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Research Article

On Some Classes of Double Difference Sequences of Interval Numbers

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The aim of this paper is to introduce some interval valued double difference sequence spaces by means of Musielak-Orlicz function $\mathscr{M} = (M_{ij})$. We also determine some topological properties and inclusion relations between these double difference sequence spaces.

1. Introduction

Interval arithmetic was first suggested by Dwyer [1] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [2] in 1959 and also by Moore and Yang [3] in 1962. Further works on interval numbers can be found in Dwyer [4] and Markov [5]. Furthermore, Moore and Yang [6] have developed applications of interval number sequences to differential equations. Chiao in [7] introduced sequences of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [8] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces. Recently, Esi in [9, 10] introduced and studied strongly almost λ-convergence and statistically almost λ-convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively (also see [11–17]).

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance, arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $\mathbb{R}$. Any elements of $\mathbb{R}$ are called closed interval and denoted by $\overline{x}$. That is, $\overline{x} = \{x \in \mathbb{R} : a \leq x \leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers [7]. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in \mathbb{R}$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}$, $x_{1_r} = x_{2_r}$. Consider $\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{x}_1 \cdot \overline{x}_2 = \left\{ \begin{array}{l} x \in \mathbb{R} : \min\left\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\right\} \\ \quad \leq x \leq \min\left\{x_{1_l} \cdot x_{2_l}, x_{1_l} \cdot x_{2_r}, x_{1_r} \cdot x_{2_l}, x_{1_r} \cdot x_{2_r}\right\} \end{array} \right\}. \quad (1)$$

In [2], Moore proved that the set of all interval numbers $\mathbb{R}$ is a complete metric space defined by $d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} \cdot x_{2_r}|\}$. In the special cases $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$. Let us define transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}, \overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called kth term of sequence $\overline{x} = (\overline{x}_k)$. We denote the set of all interval numbers with real terms as w^i. The algebraic properties of w^i can be found in [7]. Now we give the basic definitions used in this paper.

Definition 1 (see [7]). A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_0$ if for each $\epsilon > 0$ there exists a positive integer k_0 such that

$d(\overline{x}_k, \overline{x}_0) < \epsilon$ for all $k \geq k_0$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_0$. Thus, $\lim_k \overline{x}_k = \overline{x}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$.

Definition 2. A transformation f from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$ is defined by $i, j \rightarrow f(i, j) = \overline{x}, \overline{x} = (\overline{x}_{ij})$. Then $\overline{x} = (\overline{x}_{ij})$ is called sequence of double interval numbers. Then $\overline{x}_{ij}$ is called ij^{th} term of sequence $\overline{x} = (\overline{x}_{ij})$.

Definition 3. An interval valued double sequence $\overline{x} = (\overline{x}_{ij})$ is said to be convergent in Pringsheim's sense or P-convergent to an interval number $\overline{x}_0$, if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d\left(\overline{x}_{ij}, \overline{x}_0\right) < \epsilon \quad \forall i, j > N, \tag{2}$$

where $\mathbb{N}$ is the set of natural numbers, and we denote it also by $P - \lim \overline{x}_{ij} = \overline{x}_0$. The interval number $\overline{x}_0$ is called the Pringsheim limit of $\overline{x} = (\overline{x}_{ij})$.

More exactly, we say that a double sequence $\overline{x} = (\overline{x}_{ij})$ converges to a finite interval number $\overline{x}_0$ if $\overline{x}_{ij}$ tend to $\overline{x}_0$ as both i and j tend to ∞ independently of one another. We denote by $\overline{c}^2$ the set of all double convergent interval numbers of double interval numbers.

Definition 4. An interval valued double sequence $\overline{x} = (\overline{x}_{ij})$ is bounded if there exists a positive number M such that $d(\overline{x}_{ij}, \overline{x}_0) \leq M$ for all $i, j \in \mathbb{N}$. We will denote all bounded double interval number sequences by $\overline{l}^2_\infty$. It should be noted that, similar to the case of double sequences, $\overline{c}^2$ is not the subset of $\overline{l}^2_\infty$.

