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*Research Article*

# The Wigner-Ville Distribution Based on the Linear Canonical Transform and Its Applications for QFM Signal Parameters Estimation

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The Wigner-Ville distribution (WVD) based on the linear canonical transform (LCT) (WDL) not only has the advantages of the LCT but also has the good properties of WVD. In this paper, some new and important properties of the WDL are derived, and the relationships between WDL and some other time-frequency distributions are discussed, such as the ambiguity function based on LCT (LCTAF), the short-time Fourier transform (STFT), and the wavelet transform (WT). The WDLs of some signals are also deduced. A novel definition of the WVD based on the LCT and generalized instantaneous autocorrelation function (GWDL) is proposed and its applications in the estimation of parameters for QFM signals are also discussed. The GWDL of the QFM signal generates an impulse and the third-order phase coefficient of QFM signal can be estimated in accordance with the position information of such impulse. The proposed algorithm is fast because it only requires 1-dimensional maximization. Also the new algorithm only has fourth-order nonlinearity thus it has accurate estimation and low signal-to-noise ratio (SNR) threshold. The simulation results are provided to support the theoretical results.

## 1. Introduction

The main elements of the modern signal processing are nonstationary, non-Gaussian, and nonlinear signals. Among these signals, the development of the nonstationary signal processing theory is especially remarkable. There are many time-frequency analysis tools for nonstationary signals, such as short-time Fourier transform (STFT), fractional Fourier transform (FRFT), Gabor transform (GT), Wigner-Ville distribution function (WVD), ambiguity function (AF), linear canonical transform (LCT), and so forth [1]. The WVD is regarded as the mother of all the time-frequency distribution and has become an important distribution in signal analysis and processing, especially in the nonstationary signal analysis and processing [2–4]. The LCT as the generalization of the Fourier transform (FT) and the FRFT was first introduced by Moshinsky and Quesne [5] and Collins and Stuart [6]. Now it has been applied for filter designing, time-frequency signal separating, signal synthesis, and signal encryption [7–9].

The quadratic frequency modulated (QFM) signal exists widely in nature and is an important nonlinear module in the signal processing field. It is applied widely in radar, sonar, speech, and communication fields, mostly in radar systems [10]. There are many algorithms for estimating the parameters of QFM signal, such as the maximum likelihood (ML) method [11], the adaptive short-time Fourier transform method [12], the polynomial Wigner-Ville distributions (PWVDs) [13], the product high-order matched-phase transform (PHMT) [14], and the ambiguity function based on the LCT method (LCTAF) [15]. Because the ML method and LCTAF method need 3-dimensional (3D) and 2D maximizations, respectively, these methods suffer from computational burden. The adaptive STFT method has lower resolution. The PWVDs and PHMT algorithms need high order of nonlinearity (sixth-order to be exact) and this leads to high signal-to-noise ratio (SNR) threshold. Therefore, methods to estimate the QFM signal parameters quickly and accurately are still an important issue to be solved.

In [15, 16], Tao et al. and Bai et al. have defined the Wigner-Ville distribution based on linear canonical transform (WDL) separately. For the WDL, Bai has derived some properties and used them to detect the linear frequency modulated (LFM) signal. The WDL is a new and important signal processing tool, but they have not discussed the WDL in depth enough. In this paper we deduce some new properties of WDL and investigate the relationship between WDL and other transforms. We also derive WDLs of some common signals. In order to estimate QFM signal parameters, we define a new kind of Wigner-Ville distribution—the generalized Wigner-Ville distribution based on the linear canonical transform (GWDL). The GWDL algorithm just needs 1D maximization, so the amount of calculations is smaller compared to ML method and LCTAF method. And the new algorithm only needs fourth-order nonlinearity, so it has lower SNR threshold than the PHMT algorithm and PWVDs algorithm. The simulation results are provided to support the theoretical results.

The remainder of this paper is organized as follows. Section 2 reviews the preliminaries about the WVD and the LCT. In Section 3, some new properties of WDL are deduced and the relationship between WDL and other transforms is investigated. The GWDL is defined and its application to QFM signal parameter estimation is illustrated in Section 4. Finally, Section 5 gives the conclusion.

