Research Article

Three-Dimensional Flow and Heat Transfer Past a Permeable Exponentially Stretching/Shrinking Sheet in a Nanofluid

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The three-dimensional flow and heat transfer of a nanofluid over a stretching/shrinking sheet is investigated. Numerical results are obtained using bvp4c in MATLAB. The results show nonunique solutions for the shrinking case. The effects of the stretching/shrinking parameter, suction parameter, Brownian motion parameter, thermophoresis parameter, and Lewis number on the local skin friction coefficient and the local Nusselt number are studied. Suction increases the solution domain. Furthermore, as the sheet is shrunk in the x-direction, suction increases the skin friction coefficient in the same direction while decreasing the skin friction coefficient in the y-direction. The local Nusselt number is consistently lower for higher values of thermophoresis parameter and Lewis number. On the other hand, the local Nusselt number increases as the Brownian motion parameter increases.

1. Introduction

Nanofluids are dispersions of nanometer-sized particles in a base fluid such as water, ethylene glycol, and propylene glycol, to increase their thermal conductivities. Choi and Eastman [1] showed that the addition of a small amount (less than 1% by volume) of nanoparticles to conventional heat transfer liquids increased the thermal conductivity of the fluid up to approximately two times. In his paper, Buongiorno [2] developed a model for convective transport in nanofluids which takes into account the Brownian diffusion and thermophoresis effects. Buongiorno's nanofluid model was used in many recent papers, for example, Nield and Kuznetsov [3–5], Khan and Pop [6], Bachok et al. [7–9], Mansur and Ishak [10, 11], and Zaimi et al. [12] among others.

The boundary layer flow over a stretching sheet is significant in applications such as extrusion, wire drawing, metal spinning, and hot rolling [13]. Wang [14, 15], Mandal

and Mukhopadhyay [16], P. S. Gupta and A. S. Gupta [17], Andersson [18], Ishak et al. [19], and Makinde and Aziz [20] are among various names who published their papers on a stretching sheet. Miklavčič and Wang [21] studied flow over a shrinking sheet in which they observed that the vorticity is not confined within a boundary layer and the steady flow cannot exist without exerting adequate suction at the boundary. As the studies of shrinking sheet garner considerable attention, this finding proves to be crucial to these researches. In response to Miklavčič and Wang, numerous studies on these problems have been conducted by researches, namely, Wang [22], Fang et al. [23], Bachok et al. [24], Bhattacharyya et al. [25], Zaimi et al. [26], and Roşca and Pop [27] among others.

All the above-mentioned studies dealt with problems involving linear stretching/shrinking sheet. The boundary layer flow induced by a stretching/shrinking sheet is very important in engineering processes [28] and has attracted many researchers to delve into this study such as Bachok et al.
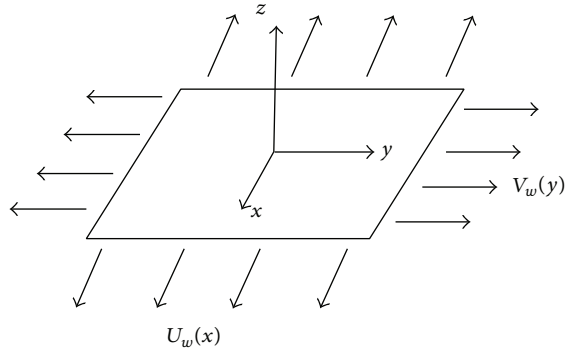
FIGURE 1: Geometry of the problem.

[29], Bhattacharyya and Vajravelu [30], and Rohni et al. [31]. However, most studies revolved around two-dimensional flows. Motivated by this, the objective of this paper is to solve the problem of three-dimensional flow and heat transfer past a permeable exponentially stretching/shrinking sheet in a nanofluid. The dependency of the local skin friction coefficient and the local Nusselt number on several parameters, namely, the stretching/shrinking, Brownian motion, and thermophoresis parameters is the main focus of the present investigation. Numerical solutions are presented graphically and in tabular forms to show the effects of these parameters on the local skin friction coefficient and the local Nusselt number.