Definition 5. Let $A = (a_{mnij})$ denote a four-dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mnth term to Ax is as follows:

$$(Ax)_{mn} = \sum_{i,j=1,1}^{\infty,\infty} a_{mnij} x_{ij}. \tag{3}$$

Such a transformation is said to be nonnegative if a_{mnij} is nonnegative for all m, n, i and j.

The notion of difference sequence spaces was introduced by Kizmaz [18] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [19] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$, and $c_0(\Delta^n)$. Let w denote the set of all real and complex sequences and let n be a nonnegative integer; then for $Z = c, c_0$, and l_∞, we have sequence spaces

$$Z\left(\Delta^n\right) = \left\{x = \left(x_k\right) \in w : \left(\Delta^n x_k\right) \in Z\right\}, \tag{4}$$

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}. \tag{5}$$

Taking $n = 1$, we get the spaces studied by Et and Çolak [19]. For more details about sequence spaces see [20–32] and references therein. Quite recently, Et et al. [33] defined and studied the concept of statistical convergence of order α involving the notions of Δ and ideal I.

Definition 6. An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is a continuous, nondecreasing, and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function. Lindenstrauss and Tzafriri [34] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{x = \left(x_k\right) \in w : \sum_{k=1}^{\infty} M\left(\frac{\left|x_k\right|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\} \tag{6}$$

which is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\left|x_k\right|}{\rho}\right) \leq 1\right\}. \tag{7}$$

Also it was shown in [34] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x \eta(t)\, dt, \tag{8}$$

where η is known as the kernel of M and is a right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, and η is nondecreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Definition 7. A sequence $\mathscr{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [35, 36]). A sequence $\mathscr{N} = (N_k)$ is defined by

$$N_k(v) = \sup\left\{|v| u - M_k(u) : u \geq 0\right\}, \quad k = 1, 2, \ldots, \tag{9}$$

and is called the complementary function of a Musielak-Orlicz function $\mathscr{M}$. For a given Musielak-Orlicz function $\mathscr{M}$, the Musielak-Orlicz sequence space $t_{\mathscr{M}}$ and its subspace $h_{\mathscr{M}}$ are defined as follows:

$$\begin{gathered} t_{\mathscr{M}} = \left\{x \in w : I_{\mathscr{M}}(cx) < \infty \text{ for some } c > 0\right\}, \\ h_{\mathscr{M}} = \left\{x \in w : I_{\mathscr{M}}(cx) < \infty\ \forall c > 0\right\}, \end{gathered} \tag{10}$$

where $I_{\mathscr{M}}$ is a convex modular defined by

$$I_{\mathscr{M}}(x) = \sum_{k=1}^{\infty} M_k\left(x_k\right), \quad x = \left(x_k\right) \in t_{\mathscr{M}}. \tag{11}$$

We consider $t_{\mathscr{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_{\mathscr{M}}\left(\frac{x}{k}\right) \leq 1\right\} \tag{12}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0\right\}. \quad (13)$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-condition if there exist constants $a,\ K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \le KM_k(u) + c_k \quad (14)$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \le a$.

Definition 8. Let X be a linear metric space. A function p: $X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \ge 0$ for all $x \in X$;

(2) $p(-x) = p(x)$ for all $x \in X$;

(3) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$;

(4) (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

Let $\mathcal{M} = (M_{ij})$ be a Musielak-Orlicz function and let $A = (a_{mnij})$ be a nonnegative four-dimensional bounded regular matrix (see [37, 38]). Let $p = (p_{ij})$ be a bounded double sequence of positive real numbers and $u = (u_{ij})$ be a double sequence of strictly positive real numbers. In the present paper we define the following new double sequence spaces for interval sequences:

$$\begin{aligned}
&{}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A)\\
&= \Bigg\{\overline{x} = (\overline{x}_{ij}) : P - \lim_{mn}\frac{1}{mn} \times \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} = 0, \text{ for some } \rho > 0\Bigg\},\\
&{}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r, A)\\
&= \Bigg\{\overline{x} = (\overline{x}_{ij}) : P - \lim_{mn}\frac{1}{mn} \times \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} = 0, \text{ for some } \rho > 0\Bigg\},\\
&{}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r, A)\\
&= \Bigg\{\overline{x} = (\overline{x}_{ij}) : \sup_{mn}\frac{1}{mn} \times \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} < \infty, \text{ for some } \rho > 0\Bigg\}.
\end{aligned} \quad (15)$$

Remark 9. Let us consider a few special cases of the above sequence spaces.

