$u_3 = (x - \sin x) \times \Bigg((2-\alpha) - \alpha t + (\alpha-1)(\alpha^2 - 5\alpha + 7)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (\alpha-1)(\alpha^2 - 5\alpha + 5)\frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + (\alpha-1)^2(2-\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\alpha-1)^2(5-2\alpha)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + (\alpha-1)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - (\alpha-1)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+2)}\Bigg)$$

$$u_0(x,t) = (x - \sin x) - (x - \sin x)t$$

$$u_1(x,t) = u_0(x,t) - (\alpha - 1) J_t^{\alpha} \times \left(\frac{\partial^{\alpha} u_0(x,t)}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_0}{\partial x^4}\right)$$
$$= (x - \sin x) - (x - \sin x)t + (\alpha-1)(x-\sin x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - (\alpha-1)(x-\sin x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u_2(x,t) = u_1(x,t) - (\alpha - 1) J_t^{\alpha} \times \left(\frac{\partial^{\alpha} u_1(x,t)}{\partial t^{\alpha}} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_1}{\partial x^4}\right)$$
$$= (x - \sin x) - (x - \sin x)t - (\alpha-3)(\alpha-1)(x-\sin x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (\alpha-3)(\alpha-1)(x-\sin x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\alpha-1)^2(x-\sin x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\alpha-1)^2(x-\sin x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$

$$u_3(x,t) = (x - \sin x) \times \Bigg((2-\alpha) - (\alpha-2)t + (\alpha-1)(\alpha^2 - 5\alpha + 7)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - (\alpha-1)(\alpha^2 - 5\alpha + 7)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - (\alpha-1)^2(2\alpha-5)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (\alpha-1)^2(2\alpha-5)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + (\alpha-1)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - (\alpha-1)^3\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\Bigg)$$
$$\vdots \tag{51}$$

Table 3 shows the absolute error of VIM solution of example (37) (when $\alpha = 1.5$, $x = 0.1$, and $n = 2$), while Table 4 shows the maximum absolute truncated error of VIM solution (using Theorem 8) at different values of n (when $t = 2$).

Example 3. Consider the following singular two-dimensional partial differential equation of fractional order:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4} = 0, \quad 0 < x, \quad y < 1, \quad t > 0, \quad 1 < \alpha \le 2. \tag{52}$$

With initial conditions

$$u(x,y,0) = 0, \quad 0 < x < 1$$
$$\frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \quad 0 < x < 1 \tag{53}$$

and boundary conditions

$$u(0.5,y,t) = \left(2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!}\right)\sin t,$$
$$u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)\sin t, \quad t > 0,$$

Table 3: Absolute error.

t	Error of VIM ($n = 2$)
0.2	2.9199×10^{-6}
0.4	6.85596×10^{-6}
0.6	7.79651×10^{-6}
0.8	5.12596×10^{-6}
1	1.42135×10^{-6}
1.2	1.19864×10^{-5}
1.4	2.66612×10^{-5}
1.6	4.5519×10^{-5}
1.8	6.86269×10^{-5}
2	9.60514×10^{-5}

Table 4: Maximum absolute error.

n	Maximum error VIM
2	9.60514×10^{-5}
3	1.82927×10^{-6}
4	5.93438×10^{-9}

$$\begin{gathered}
\frac{\partial^2 u}{\partial x^2}(0.5, y, t) = \frac{(0.5)^4}{6!}\sin t, \\
\frac{\partial^2 u}{\partial x^2}(1, y, t) = \frac{1}{6!}\sin t, \quad t > 1, \\
\frac{\partial^2 u}{\partial y^2}(x, 0.5, t) = \frac{(0.5)^4}{6!}\sin t, \\
\frac{\partial^2 u}{\partial y^2}(x, 1, t) = \frac{1}{6!}\sin t, \quad t > 1,
\end{gathered} \tag{54}$$

the exact solution in special case $\alpha = 2$ is

$$u(x, y, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\sin t. \tag{55}$$

According to variational iteration method, formula (14) for (52) can be expressed in the following form:

$$\begin{aligned}
u_{k+1}(x, t) &= u_k(x, t) - (\alpha - 1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_k(x, t)}{\partial t^{\alpha}} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} \right. \\
&\qquad \left. + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}\right).
\end{aligned} \tag{56}$$

Suppose that an initial approximation has the following form which satisfies the initial conditions:

$$u_0(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t. \tag{57}$$

Now by iteration formula (56), we obtain the following approximations:

$$\begin{aligned}
u_1(x, t) &= u_0(x, t) - (\alpha - 1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_0(x, t)}{\partial t^{\alpha}} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u_0}{\partial x^4} \right. \\
&\qquad \left. + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u_0}{\partial y^4}\right) \\
&= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t - 2(\alpha - 1)\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}.
\end{aligned} \tag{58}$$

The second approximation takes the following form:

$$\begin{aligned}
u_2(x, t) &= u_1(x, t) - (\alpha - 1) J_t^{\alpha} \\
&\quad \times \left(\frac{\partial^{\alpha} u_1(x, t)}{\partial t^{\alpha}} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u_1}{\partial x^4} \right. \\
&\qquad \left. + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u_1}{\partial y^4}\right) \\
&= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t - 2(\alpha - 1) \\
&\quad \times \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \\
&\quad + (\alpha - 1)^2\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha + 1)} \\
&= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) \\
&\quad \times \left(t - 2(\alpha - 1)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + (\alpha - 1)^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)}\right) \\
u_3 &= \left(2 + \frac{x^2}{6!} + \frac{y^2}{6!}\right) \\
&\quad \times \left((2 - \alpha)t - \alpha(2\alpha - 1)\frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} \right. \\
&\qquad \left. - 2(\alpha - 1)\frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - (\alpha - 1)^3\frac{t^{3\alpha+1}}{\Gamma(3\alpha + 2)}\right) \\
&\ \ \vdots
\end{aligned} \tag{59}$$

Table 5 shows the absolute error of VIM solution of example (38) (when $\alpha = 1.999$, $x = y = 0.1$, and $n = 2$), while Table 6 shows the maximum absolute truncated error of VIM solution (using Theorem 8, resp.) at different values of n (when $t = 2$).

Table 5: Absolute error.

t	Error of VIM ($n = 2$)
0.2	4.94792×10^{-6}
0.4	2.38092×10^{-5}
0.6	4.09852×10^{-5}
0.8	1.0933×10^{-5}
1	3.29725×10^{-4}
1.2	0.0013951
1.4	0.00421146
1.6	0.0106573
1.8	0.0239528
2	0.0492518

Table 6: Maximum absolute error.

n	Maximum error VIM
2	0.0492518
3	0.00159092
4	0.00124009

5. Conclusion

The variational iteration method has been known as powerful tools for solving many equations in fractional calculus such as ordinary equations, partial differential equations, integrodifferential equations, and so many other equations. In this paper, this method has been analyzed with an aim to investigate the conditions which result in the convergence of generated series solutions of the singular partial differential equations of fractional order. The theorems outlined in the paper have proved that the approximate solutions successfully converge to the exact solution. We consider three examples to verify convergence hypothesis simplicity of the method. From the results we see that the exact error coincides with the approximate error obtained from using the theorems; for example, see Tables 1, 2, 3, and 4. Further, the high agreement of the numerical results so obtained between the variational iteration method and the exact solution in all examples reinforces the conclusion that the efficiency of this method and related phenomena give the method much wider applicability. Furthermore, the results obtained by proposed method confirm the robustness and efficiency of it. And we hope that the work in this paper is a step in this direction.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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