## 2. Preliminary

*2.1. The Winger-Ville Distribution.* The instantaneous autocorrelation function $k_f(t,\tau)$ of signal $f(t)$ is

$$k_f(t,\tau) = f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right). \tag{1}$$

The WVD of $f(t)$ is defined as the Fourier transform of $k_f(t,\tau)$ for $\tau$:

$$\begin{aligned} W_f(t,\omega) &= \int_{-\infty}^{+\infty} k_f(t,\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau} d\tau. \end{aligned} \tag{2}$$

Another definition of WVD is

$$W_F(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F\left(\omega+\frac{\nu}{2}\right) F^*\left(\omega-\frac{\nu}{2}\right) e^{j\nu t} d\nu, \tag{3}$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

The WVD has many important properties, such as conjugation symmetry property, time marginal property, and energy distribution property. For more results about the WVD, one can refer to [17–19].

*2.2. Linear Canonical Transform (LCT).* The LCT of a signal $f(t)$ with parameter matrix $A = (a,b,c,d)$ is defined as

$$\begin{aligned} F_A(u) &= L_A[f](u) \\ &= \begin{cases} \int_{-\infty}^{+\infty} f(t) K_A(u,t)\, dt, & b \neq 0, \\ \sqrt{d} e^{(jcd/2)u^2} f(du), & b = 0, \end{cases} \end{aligned} \tag{4}$$

where the kernel function $K_A(u,t)$ is

$$K_A(u,t) = \frac{1}{\sqrt{j2\pi b}} \exp\left(j\frac{d}{2b}u^2 - j\frac{ut}{b} + j\frac{a}{2b}t^2\right) \tag{5}$$

and parameters $a,b,c,d \in R$ and satisfy $ad - bc = 1$. The LCT has additive property

$$L_{(a_2,b_2,c_2,d_2)}\left[L_{(a_1,b_1,c_1,d_1)}[f(t)]\right] = L_{(e,f,g,h)}[f(t)], \tag{6}$$

where $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$, and reversible property

$$L_A\left[L_{A^{-1}}[f(t)]\right] = f(t). \tag{7}$$

Other properties of LCT such as sampling and discretization, uncertainty principles, product and convolution theorems, and Hilbert Transform are discussed in detail in [20–23].

When the parameter matrix $A$ is with some special cases, the LCT reduces to FT, FRFT, Fresnel transform, and scaling operation [24]. So the LCT is the generalization of these transforms. From (4) we can see that when the parameter $b = 0$, the LCT is a scaling transform operation multiplying a linear frequency modulation signal, so we suppose that $b \neq 0$ in the following discussion.

## 3. The Wigner-Ville Distribution Based on Linear Canonical Transform

In [16], the authors have defined the WDL according to the actual needs, but they have missed some important properties. In this section, we deduce some new properties of WDL and investigate the relationship between WDL and other transforms.

*3.1. The Definition of WDL*

*Definition 1.* Keeping the instantaneous autocorrelation function $k_f(t,\tau)$ unchanged and replacing the kernel function $e^{-j\omega\tau}$ of FT by the kernel function $K_A(u,\tau)$ of LCT in the WVD definition, the WDL is defined as [15]

$$\mathrm{WD}_f^A(t,u) = \int_{-\infty}^{+\infty} k_f(t,\tau) K_A(u,\tau)\, d\tau, \tag{8}$$

where the $k_f(t,\tau)$ is given by (1) and $K_A(u,\tau)$ is shown by (5).

Obviously, when $A = (0,1,-1,0)$, the LCT reduces to FT. Accordingly, the WDL reduces to classical WVD:

$$\mathrm{WD}_f^A(t,u) = \sqrt{\frac{1}{j2\pi}} W_f(t,u). \tag{9}$$

When $A = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$,

$$\mathrm{WD}_f^A(t,u) = \sqrt{e^{-j\alpha}}W_f^\alpha(t,u). \tag{10}$$

From (9) and (10) one can see that the WDL is a generalization of WVD based on the Fourier transform and the fractional Fourier transform.