(i) If $\mathcal{M} = M_{ij}(x) = x$ for all $i, j \in \mathbb{N}$, then we have

$$\begin{aligned}
{}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}(p, u, \Delta^r, A),\\
{}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_0(p, u, \Delta^r, A),\\
{}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_\infty(p, u, \Delta^r, A).
\end{aligned} \quad (16)$$

(ii) If $p = (p_{ij}) = 1$, for all i, j, then we have

$$\begin{aligned}
{}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}(\mathcal{M}, u, \Delta^r, A),\\
{}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_0(\mathcal{M}, u, \Delta^r, A),\\
{}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_\infty(\mathcal{M}, u, \Delta^r, A).
\end{aligned} \quad (17)$$

(iii) If $u = (u_{ij}) = 1$, for all i, j, then we have

$$\begin{aligned}
{}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}(\mathcal{M}, p, \Delta^r, A),\\
{}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_0(\mathcal{M}, p, \Delta^r, A),\\
{}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_\infty(\mathcal{M}, p, \Delta^r, A).
\end{aligned} \quad (18)$$

(iv) If $A = (C, 1, 1) = 1$, that is, the double Cesàro matrix, then the above classes of sequences reduce to the following sequence spaces:

$$\begin{aligned}
{}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}(\mathcal{M}, p, u, \Delta^r),\\
{}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_0(\mathcal{M}, p, u, \Delta^r),\\
{}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r, A) &= {}_2\overline{w}_\infty(\mathcal{M}, p, u, \Delta^r).
\end{aligned} \quad (19)$$

(v) Let $A = (C, 1, 1) = 1$ and $u_{ij} = 1$ for all i, j. If, in addition, $\mathscr{M}(x) = M(x)$ and $r = 0$, then the spaces ${}_2\overline{w}(\mathscr{M}, p, u, \Delta^r, A)$, ${}_2\overline{w}_0(\mathscr{M}, p, u, \Delta^r, A)$, and ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$ are reduced to ${}_2\overline{w}(M, p)$, ${}_2\overline{w}_0(M, p)$, and ${}_2\overline{w}_\infty(M, p)$ which were introduced and studied by Esi and Hazarika [39].

The following inequality will be used throughout the paper. If $0 \le p_{ij} \le \sup p_{ij} = H$, $K = \max(1, 2^{H-1})$ then

$$\left|a_{ij} + b_{ij}\right|^{p_{ij}} \le K\left(\left|a_{ij}\right|^{p_{ij}} + \left|b_{ij}\right|^{p_{ij}}\right) \tag{20}$$

for all i, j and $a_{ij}, b_{ij} \in \mathbb{C}$. Also $|a|^{p_{ij}} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce interval valued double difference sequence spaces ${}_2\overline{w}(\mathscr{M}, p, u, \Delta^r, A)$, ${}_2\overline{w}_0(\mathscr{M}, p, u, \Delta^r, A)$, and ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$ and to study different properties of these spaces like linearity, paranorm, solidity, monotone, and so forth. Some inclusion relations between theses spaces are also established.

2. Main Results

Theorem 10. *If $0 < p_{ij} < q_{ij}$ for each i and j, then we have ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A) \subset {}_2\overline{w}_\infty(\mathscr{M}, q, u, \Delta^r, A)$.*

Proof. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$. Then there exists $\rho > 0$ such that

$$\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} < \infty. \tag{21}$$

This implies that

$$a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} < 1, \tag{22}$$

for sufficiently large values of i and j. Since M_{ij} is nondecreasing, we get

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{q_{ij}} \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} < \infty.
\end{aligned} \tag{23}$$

Thus $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, q, u, \Delta^r, A)$. This completes the proof. □