2. Problem Formulation

We consider the steady three-dimensional boundary layer flow of a viscous nanofluid past a permeable stretching/shrinking flat surface in a quiescent fluid. A locally orthogonal set of coordinates (x, y, z) is chosen with the origin O in the plane of the stretching/shrinking sheet. The x- and y-coordinates are in the plane of the sheet, while the coordinate z is measured in the perpendicular direction to the stretching/shrinking surface as shown in Figure 1. It is assumed that the flat surface is stretched/shrunk continuously in the both x- and y-directions with the velocities $u(x) = u_w(x)$ and $v(y) = v_w(y)$, respectively. It is also assumed that the mass flux velocity is $w_w(x, y)$, where $w_w(x, y) < 0$ is for suction and $w_w(x, y) > 0$ is for injection or withdrawal of the fluid. Further, we assume that the constant surface temperature and the constant surface volume fraction are T_w and C_w, while the constant temperature and the constant surface volume fraction of the ambient (inviscid) fluid are T_∞ and C_∞, respectively. Under these conditions, the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = v\frac{\partial^2 u}{\partial z^2}, \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = v\frac{\partial^2 v}{\partial z^2}, \tag{3}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = v\frac{\partial^2 w}{\partial z^2}, \tag{4}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$
$$= \alpha\frac{\partial^2 T}{\partial z^2} + \tau\left[D_B\frac{\partial C}{\partial z}\frac{\partial T}{\partial z} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right], \tag{5}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_B\frac{\partial^2 C}{\partial z^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2}, \tag{6}$$

along with the boundary conditions

$$u = u_w(x) = \lambda_1 U_w(x), \qquad v = v_w(y) = \lambda_2 V_w(y),$$
$$w = w_w = w_0, \qquad T = T_w,$$
$$D_B\frac{\partial C}{\partial z} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0$$
$$u \longrightarrow 0, \quad w \longrightarrow 0, \quad T \longrightarrow T_\infty, \quad C \longrightarrow C_\infty \quad \text{as } z \longrightarrow \infty. \tag{7}$$

Here u, v, and w are the velocity components along x-, y-, and z-axes, respectively; v is the kinematic viscosity of the fluid, λ_1 is the constant stretching ($\lambda_1 > 0$) or shrinking ($\lambda_1 < 0$) parameter in the x-direction, and λ_2 is the constant stretching ($\lambda_2 > 0$) or shrinking ($\lambda_2 < 0$) parameter in the y-direction, respectively. Further, we assume that $U_w(x, y)$ and $V_w(x, y)$ are of the following form:

$$U_w(x, y) = V_w(x, y) = U_0\, e^{(x+y)/L}, \tag{8}$$

where L is the characteristic length and U_0 is the characteristic velocity of the stretching/shrinking sheet.

We introduce now the following similarity variables:

$$u = U_0 e^{(x+y)/L} f'(\eta), \qquad v = U_0 e^{(x+y)/L} g'(\eta),$$
$$w = -\left(\frac{vU_0}{2L}\right)^{1/2} e^{(x+y)/2L}\left[f(\eta) + \eta f'(\eta) + g(\eta) + \eta g'(\eta)\right],$$
$$\theta(\eta) = \frac{(T - T_\infty)}{(T_w - T_\infty)}, \qquad \phi(\eta) = \frac{(C - C_\infty)}{(C_w - C_\infty)},$$
$$\eta = \left(\frac{U_0}{2vL}\right)^{1/2} e^{(x+y)/2L} z, \tag{9}$$

where primes denote differentiation with respect to η. Next we take

$$w_w(x, y) = -\left(\frac{vU_0}{2L}\right)^{1/2} e^{(x+y)/2L} S, \tag{10}$$

where S is the surface mass transfer parameter with $S > 0$ for suction and $S < 0$ for injection. Substituting the similarity variables (9) into (1) to (6), it is found that the continuity equation (1) is automatically satisfied, and (2) to (6) are

