Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 513051, 7 pages
http://dx.doi.org/10.1155/2014/513051

Research Article

Existence and Nonexistence of Positive Solutions for a Higher-Order Three-Point Boundary Value Problem

Yongping Sun,[1] Qian Sun,[2] and Xiaoping Zhang[1]

[1] *College of Electron and Information, Zhejiang University of Media and Communications, Hangzhou, Zhejiang 310018, China*
[2] *Department of Mathematics and Statistics, Curtin University, Perth, WA 6845, Australia*

Correspondence should be addressed to Yongping Sun; sunyongping@126.com

Received 3 November 2013; Accepted 27 December 2013; Published 12 January 2014

Academic Editor: Simeon Reich

This paper is concerned with the existence and nonexistence of positive solutions for a nonlinear higher-order three-point boundary value problem. The existence results are obtained by applying a fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson, and O'Regan.

1. Introduction

We consider the existence and nonexistence of positive solution to the nonlinear higher-order three-point boundary value problem (BVP for short):

$$\begin{gathered} u^{(n)}(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \\ u^{(p)}(1) = \xi u^{(p)}(\eta), \quad \left(p \in \{1, 2, \ldots, n-2\} \text{ but fixed}\right) \end{gathered} \tag{1}$$

with $n \geqslant 3$, $\xi \in (0,\infty)$, $\eta \in (0,1)$ are constants with $0 < \xi\eta^{n-p-1} < 1$. $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Here, by a positive solution u of BVP (1) we mean a function u satisfying (1) and $u(t) > 0$ for $t \in (0,1)$. If $\xi = 0$, BVP (1) reduces to so-called (n, p) boundary value problem which has been considered by many authors. For example, Agarwal et al. [1] considered the existence of positive solutions for the singular (n, p) boundary value problem. Baxley and Houmand [2] considered the existence of multiple positive solutions for the (n, p) boundary value problem. Yang [3] obtained some new upper estimates to positive solutions for the problem. Existence and nonexistence results for positive solutions of the problem were obtained by using the Krasnosel'skii fixed point theorem.

In recent years, the existence and multiplicity of positive solutions for nonlinear higher-order ordinary differential equations with three-point boundary conditions have been studied by several authors; we can refer to [4–19] and the references therein. For example, Eloe and Ahmad in [4] discussed the existence of positive solutions of a nonlinear nth-order three-point boundary value problem

$$\begin{gathered} u^{(n)}(t) + a(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \\ u(1) = \alpha u(\eta), \end{gathered} \tag{2}$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$. The existence of at least one positive solution if f is either superlinear or sublinear was established by applying the fixed point theorem in cones due to Krasnosel'skii. Under conditions different from those imposed in [4], Graef and Moussaoui in [5] studied the existence of both sign changing solutions and positive solutions for BVP (2). Hao et al. [6] are devoted to the existence and multiplicity of positive solutions for BVP (2) under certain suitable weak conditions, where $a(t)$ may be singular at $t = 0$ and/or $t = 1$. The main tool used is also the Krasnosel'skii fixed point theorem. Graef et al.

in [7, 8] considered the higher-order three-point boundary value problem

$$\begin{aligned} u^{(n)}(t) &= f(t,u(t)), \quad t \in [0,1], \\ u^{(i-1)}(0) &= 0, \quad 1 \leqslant i \leqslant n-2, \\ u^{(n-2)}(p) &= u^{(n-1)}(1) = 0, \end{aligned} \tag{3}$$

where $n \geqslant 4$ is an integer and $p \in (1/2, 1)$ is a constant. Sufficient conditions for the existence, nonexistence, and multiplicity of positive solutions of this problem are obtained by using Krasnosel'skii's fixed point theorem, Leggett-Williams' fixed point theorem, and the five-functional fixed point theorem.

Let E be a real Banach space with norm $\|\cdot\|$ and $P \subset E$ be a cone of E. Recently, Zhang et al. [9] studied a higher order three-point boundary value problem

$$\begin{aligned} u^{(n)}(t)+f\left(t,u(t),u'(t),u''(t),\ldots,u^{(n-2)}(t)\right) &= \theta, \quad t \in J, \\ u^{(i)}(0) &= \theta, \quad 0 \leqslant i \leqslant n-2, \\ u^{(n-2)}(1) &= \rho u^{(n-2)}(\eta), \end{aligned} \tag{4}$$

where $J = [0,1]$, $f \in C(J \times P^{n-1}, P)$, $\rho, \eta \in (0,1)$, and θ is the zero element of E. By using the fixed-point principle in cone and the fixed-point index theory for strict-set-contraction operator, the authors obtained the existence, nonexistence, and multiplicity of positive solutions for the nonlinear three-point boundary value problems of nth-order differential equations in ordered Banach spaces.