We know that some nonbandlimited signals in the classical Fourier domain, especially some nonstationary signals and non-Gaussian signals in the Fourier domain, can be bandlimited in the LCT domain [20]. The WVD can be seen as a FT of the instantaneous autocorrelation function and the WDL can be seen as a LCT of the instantaneous autocorrelation function. So signals which are nonbandlimited after WVD may be bandlimited in the WDL domain, and then we can use bandlimited theory to process this kind of signals. The traditional nonbandlimited signals processing problems in the Fourier domain can be solved in the LCT domain. This is one of the reasons why we discuss the WVD based on the LCT.

The cross-WVD is the FT of crosscorrelation function for $\tau$. It has more information and making full use of this information can improve signal processing ability. So we provide the definition of the cross-WVD based on the LCT here.

*Definition 2.* The cross-WVD based on LCT of signal $f(t)$ and signal $g(t)$ is defined as

$$\mathrm{WD}_{f,g}^A(t,u) = \int_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) g^*\left(t-\frac{\tau}{2}\right) K_A(u,\tau)\,d\tau. \tag{11}$$

*3.2. Basic Properties of WDL.* In this subsection some new properties of WDL which are different from those in [16] are investigated and the proofs of some complex properties will be given in detail. All the properties are based on the fact that the WDL of $f(t)$ is $\mathrm{WD}_f^A(t,u)$.

*Property 1* (symmetry and conjugation property). The WDL of signal $f^*(-t)$ is

$$\mathrm{WD}_{f^*(-t)}^A(t,u) = \left[\mathrm{WD}_f^{A^{-1}}(-t,u)\right]^*. \tag{12}$$

*Property 2* (shifting properties). The WDL of signal $f(t)e^{j\omega t^2}$ is

$$\mathrm{WD}_{f(t)e^{j\omega t^2}}^A(t,u) = e^{j2d\omega tu - j2bd\omega^2\tau^2}\mathrm{WD}_f^A(t,u-2b\omega t). \tag{13}$$

The WDL of signal $f(t-t_0)e^{j\omega t}$ is

$$\begin{aligned}\mathrm{WD}_{f(t-t_0)e^{j\omega t}}^A(t,u) &= e^{j(ud\omega - bd\omega^2/2)}\\ &\quad\times \mathrm{WD}_f^A(t-t_0, u-\omega_1 b).\end{aligned} \tag{14}$$

The WDL of signal $g(t) = f(t)e^{j\omega_1 t + j\omega_2 t^2}$ is

$$\begin{aligned}\mathrm{WD}_{g(t)}^A(t,u) &= e^{j(ud(\omega_1+2\omega_2 t) - bd(\omega_1+2\omega_2 t)^2/2)}\\ &\quad\times \mathrm{WD}_f^A(t, u-\omega_1 b - 2b\omega_2 t).\end{aligned} \tag{15}$$

Especially when $f(t) = 1$, $g(t) = e^{j\omega_1 t + j\omega_2 t^2}$ is the LFM signal, and its WDL is

$$\mathrm{WD}_{g(t)}^A(t,u) = \begin{cases}\sqrt{\dfrac{2\pi b}{j}}e^{j(d/2b)u^2}\delta(2\omega_2 bt + \omega_1 b - u), & a = 0,\\[2ex] \sqrt{\dfrac{1}{a}}e^{j(d/2b)u^2 - j((u-\omega_1 b - 2b\omega_2 t)^2/2ab)}, & a \neq 0.\end{cases} \tag{16}$$

*Proof.* Firstly, a useful formula is given below that will be used in this paper:

$$\int_{-\infty}^{+\infty} e^{(-Ot^2 \pm 2Pt + Q)}dt = \sqrt{\frac{\pi}{O}}e^{(P^2/O + Q)}, \tag{17}$$

where $O, P, Q \in \mathbb{C}$, $O \neq 0$ and $\mathrm{Re}(O) \geq 0$.

Equation (15) is easy to get, so we only prove (16):

$$\begin{aligned}\mathrm{WD}_{g(t)}^A(t,u) &= \int_{-\infty}^{+\infty} g\left(t+\frac{\tau}{2}\right) g^*\left(t+\frac{\tau}{2}\right) K_A(u,\tau)\,d\tau\\ &= \sqrt{\frac{1}{j2\pi b}}e^{j(d/2b)u^2}\\ &\quad\times \int_{-\infty}^{+\infty} e^{j(\omega_1 + 2\omega_2 t - u/b)\tau - (-j(a/2b))\tau^2}d\tau.\end{aligned} \tag{18}$$

So $O = -j(a/2b)$, $P = (\omega_1 b - u)/2b + \omega_2 t$.