reduced to the following ordinary (similarity) differential equations:

$$f''' + (f + g) f'' - 2 (f' + g') f' = 0,$$

$$g''' + (f + g) g'' - 2 (f' + g') g' = 0,$$

$$\frac{1}{\text{Pr}} \theta'' + (f + g) \theta' + \text{Nb}\phi'\theta' + \text{Nt}\theta'^2 = 0, \tag{11}$$

$$\phi'' + \text{Le} (f + g) \phi' + \frac{\text{Nt}}{\text{Nb}} \theta'' = 0$$

subject to the boundary conditions

$$f (0) = S, \qquad g (0) = 0, \qquad f' (0) = \lambda_1,$$

$$g' (0) = \lambda_2, \qquad \theta (0) = 1, \qquad \text{Nt}\phi' (0) + \text{Nb}\theta' (0) = 0$$

$$f' (\eta) \longrightarrow 0, \quad g' (\eta) \longrightarrow 0, \quad \theta (\eta) \longrightarrow 0, \quad \phi (\eta) \longrightarrow 0$$

$$\text{as } \eta \longrightarrow \infty, \tag{12}$$

where Pr is the Prandtl number, Le is the Lewis number, Nb is the Brownian motion parameter, and Nt is the thermophoresis parameter, which are defined as follows:

$$\text{Pr} = \frac{\nu}{\alpha}, \qquad \text{Le} = \frac{\nu}{D_B}, \qquad \text{Nb} = \frac{\beta D_B (C_w - C_\infty)}{\nu},$$

$$\text{Nt} = \frac{\beta D_T (T_w - T_\infty)}{T_\infty \nu}. \tag{13}$$

The physical quantities of practical interest are the local skin friction coefficients, C_{fx} and C_{fy}, and the local Nusselt number Nu_x, which are defined as follows:

$$C_{fx} = \frac{2\tau_{wx}}{\rho U_w^2}, \qquad C_{fy} = \frac{2\tau_{wy}}{\rho U_w^2},$$

$$\text{Nu}_x = \frac{q_w}{k (T_w - T_\infty)}, \tag{14}$$

where τ_{wx} and τ_{wy} are the shear stresses in the x- and y-directions of the stretching/shrinking sheet and q_w is the heat flux from the surface of the sheet, which are given by

$$\tau_{wx} = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}, \qquad \tau_{wy} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=0},$$

$$q_w = - k\left(\frac{\partial T}{\partial z}\right)_{z=0}. \tag{15}$$

Substituting (9) into (14) and using (15), we obtain

$$\text{Re}_x^{1/2}C_{fx} = 2f'' (0), \qquad \text{Re}_y^{1/2}C_{fy} = 2g'' (0),$$

$$\text{Re}_x^{- 1/2}\text{Nu}_x = - \theta' (0), \tag{16}$$

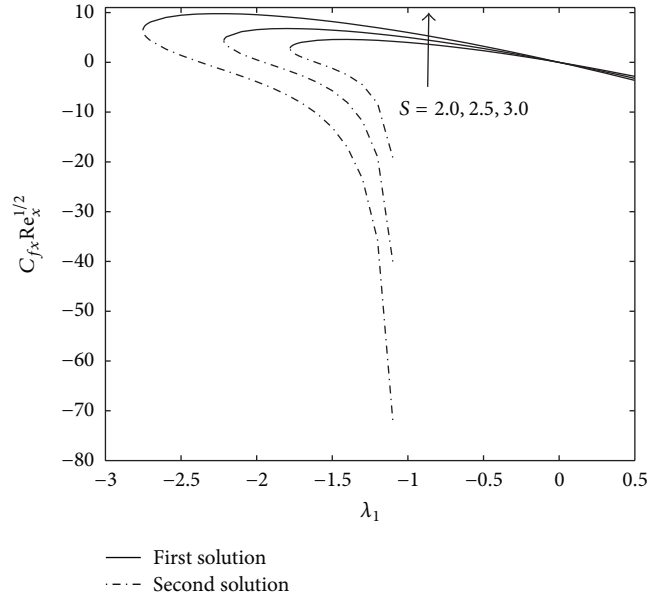where $\text{Re}_x = U_w L/\nu$ and $\text{Re}_y = V_w L/\nu$ are the local Reynolds numbers.