Theorem 11. *Suppose that $\mathscr{M} = (M_{ij})$ is a Musielak-Orlicz function, $p = (p_{ij})$ a bounded double sequence of positive real numbers, and $u = (u_{ij})$ a double sequence of strictly positive real numbers. Then the following hold.*

(i) *If $0 < \inf p_{ij} < p_{ij} \le 1$, then ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A) \subset {}_2\overline{w}_\infty(\mathscr{M}, u, \Delta^r, A)$.*

(ii) *If $1 \le p_{ij} \le \sup p_{ij} < \infty$, then ${}_2\overline{w}_\infty(\mathscr{M}, u, \Delta^r, A) \subset {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$.*

Proof. (i) Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$. Since $0 < \inf p_{ij} \le 1$, we obtain the following:

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right] \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} < \infty,
\end{aligned} \tag{24}$$

and hence $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, u, \Delta^r, A)$.

(ii) Let $p_{ij} \ge 1$ for each i and j and $\sup p_{ij} < \infty$. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, u, \Delta^r, A)$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right] \\
&\quad \le \epsilon < 1 \quad \forall n, m \ge N.
\end{aligned} \tag{25}$$

This implies that

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right]^{p_{ij}} \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho}\right)\right] < \infty.
\end{aligned} \tag{26}$$

Therefore, $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$. This completes the proof. □

Theorem 12. *Let $0 < p_{ij} \le q_{ij}$ for all $i, j \in \mathbb{N}$ and (q_{ij}/p_{ij}) be bounded. Then we have ${}_2\overline{w}_\infty(\mathscr{M}, q, u, \Delta^r, A) \subset {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$.*

Proof. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, q, u, \Delta^r, A)$. Then

$$\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r x_{ij}, \overline{0}\right)}{\rho}\right)\right]^{q_{ij}} < \infty, \quad \text{for some } \rho > 0. \tag{27}$$

Let $s_{ij} = \sup_{mn}(1/mn)\sum_{i,j=1,1}^{m,n} a_{mnij}[M_{ij}(u_{ij} d(\Delta^r \overline{x}_{ij}, \overline{0})/\rho)]^{q_{ij}}$ and $\lambda_{ij} = p_{ij}/q_{ij}$. Since $p_{ij} \le q_{ij}$, we have $0 \le \lambda_{ij} \le 1$. Take $0 < \lambda < \lambda_{ij}$.

Define

$$\begin{aligned}
u_{ij} &= \begin{cases} s_{ij} & \text{if } s_{ij} \ge 1 \\ 0 & \text{if } s_{ij} < 1, \end{cases} \\
v_{ij} &= \begin{cases} 0 & \text{if } s_{ij} \ge 1 \\ s_{ij} & \text{if } s_{ij} < 1, \end{cases}
\end{aligned} \tag{28}$$

$s_{ij} = u_{ij} + v_{ij}, s_{ij}^{\lambda_{ij}} = u_{ij}^{\lambda_{ij}} + v_{ij}^{\lambda_{ij}}$. It follows that $u_{ij}^{\lambda_{ij}} \le u_{ij} \le s_{ij}$, $v_{ij}^{\lambda_{ij}} \le v_{ij}^{\lambda}$. since $s_{ij}^{\lambda_{ij}} = u_{ij}^{\lambda_{ij}} + v_{ij}^{\lambda_{ij}}$, then $s_{ij}^{\lambda_{ij}} \le s_{ij} + v_{ij}^{\lambda}$

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right)^{q_{ij}} \right]^{\lambda_{ij}} \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{q_{ij}} \\
&\quad \Longrightarrow \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right)^{q_{ij}} \right]^{p_{ij}/q_{ij}} \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{q_{ij}} \\
&\quad \Longrightarrow \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&\quad \le \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{q_{ij}},
\end{aligned} \tag{29}$$

but

$$\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{q_{ij}} < \infty \quad \text{for some } \rho > 0. \tag{30}$$

Therefore,

$$\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} < \infty \quad \text{for some } \rho > 0. \tag{31}$$