When $a = 0$, (18) is

$$\begin{aligned}\mathrm{WD}_{g(t)}^A(t,u) &= \sqrt{\frac{1}{j2\pi b}}e^{j(d/2b)u^2}\int_{-\infty}^{+\infty} e^{j(\omega_1 + 2\omega_2 t - u/b)\tau}d\tau\\ &= \sqrt{\frac{2\pi b}{j}}e^{j(d/2b)u^2}\delta(2\omega_2 tb + \omega_1 b - u).\end{aligned} \tag{19}$$

When $a \neq 0, O \neq 0$ and the condition $\mathrm{Re}(O) \geq 0$ is met. According to (24), we can get

$$\mathrm{WD}_{g(t)}^A(t,u) = \sqrt{\frac{1}{a}}e^{j(d/2b)u^2 - j((2b\omega_2 t + \omega_1 b - u)^2/2ab)}. \tag{20}$$

Then we obtain (16). □

*Property 3* (scaling property). If $\hat{f}(t) = \sqrt{|\lambda|}f(\lambda t)$ and $\lambda \neq 0$, then the WDL of $\hat{f}(t)$ is

$$\mathrm{WD}_h^A(t,u) = \sqrt{\frac{1}{|\lambda|}}\mathrm{WD}_f^B(\lambda t, u), \tag{21}$$

where $B = (a/\lambda, b\lambda, c/\lambda, d\lambda)$.

*3.3. The Relationships between WDL and Other Time-Frequency Analysis Tools*

*3.3.1. The Relationship between WDL and the Ambiguity Function Based on LCT (LCTAF).* The LCTAF is defined as [15]

$$\mathrm{AF}_f^A(\tau,u)=\int_{-\infty}^{+\infty}k_f(t,\tau)K_A(u,t)\,dt. \quad (22)$$

**Theorem 3.** *The relationship between WDL and LCTAF is [15]*

$$WD_f^A(t,u)=\iint_{-\infty}^{+\infty}AF_f^A(\tau,v)K_{A,A^{-1}}(\tau,v,u,t)\,d\tau\,dv, \quad (23)$$

*where $K_{A,A^{-1}}(\tau,v,u,t)=K_A(\tau,u)K_{A^{-1}}(v,t)$ is the kernel function of 2-D LCT [25].*

Equation (23) indicates that the WDL is the 2D LCT of LCTAF, and two parameter matrixes corresponding to the kernel function of 2D LCT are reciprocal matrixes. This indicates that not only classical AF and WVD but also the LCTAF and the WDL have close relationships.

*3.3.2. The Relationship between WDL and the STFT.* The STFT of signal $f(t)$ is defined as

$$S_f^w(t,\omega)=\int_{-\infty}^{+\infty}f(u)g^*(u-t)e^{-j\omega u}du. \quad (24)$$

**Theorem 4.** *The WDL can be expressed by STFT:*

$$\begin{aligned}WD_f^A(t,u)&=\sqrt{\frac{1}{j2\pi b}}\iint_{-\infty}^{+\infty}e^{j((d/2b)v^2+v\tau/2b-3a\tau^2/8b)}\\&\quad\cdot S_f^W\left(\tau,\frac{2v-a\tau}{b}\right)\\&\quad\times K_{A,A^{-1}}(\tau,v,u,t)\,d\tau\,dv,\end{aligned} \quad (25)$$

*where $S_f^W(\tau,(2v-a\tau)/2b)=\int_{-\infty}^{+\infty}f(t')[f(t'-\tau)e^{-j(a/2b)(t'-\tau)^2}]^*e^{-jt'(v/b-(a/2b)\tau)}dt'$.*

*Proof.* From [15] we know that the LCTAF can be expressed by STFT as

$$\begin{aligned}\mathrm{AF}_f^A(\tau,u)&=\sqrt{\frac{1}{j2\pi b}}e^{j((d/2b)u^2+u\tau/2b-3a\tau^2/8b)}\\&\quad\cdot S_f^W\left(\tau,\frac{2u-a\tau}{b}\right).\end{aligned} \quad (26)$$

Taking (26) into (23), we can get (25). □

*3.3.3. The Relationship between WDL and Wavelet Transform (WT).* The WT of signal $f(t)$ is defined as

$$\mathrm{WT}_f(a,b)=\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}f(t)\psi^*\left(\frac{t-b}{a}\right)dt, \quad (27)$$

where $\psi_{ab}(t)=(1/\sqrt{a})\psi((t-b)/a)$ is the basic function of WT.