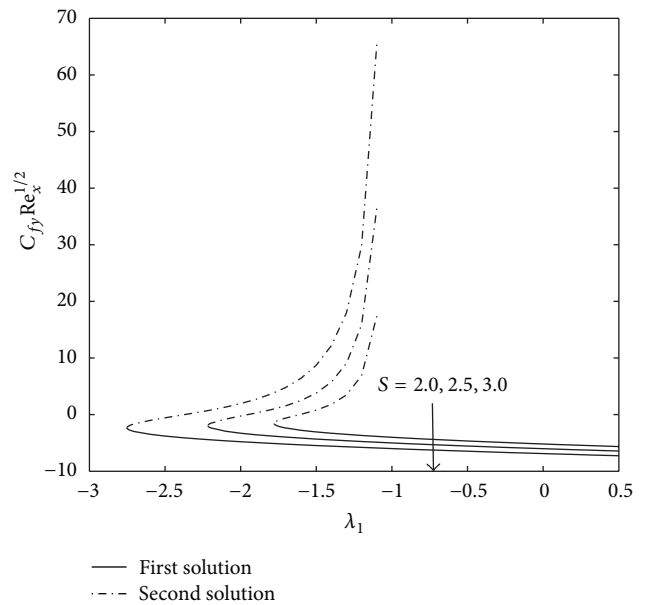FIGURE 2: Variation of the skin friction coefficient in x-direction with λ_1 for different values of S when $\lambda_2 = 1$.

FIGURE 3: Variation of the skin friction coefficient in y-direction with λ_1 for different values of S when $\lambda_2 = 1$.

3. Results and Discussions

The system of ordinary differential equations (11) subject to the boundary conditions (12) was solved numerically using the package bvp4c in MATLAB for different values of parameters: the stretching/shrinking parameter in x-direction λ_1, suction S, Brownian motion parameter Nb, thermophoresis parameter Nt, and Lewis number Le. We fixed the Prandtl number to be equal to 6.8 and the stretching/shrinking parameter in the y-direction λ_2 to be 1 ($\lambda_2 = 1$)

TABLE 1: Values of λ_{1c}.

S	λ_c
2.0	−1.7785
2.5	−2.2164
3.0	−2.7157

throughout the paper. The relative tolerance is set to 10^{-10} and the boundary conditions (12) at $\eta = \infty$ are replaced by $\eta = 10$. This choice is sufficient for the velocity and the temperature profiles to reach the far field boundary conditions asymptotically.

In this paper, we intend to study the three-dimensional flow and heat transfer of a nanofluid over a stretching/shrinking sheet. The analysis shows that the existence of solution depends on the suction parameter S and the stretching/shrinking parameter λ_1. Figures 2 and 3 show that the skin friction coefficient in the x-direction and the y-direction, respectively, decreases as λ_1 increases. From these figures, we can see that dual solutions exist for the problem. However, based on the previous studies [27, 32, 33], only the first solution is physically realizable and thus relevant to that studies. It is portrayed in Figures 2 and 3 that unique solution exists for $\lambda_1 \geq -1$ and $\lambda_1 = \lambda_c$, where λ_c is the critical values of λ_1. Furthermore, it is seen that the range of λ_1, where solutions exist, increases as S increases, as shown in Table 1. In addition, in Figure 2, it is shown that when the sheet is shrunk in the x-direction, the skin friction coefficient parallel to the direction increases as S increases. However, the skin friction coefficient in the y-direction decreases with increasing S as illustrated in Figure 3. Moreover, it is interesting to note that the shear stress in the x-direction is prominently higher than the shear stress in the y-direction.