Hence $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{W}_{\infty}(\mathscr{M}, p, u, \Delta^r, A)$. Thus, we get ${}_2\overline{W}_{\infty}(\mathscr{M}, q, u, \Delta^r, A) \subset {}_2\overline{w}_{\infty}(\mathscr{M}, p, u, \Delta^r, A)$. □

Theorem 13. *Let $\mathscr{M}' = (M'_{ij})$ and $\mathscr{M}'' = (M''_{ij})$ be two Musielak-Orlicz functions,*

$$\begin{aligned}
&{}_2\overline{w}_{\infty}\left(\mathscr{M}', p, u, \Delta^r, A\right) \cap {}_2\overline{w}_{\infty}\left(\mathscr{M}'', p, u, \Delta^r, A\right) \\
&\quad \subset {}_2\overline{w}_{\infty}\left(\mathscr{M}' + \mathscr{M}'', p, u, \Delta^r, A\right).
\end{aligned} \tag{32}$$

Proof. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_{\infty}(\mathscr{M}', p, u, \Delta^r, A) \cap {}_2\overline{w}_{\infty}(\mathscr{M}'', p, u, \Delta^r, A)$. Then

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M'_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_1} \right) \right]^{p_{ij}} < \infty, \\
&\qquad\qquad \text{for some } \rho_1 > 0, \\
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M''_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_2} \right) \right]^{p_{ij}} < \infty, \\
&\qquad\qquad \text{for some } \rho_2 > 0.
\end{aligned} \tag{33}$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[\left(M'_{ij} + M''_{ij} \right) \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&\quad = \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M'_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_1} \right) \right]^{p_{ij}} \\
&\qquad + \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M''_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_2} \right) \right]^{p_{ij}} \\
&\quad \le K \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M'_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_1} \right) \right]^{p_{ij}} \\
&\qquad + K \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M''_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_2} \right) \right]^{p_{ij}} \\
&\quad < \infty.
\end{aligned} \tag{34}$$

Thus, $\sup_{mn}(1/mn) \sum_{i,j=1,1}^{m,n} a_{mnij}[(M'_{ij} + M''_{ij})(u_{ij} d(\Delta^r \overline{y}_{ij}, \overline{0})/\rho)]^{p_{ij}} < \infty$. Therefore, $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_{\infty}(\mathscr{M}' + \mathscr{M}'', p, u, \Delta^r, A)$. □

Theorem 14. *Let $\mathscr{M} = (M_{ij})$ be a Musielak-Orlicz function and let $A = (a_{nmij})$ be a nonnegative four-dimensional regular summability method. Suppose that $\beta = \lim_{t \to \infty}(M_{ij}(t)/t) < \infty$. Then ${}_2\overline{w}(p, u, \Delta^r, A) = {}_2\overline{w}(\mathscr{M}, p, u, \Delta^r, A)$.*

Proof. In order to prove that ${}_2\overline{W}(p, u, \Delta^r, A) = {}_2\overline{w}(\mathscr{M}, p, u, \Delta^r, A)$, it is sufficient to show that ${}_2\overline{W}(\mathscr{M}, p, u, \Delta^r, A) \subset {}_2\overline{w}(p, u, \Delta^r, A)$. Now, let $\beta > 0$. By definition of β, we have $M_{ij}(t) \ge \beta t$ for all $t \ge 0$. Since $\beta > 0$, we have $t \le (1/\beta) M_{ij}(t)$ for all $t \ge 0$. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}(\mathscr{M}, p, u, \Delta^r, A)$. Thus, we have

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho} \right) \right]^{p_{ij}} \\
&\quad \le \frac{1}{\beta} \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho} \right) \right]^{p_{ij}} < \infty
\end{aligned} \tag{35}$$

which implies that $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}(p, u, \Delta^r, A)$. This completes the proof. □

Theorem 15. *Let $0 < h = \inf p_{ij} \le p_{ij} \le \sup p_{ij} = H < \infty$. Then for a Musielak-Orlicz function $\mathcal{M} = (M_{ij})$ which satisfies the Δ_2-condition, we have ${}_2\overline{w}(p, u, \Delta^r, A) = {}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A)$.*