**Theorem 5.** *The WDL can be expressed by WT as*

$$\begin{aligned}WD_f^A(t,u)&=\sqrt{\frac{1}{j2\pi b}}\\&\quad\times\iint_{-\infty}^{+\infty}e^{j((d/2b)v^2-v\tau/2b+a\tau^2/8b)}\cdot WT_f(1,\tau)\\&\quad\times K_{A,A^{-1}}(\tau,v,u,t)\,d\tau\,dv,\end{aligned} \quad (28)$$

*where $WT_f(1,\tau)=f(t-\tau)e^{-j(a/2b)(t-\tau)^2+j(t-\tau)((2u-a\tau)/2b)}$.*

*Proof.* From [15] we know that the LCTAF can be expressed by STFT as

$$\mathrm{AF}_f^A(\tau,u)=\sqrt{\frac{1}{j2\pi b}}e^{j((d/2b)u^2-u\tau/2b+a\tau^2/8b)}\cdot\mathrm{WT}_f(1,\tau). \quad (29)$$

Taking (29) into (23), we can get (28). □

*3.4. WDLs of Some Common Signals.* Table 1 gives WDLs of some common signals. They are easy to get, so we do not prove them here.

## 4. The QFM Signal Parameter Estimation Algorithm Based on the Generalized WDL

*4.1. The Algorithm for the Parameter Estimation of QFM Signal.* In [16], the authors have used the WDL for estimating LFM signals, but is it appropriate for dealing with QFM signal? Now let us discuss it. The QFM signal considered in this paper confirms the model $f(t)=A_0e^{j(a_1t+a_2t^2+a_3t^3)}$, where $A_0$ is the amplitude and $a_1$, $a_2$, and $a_3$ $(a_3\neq 0)$ are the phase coefficients to be determined. For the QFM signal, its WDL is

$$\begin{aligned}\mathrm{WD}_f^A(t,u)&=\sqrt{\frac{1}{j2\pi b}}A_0^2e^{j(d/2b)u^2}\\&\quad\times\int_{-\infty}^{+\infty}e^{j[(a_1+2a_2t+3a_3t^2-u/b)\tau+(a/2b)\tau^2+a_3\tau^3/4]}d\tau.\end{aligned} \quad (30)$$

Formula (30) shows that we cannot get any information about phase coefficients, so it is difficult to estimate parameters of QFM signals. But we can change the instantaneous autocorrelation function $k_f(t,\tau)$ of WDL to achieve this

Table 1: WDLs of some common signals.

| Signals | WDLs |
|---|---|
| $1$ | $\sqrt{a^{-1}}e^{(jc/2a)\,u^2}$ |
| $\delta(t-t_0)$ | $\delta(2(t-t_0))K_A(u,2t-2t_0)$ |
| $e^{j\lambda t}$ | $\begin{cases}\sqrt{\frac{2\pi b}{j}}e^{j(d/2b)\,u^2}\delta(u-\lambda b), & a=0\\ \sqrt{\frac{1}{a}}e^{j(d/2b)u^2-j((u-\omega_1 b)^2/2ab)}, & a\neq 0\end{cases}$ |
| $e^{j\mu t^2/2}$ | $\begin{cases}\sqrt{\frac{2\pi b}{j}}e^{j(d/2b)u^2}\delta(u-2\mu bt), & a=0\\ \sqrt{\frac{1}{a}}e^{j(d/2b)u^2-j((u-2b\mu t)^2/2ab)}, & a\neq 0\end{cases}$ |
| $e^{j\mu t^2/2+j\lambda t+j\gamma}$ | $\begin{cases}\sqrt{\frac{2\pi b}{j}}e^{j(d/2b)u^2}\delta(u-\lambda b-2\mu bt), & a=0\\ \sqrt{\frac{1}{a}}e^{j(d/2b)u^2\,-j((u-\lambda b-2b\mu t)^2/2ab)}, & a\neq 0\end{cases}$ |
| $e^{-t^2/2}$ | $\begin{cases}\sqrt{\frac{2\pi b}{j}}e^{j(d/2b)u^2}\delta(jbt-u), & a=0\\ \sqrt{\frac{1}{a}}e^{(jc/2a)u^2-(tu/a)+j(t^2b/2a)}, & a\neq 0\end{cases}$ |

goal. We define this new WDL as the generalized Wigner-Ville distribution based on the linear canonical transform (GWDL).