Figure 4 shows that the local Nusselt number increases with λ_1. However, the local Nusselt number decreases as thermophoresis parameter increases. This phenomenon may be caused by the thermal boundary layer that thickens as the thermophoresis parameter is increased. As opposed to this occurrence, the thermal boundary layer becomes thinner as the Brownian motion parameter increases. This leads to the increase of the local Nusselt number as Brownian motion parameter increases as shown in Table 2. The table also shows that the Lewis number lowers the local Nusselt number.

Figures 5–7 show the velocity profiles for the flow in the x- and y-directions for different values S and λ_1. These profiles show that the far field boundary conditions are satisfied which validates the numerical result. Furthermore, these profiles also support the existence of dual solutions. The effect of S on both $f'(\eta)$ and $g'(\eta)$ is shown in Figures 5 and 6, respectively. From the two figures, it is noted that while S increases the velocity $f'(\eta)$, it decreases the velocity $g'(\eta)$. Figure 7 then shows the effect of λ on the velocity profiles $f'(\eta)$ and $g'(\eta)$. Increasing the stretching parameter in the x-direction causes $f'(\eta)$ to increase. On the other hand, the velocity $g'(\eta)$ is consistently lower for higher λ_1 although it is seen that the changes are minuscule.
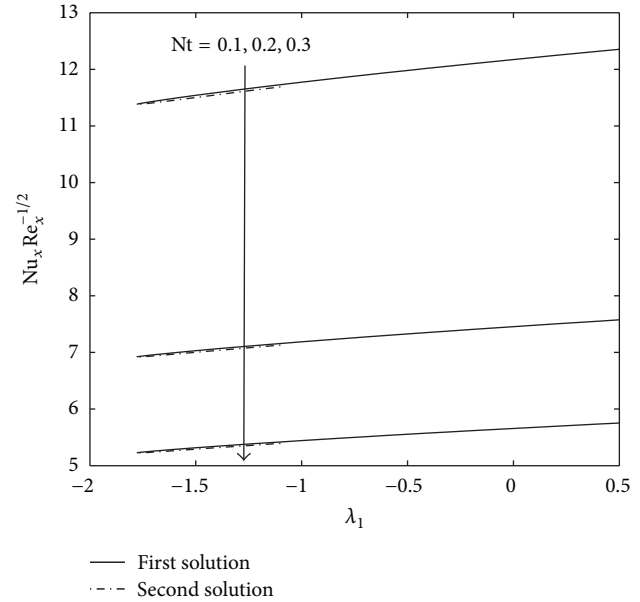

FIGURE 4: Variation of the local Nusselt number with λ_1 for different values of Nt when $\lambda_2 = 1$, Pr = 6.8, Nb = 0.1, and Le = 5.
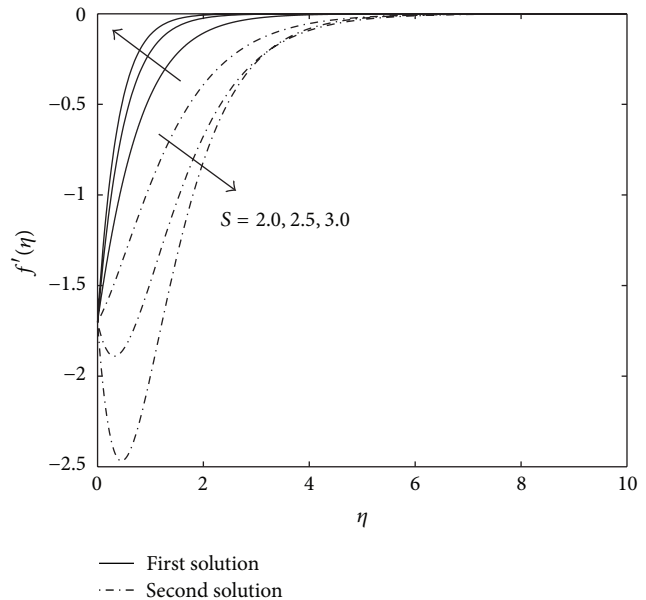

FIGURE 5: Velocity profiles in x-direction for different values of S when $\lambda_1 = -1.7$ and $\lambda_2 = 1$.