Proof. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}(p, u, \Delta^r, A)$; that is,

$$\frac{1}{mn}\sum_{i,j=1,1}^{m,n} a_{mnij}\left[\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} = 0, \quad \text{for some } \rho > 0. \tag{36}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{ij}(t) < \epsilon$ for $0 \le t \le \delta$. Then

$$\begin{aligned}
&\frac{1}{mn}\sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} \\
&\quad = \frac{1}{mn}\sum_{\substack{i,j=1,1 \\ d(\Delta^r \overline{x}_{ij}, \overline{x}_0) \le \delta}}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} \\
&\qquad + \frac{1}{mn}\sum_{\substack{i,j=1,1 \\ d(\Delta^r \overline{x}_{ij}, \overline{x}_0) > \delta}}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} \\
&\quad = \sum_1 + \sum_2,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\sum_1 &= \frac{1}{mn}\sum_{\substack{i,j=1,1 \\ d(\Delta^r \overline{x}_{ij}, \overline{x}_0) \le \delta}}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right]^{p_{ij}} \\
&< \max\left(\epsilon, \epsilon^H\right)
\end{aligned} \tag{38}$$

by using continuity of (M_{ij}). For the second summation, we will make the following procedure. Thus we have

$$\frac{d\left(\Delta^r x_{ij}, \overline{x}_0\right)}{\rho} < 1 + \frac{d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}. \tag{39}$$

Since $\mathcal{M} = (M_{ij})$ is nondecreasing and convex, so we have

$$\begin{aligned}
&a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right] \\
&\quad < a_{mnij}\left[M_{ij}\left\{1 + \frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}\right\}\right] \\
&\quad \le \frac{1}{2} a_{mnij}\left[\left(u_{ij}\right) M_{ij}(2)\right] \\
&\qquad + \frac{1}{2} a_{mnij}\left[M_{ij}\left\{2\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}\right\}\right].
\end{aligned} \tag{40}$$

Again, since $\mathcal{M} = (M_{ij})$ satisfies the Δ_2-condition, it follows that

$$\begin{aligned}
&a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right)\right] \\
&\quad \le \frac{1}{2} K\left\{\frac{d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}\right\} a_{mnij}\left[\left(u_{ij}\right) M_{ij}(2)\right] \\
&\qquad + \frac{1}{2} K\left\{\frac{d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}\right\} a_{mnij}\left[\left(u_{ij}\right) M_{ij}(2)\right] \\
&\quad = K\left\{\frac{d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)/\rho}{\delta}\right\} a_{mnij}\left[\left(u_{ij}\right) M_{ij}(2)\right].
\end{aligned} \tag{41}$$

Thus, it follows that

$$\begin{aligned}
\sum_2 = {} & \max\left\{1, \left[\frac{K a_{mnij}\left[\left(u_{ij}\right) M_{ij}(2)\right]}{\delta}\right]^H\right\} \\
&\times \frac{1}{mn}\sum_{i,j=1,1}^{m,n}\left[\frac{d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho}\right]^{p_{ij}}.
\end{aligned} \tag{42}$$

Taking the limit as $\epsilon \to 0$ and $m, n \to \infty$, it follows that $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A)$. □

Theorem 16. *Suppose that $\mathcal{M} = (M_{ij})$ is a Musielak-Orlicz function, $p = (p_{ij})$ a bounded double sequence of positive real numbers, and $u = (u_{ij})$ a double sequence of strictly positive real numbers. If $\sup_{i,j}(M_{ij}(x))^{p_{ij}} < \infty$ for all fixed $x > 0$, then*

$${}_2\overline{w}\left(\mathcal{M}, p, u, \Delta^r, A\right) \subset {}_2\overline{w}_\infty\left(\mathcal{M}, p, u, \Delta^r, A\right). \tag{43}$$

Proof. Let $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}(\mathcal{M}, p, u, \Delta^r, A)$. Then there exists a positive number $\rho_1 > 0$ such that

$$\frac{1}{mn}\sum_{i,j=1,1}^{m,n} a_{mnij}\left[M_{ij}\left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho_1}\right)\right]^{p_{ij}} = 0, \quad \text{for some } \rho_1 > 0. \tag{44}$$