*Definition 6.* Given $k'_f(t,\tau) = f(t+\tau/2)f(t-\tau/2)f^*(-t+\tau/2)f^*(-t-\tau/2)$, one defines the GWDL of signal $f(t)$ as follows:

$$\mathrm{GWD}_f^A(t,u) = \int_{-\infty}^{+\infty} k'_f(t,\tau)\,K_A(u,\tau)\,d\tau, \tag{31}$$

where the superscript $*$ denotes the conjugate and $K_A(u,\tau)$ is given by (5). The definition of GWDL in (31) indicates that the GWDL has fourth-order nonlinearity. Substituting the QFM signal $f(t) = A_0e^{j(a_1t+a_2t^2+a_3t^3)}$ into $k'_f(t,\tau)$, we obtain $k'_f(t,\tau) = A_0^4e^{j(4(a_1t+a_3t^3)+3a_3t\tau^2)}$. So the phase of $k'_f(t,\tau)$ is quadratic for $\tau$, while the phase of $k_f(t,\tau)$ is cubic for $\tau$. By this change we can estimate the QFM signal parameters using GWDL.

**Theorem 7.** *The GWDL of QFM signal is*

$$GWD_f^A(t,u) = \begin{cases} A_0^4\sqrt{\dfrac{2\pi b}{j}}e^{j((d/2b)u^2+4(a_1t+a_3t^3))}\delta(u), & a+6ba_3t=0,\\ \dfrac{A_0^4}{\sqrt{a+6ba_3t}} \\ \quad\times e^{j((d/2b)u^2+4(a_1t+a_3t^3)-u^2/(2ab+24b^2ta_3))}, & a+6ba_3t\neq 0.\end{cases} \tag{32}$$

*Proof.* Substituting $k'_f(t,\tau) = A_0^4e^{j(4(a_1t+a_3t^3)+3a_3t\tau^2)}$ into the definition of GWDL, we obtain

$$\begin{aligned}\mathrm{GWD}_f^A(t,u) &= \frac{A_0^4}{\sqrt{j2\pi b}}e^{j((d/2b)u^2+4(a_1t+a_3t^3))}\\ &\quad\times\int_{-\infty}^{+\infty} e^{-j(u\tau/b)+j(a/2b+3a_3t)\tau^2}d\tau.\end{aligned} \tag{33}$$

According to (17), it is easy to obtain (32) by (33). □

Formula (32) indicates that the GWDL of QFM signal will generate an impulse at the point $(-a/6ba_3, 0)$ in the $(t,u)$ plane and the energy will gather along the line $u = 0$. So we can get the location information $(t', 0)$ of the impulse by searching the peak. According to the formula $t' = -a/6ba_3$, the parameter $a_3$ can be estimated as

$$\widehat{a}_3 = -\frac{a}{6b\,\mathrm{argmax}_t\,|\mathrm{GWDL}(t,0)|}. \tag{34}$$

Equation (34) shows that the condition $a\neq 0$ must be satisfied. After $a_3$ has been estimated, the signal $f(t)e^{-j\widehat{a}_3t^3}$ can be approximated to a LFM signal. So other parameters can be estimated by algorithms for estimating the LFM signal, such as WDL algorithm [16], FRFT algorithm [26], and the cubic phase function (CPF) algorithm [27] (in Section 4.2 we choose the CPF algorithm to estimate the LFM signal).

For the discrete QFM signal $f(n)$, the specification of the proposed algorithm is as follows.

*Step 1.* Compute the GWDL of $f(n)$ and search for the peak in the time-$u$ frequency plane to get the location information $(n', 0)$; then estimate $a_3$ according to (34).