4. Conclusions

The three-dimensional flow and heat transfer of a nanofluid over a stretching/shrinking sheet was investigated numerically. The effects of various parameters on the skin friction coefficient and the local Nusselt number were discussed. The results showed that suction parameter increases the solution domain. Furthermore, as the sheet is shrunk in the x-direction, suction increases the skin friction coefficient in the same direction while decreasing the skin friction coefficient
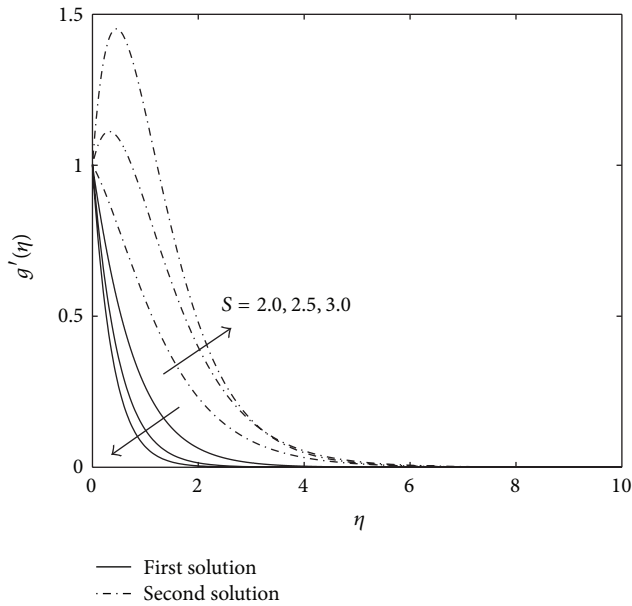
FIGURE 6: Velocity profiles in y-direction for different values of S when $\lambda_1 = -1.7$ and $\lambda_2 = 1$.
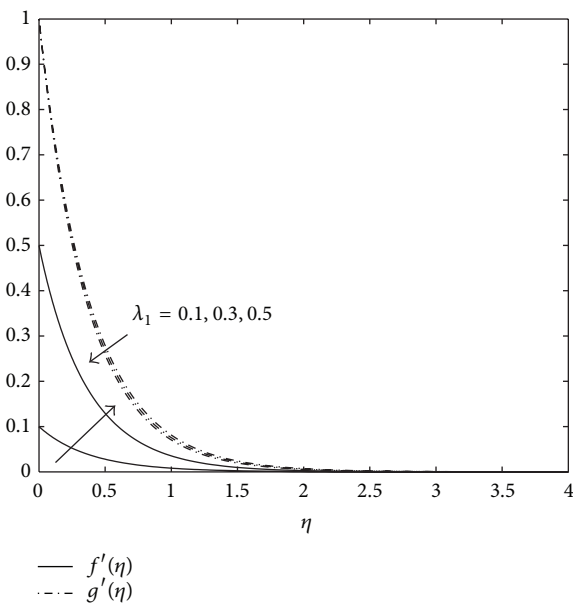

FIGURE 7: Velocity profiles for different values of λ_1 when $\lambda_2 = 1$.

in the y-direction. As thermophoresis parameter and Lewis number increase, the local Nusselt number decreases. On the other hand, the local Nusselt number increases as Brownian motion parameter increases.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

TABLE 2: The local Nusselt number for different Nb and Le when $\lambda_1 = 1$, Pr = 6.8, and Nt = 0.1.

Nb	Le	$\mathrm{Nu}_x \mathrm{Re}_x^{-1/2}$
0.10		11.689893
0.12	5	14.032429
0.14		19.445658
	5	11.689893
0.10	8	11.260672
	10	11.105908

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