Define $\rho = 2\rho_1$. Since $\mathscr{M} = (M_{ij})$ is nondecreasing and convex, for each i, j, so by using (20), we have

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&\leq \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right) + d\left(\overline{x}_0, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&\leq K \left\{ \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{x}_0\right)}{\rho_1} \right) \right]^{p_{ij}} \right. \\
&\qquad \left. + \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho_1} \right) \right]^{p_{ij}} \right\} \\
&< \infty.
\end{aligned} \tag{45}$$

Thus $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$. This completes the proof of the theorem. □

Theorem 17. *The double sequence space* ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$ *is solid.*

Proof. Suppose $\overline{x} = (\overline{x}_{ij}) \in {}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$

$$\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} < \infty, \quad \text{for some } \rho > 0. \tag{46}$$

Let (α_{ij}) be a double sequence of scalars such that $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then we get

$$\begin{aligned}
&\sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \alpha_{ij} \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&\leq \sup_{mn} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} a_{mnij} \left[M_{ij} \left(\frac{u_{ij} d\left(\Delta^r \overline{x}_{ij}, \overline{0}\right)}{\rho} \right) \right]^{p_{ij}} \\
&< \infty.
\end{aligned} \tag{47}$$

This completes the proof. □

Theorem 18. *The double sequence space* ${}_2\overline{w}_\infty(\mathscr{M}, p, u, \Delta^r, A)$ *is monotone.*

Proof. The proof is trivial so we omit it. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] P. S. Dwyer, *Linear Computation*, Wiley, New York, NY, USA, 1951.

[2] R. E. Moore, *Automatic Error Analysis in Digital Computation, LSMD-48421*, Lockheed Missiles and Space Company, 1959.

[3] R. E. Moore and C. T. Yang, "Interval analysis I," Tech. Rep. LMSD-285875, Lockheed Missiles and Space Company, 1962.

[4] P. S. Dwyer, "Errors of matrix computation, simultaneous equations and eigenvalues, National Bureu of Standarts," *Applied Mathematics Series*, vol. 29, pp. 49–58, 1953.

[5] S. Markov, "Quasilinear spaces and their relation to vector spaces," *Electronic Journal on Mathematics of Computation*, vol. 2, no. 1, pp. 1–21, 2005.

[6] R. E. Moore and C. T. Yang, *Theory of an Interval Algebra and Its Application to Numeric Analysis*, RAAG Memories II, Gaukutsu Bunken Fukeyu-kai, Tokyo, Japan, 1958.

[7] K. P. Chiao, "Fundamental properties of interval vector max-norm," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 18, no. 2, pp. 219–233, 2002.

[8] M. Şengönül and A. Eryilmaz, "On the sequence spaces of interval numbers," *Thai Journal of Mathematics*, vol. 8, no. 3, pp. 503–510, 2010.

[9] A. Esi, "Strongly almost λ-convergence and statistically almost λ-convergence of interval numbers," *Scientia Magna*, vol. 7, no. 2, pp. 117–122, 2011.

[10] A. Esi, "Lacunary sequence spaces of interval numbers," *Thai Journal of Mathematics*, vol. 10, no. 2, pp. 445–451, 2012.

[11] A. Esi and B. Hazarika, "On interval valued generalized difference classes defined by Orlicz function," *Turkish Journal of Analysis and Number Theory*, vol. 1, no. 1, pp. 48–53, 2013.

[12] A. Esi and N. Braha, "On asymptotically λ-statistical equivalent sequences of interval numbers," *Acta Scientiarum Technology*, vol. 35, no. 3, pp. 515–520, 2013.

[13] A. Esi and A. Esi, "Asymptotically lacunary statistically equivalent sequences of interval numbers," *International Journal of Applied Mathematics*, vol. 1, no. 1, pp. 43–48, 2013.

[14] A. Esi, "λ-sequence spaces of interval numbers," *Applied Mathematics & Information Sciences*, vol. 8, no. 3, pp. 1099–1102, 2014.