*Step 2.* Multiply $f(n)$ with $e^{-j\widehat{a}_3n^3}$ and do CPF for $f(n)e^{-j\widehat{a}_3n^3}$; then estimate $a_2$ according to

$$\widehat{a}_2 = \frac{\mathrm{argmax}_\Omega\,|\mathrm{CPF}(n,\Omega)|}{2}, \tag{35}$$

where $\mathrm{CPF}(n,\Omega)$ is the CPF of $f(n)e^{-j\widehat{a}_3n^3}$.

*Step 3.* Estimate $a_1$ by dechirping and finding the Fourier transform peak:

$$\widehat{a}_1 = \underset{\omega}{\mathrm{argmax}}\left|\sum_{n=-(N-1)/2}^{(N-1)/2} f(t)\,e^{-j(\widehat{a}_2n^2+\widehat{a}_3n^3)-j\omega n}\right|. \tag{36}$$

*Step 4.* Estimate $A_0$ by evaluating

$$\widehat{A}_0 = \left|\frac{1}{N}\sum_{n=-(N-1)/2}^{(N-1)/2} f(t)\,e^{-j(\widehat{a}_1n+\widehat{a}_2n^2+\widehat{a}_3n^3)}\right|. \tag{37}$$

For the discrete signal, to avoid ambiguities due to the periodicity of digital spectra, it is also assumed that [28]

$$\left|a_i\Delta^i\right| \leq \frac{\pi}{i((N-1)/2)^{i-1}}, \quad i=1,2,3, \tag{38}$$

where $N$ is the length of the discrete signal and $\Delta$ is the sampling interval.

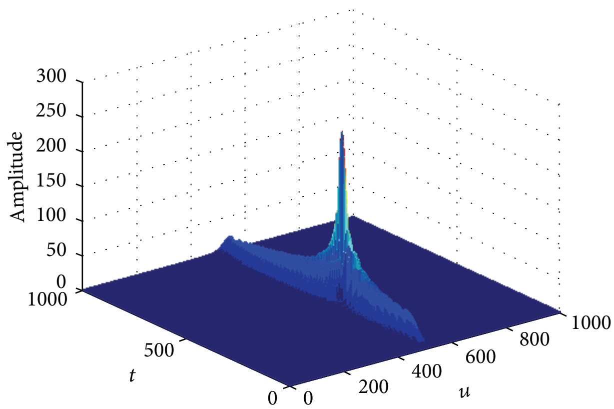


FIGURE 1: The GWDL of QFM signal.

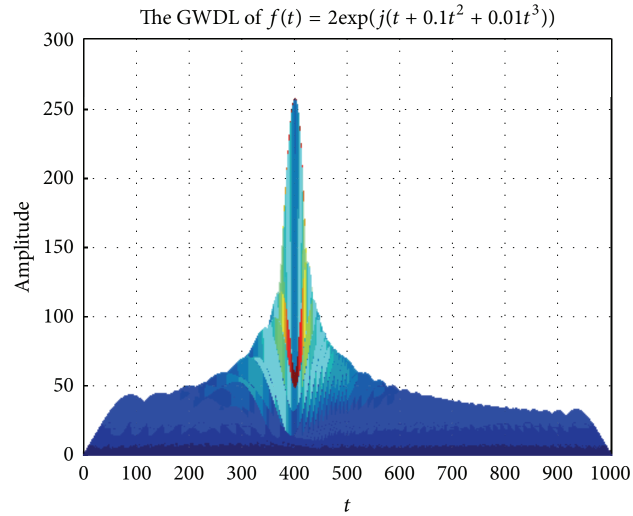


FIGURE 2: $t$-amplitude distribution of QFM signal in GWDL domain.