[15] A. Esi, "Double sequences of interval numbers defined by Orlicz functions," *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol. 17, no. 1, pp. 57–64, 2013.

[16] A. Esi and S. Yasemin, "Some spaces of sequences of interval numbers defined by a modulus function," *Global Journal of Mathematical Analysis*, vol. 2, no. 1, pp. 11–16, 2014.

[17] A. Esi, "Double lacunary sequence spaces of double sequence of interval numbers," *Proyecciones. Journal of Mathematics*, vol. 31, no. 3, pp. 299–308, 2012.

[18] H. Kizmaz, "On certain sequence spaces," *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

[19] M. Et and R. Çolak, "On some generalized difference sequence spaces," *Soochow Journal of Mathematics*, vol. 21, no. 4, pp. 377–386, 1995.

[20] M. Et, "Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 9372–9376, 2013.

[21] M. Et, "Spaces of Cesàro difference sequences of order r defined by a modulus function in a locally convex space," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 865–879, 2006.

[22] M. Et, Y. Altin, B. Choudhary, and B. C. Tripathy, "On some classes of sequences defined by sequences of Orlicz functions," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 335–342, 2006.

[23] V. Karakaya, E. Savas, and H. Polat, "Some paranormed Euler sequence spaces of difference sequences of order m," *Mathematica Slovaca*, vol. 63, no. 4, pp. 849–862, 2013.

[24] S. A. Mohiuddine and A. Alotaibi, "Some spaces of double sequences obtained through invariant mean and related concepts," *Abstract and Applied Analysis*, vol. 2013, Article ID 507950, 11 pages, 2013.

[25] S. A. Mohiuddine, K. Raj, and A. Alotaibi, "Some paranormed double difference sequence spaces for Orlicz functions and bounded-regular matrices," *Abstract and Applied Analysis*, vol. 2014, Article ID 419064, 10 pages, 2014.

[26] M. Mursaleen and S. A. Mohiuddine, "Some new double sequence spaces of invariant means," *Glasnik Matematički*, vol. 45, no. 1, pp. 139–153, 2010.

[27] M. Mursaleen, S. K. Sharma, and A. Kiliçman, "Sequence spaces defined by Musielak-Orlicz function over n-normed spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 364743, 10 pages, 2013.

[28] B. C. Tripathy, "Generalized difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex space," *Soochow Journal of Mathematics*, vol. 30, no. 4, pp. 431–446, 2004.

[29] K. Raj and S. K. Sharma, "Some difference sequence spaces defined by Musielak-Orlicz functions," *Mathematica Pannonica*, vol. 24, pp. 33–43, 2013.

[30] K. Raj and S. K. Sharma, "Some multiplier double sequence spaces," *Acta Mathematica Vietnamica*, vol. 37, no. 3, pp. 391–405, 2012.

[31] S. D. Parashar and B. Choudhary, "Sequence spaces defined by Orlicz functions," *Indian Journal of Pure and Applied Mathematics*, vol. 25, no. 4, pp. 419–428, 1994.

[32] S. Simons, "The sequence spaces $l(p_v)$ and $m(p_v)$," *Proceedings of the London Mathematical Society*, vol. 15, no. 3, pp. 422–436, 1965.

[33] M. Et, A. Alotaibi, and S. A. Mohiuddine, "On $(\triangle^m, I)$-statistical convergence of order α," *The Scientific World Journal*, vol. 2014, Article ID 535419, 5 pages, 2014.

[34] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.

[35] L. Maligranda, *Orlicz Spaces and Interpolation, Seminars in Mathematics*, vol. 5, Polish Academy of Science, Warsaw, Poland, 1989.

[36] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.

[37] H. J. Hamilton, "Transformations of multiple sequences," *Duke Mathematical Journal*, vol. 2, no. 1, pp. 29–60, 1936.

[38] G. M. Robison, "Divergent double sequences and series," *Transactions of the American Mathematical Society*, vol. 28, no. 1, pp. 50–73, 1926.

[39] A. Esi and B. Hazarika, "Some double sequence spaces of interval numbers defined by Orlicz functions," *Journal of the Egyptian Mathematical Society*, 2014.