*4.2. Simulations.* The parameter estimation algorithm proposed above is applied to a QFM signal here. The QFM signal $f(t)$ is considered and its parameter values are set to be $A_0 = 2$, $a_1 = 1$, $a_2 = 0.1$, and $a_3 = 0.01$, the sampling interval is $\Delta = 0.02$ s, and the observing time is $t = 20$ s. Obviously parameter values that we chose above meet condition (38). Figure 1 indicates the GWDL of $f(t)$ when $(a, b, c, d) = (0.02, 1/24, 0, 50)$. It shows that the energy gathers along the line $u = 0$. Figure 2 is the $t$-amplitude distribution of GWDL. Figure 3 is the $u$-amplitude distribution of GWDL and Figure 4 is the $t$-$u$ distribution of QFM signal in the GWDL domain. Using the GWDL algorithm described above we get that the estimate values of $A_0$, $a_1$, $a_2$, and $a_3$ are $\widehat{A}_0 = 1.9059$, $\widehat{a}_1 = 0.9415$, $\widehat{a}_2 = 0.1020$, and $\widehat{a}_3 = 0.0101$, respectively. The simulation results indicate that the algorithm is accurate and the GWDL applying to the parameter estimation of the QFM signal is useful and effective.

*4.3. Comparison with Other Methods.* We know that the dimension of maximization for an algorithm leads to its computational complexity and efficiency whereas the nonlinearity order of an algorithm determines its SNR threshold and accuracy. So we compare the proposed method with other methods in the aspects of the dimension of maximization and the nonlinearity order.

The algorithm we proposed above shows that it only needs three times 1D maximization for estimating all the four parameters, while the ML method requires 3D maximization and the LCTAF method needs 2D maximization. So this algorithm does not have heavy computational burden and is efficient.

The PWVDs method [13] and the PHMT method [14] both need the same dimension of maximization as the proposed method (1D to be exact), but they have sixth-order nonlinearity, which decreases the estimation accuracy and increases the SNR threshold. Our method only has fourth-order nonlinearity, so the estimation values are more accurate and have lower SNR threshold.

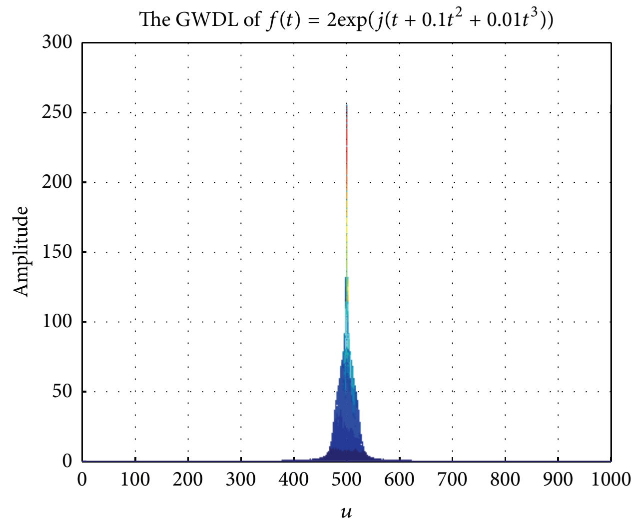


FIGURE 3: $u$-amplitude distribution of QFM signal in GWDL domain.

## 5. Conclusions

Some theories of the WDL are investigated in this paper. We first derive some new and important properties of the WDL. The relationships between WDL and other time-frequency analysis tools are also discussed, for instance, the LCTAF, STFT, and WT. The research on these theories of the WDL lays the foundation for its further application and enriches theoretical systems of the LCT and WVD. A fast and precise algorithm based on the generalized WDL for QFM signal parameter estimation is proposed, too. The new algorithm only needs 1D maximizations, so it does not have heavy computational burden and is efficient. Also the new algorithm is accurate and has a low SNR threshold because of its moderate order of nonlinearity.

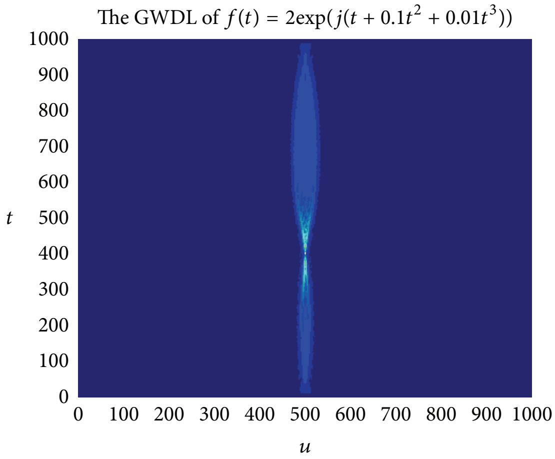


FIGURE 4: $t$-$u$ distribution of QFM signal in GWDL domain.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


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