It is well known that fixed point theorems have been applied to various boundary value problems to show the existence and multiplicity of positive solutions. Fixed point theorems and their applications to nonlinear problems have a long history; the recent book by Agarwal et al. [20] contains an excellent summary of the current results and applications. Recently, Avery et al. [21] generalized the fixed point theorem of cone expansion and compression of norm type by replacing the norms with two functionals satisfying certain conditions to produce a fixed point theorem of cone expansion and compression of functional type, and then they applied the fixed point theorem to verify the existence of a positive solution to a second order conjugate boundary value problem.

Motivated greatly by the above-mentioned works, in this paper we will try using this new fixed point theorem to consider the existence of monotone positive solution to BVP (1). The methods used to prove the existence results are standard; however, their exposition in the framework of BVP (1) is new. This paper is organized as follows. In Section 2, we present some definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson, and O'Regan. Expression and properties of Green's function will be given in Section 3. The main results will be given in Section 4.

2. Preliminaries

In this section, for the convenience of the reader, we present some definitions and background results on cones and completely continuous operators. We also state a fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson, and O'Regan.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions:

(1) $u \in P$, $\lambda > 0$ implies $\lambda u \in P$;

(2) $u \in P$, $-u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $u \leqslant v$ if and only if $v - u \in P$.

Definition 2. Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(\lambda u + (1-\lambda)v) \geqslant \lambda\alpha(u) + (1-\lambda)\alpha(v), \quad u, v \in P, \ 0 \leqslant \lambda \leqslant 1. \tag{5}$$

Similarly we said the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(\lambda u + (1-\lambda)v) \leqslant \lambda\beta(u) + (1-\lambda)\beta(v), \quad u, v \in P, \ 0 \leqslant \lambda \leqslant 1. \tag{6}$$

We say the map γ is sublinear functional if

$$\gamma(\lambda u) \leqslant \lambda\gamma(u), \quad u \in P, \ 0 \leqslant \lambda \leqslant 1. \tag{7}$$

All the concepts discussed above can be found in [22].

Property A1. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A1 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$, and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$;

(b) β is sublinear, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$, and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$;

(c) β is concave and unbounded.

Property A2. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A2 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$;

(b) β is sublinear, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$;

(c) $\beta(u+v) \geqslant \beta(u)+\beta(v)$ for all $u, v \in P, \beta(0)=0$, $\beta(u) \neq 0$ if $u \neq 0$.

The approach used in proving the existence results in this paper is the following fixed point theorem of cone expansion and compression of functional type due to Avery et al. [21].

Theorem 4. *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$ and P is a cone in E. Suppose $T: P \cap(\overline{\Omega}_2 \backslash \Omega_1) \rightarrow P$ is a completely continuous operator, α and γ are nonnegative continuous functional on P, and one of the two conditions:*

(K1) *α satisfies Property A1 with $\alpha(Tu) \geqslant \alpha(u)$, for all $u \in P \cap \partial\Omega_1$, and γ satisfies Property A2 with $\gamma(Tu) \leqslant \gamma(u)$, for all $u \in P \cap \partial\Omega_2$, or*

(K2) *γ satisfies Property A2 with $\gamma(Tu) \leqslant \gamma(u)$, for all $u \in P \cap \partial\Omega_1$, and α satisfies Property A1 with $\alpha(Tu) \geqslant \alpha(u)$, for all $u \in P \cap \partial\Omega_2$,*

is satisfied, then T has at least one fixed point in $P \cap(\overline{\Omega}_2 \setminus \Omega_1)$.

3. Expression and Properties of Green's Function

In this section we present the expression and properties of Green's function associated with BVP (1). We will consider the Banach space $E=C[0,1]$ equipped with norm $\|u\|=\max_{0 \leqslant t \leqslant 1}|u(t)|$. In arriving our result, we need the following two preliminary lemmas.

Lemma 5. *Let $h \in C[0,1]$, then boundary value problem*

$$\begin{gathered}
u^{(n)}(t)+h(t)=0, \quad 0 \leqslant t \leqslant 1, \\
u^{(i)}(0)=0, \quad i=0,1, \ldots, n-2, \\
u^{(p)}(1)=\xi u^{(p)}(\eta)
\end{gathered} \tag{8}$$

has a unique solution

$$u(t)=\int_0^1 G(t, s) h(s) d s, \tag{9}$$

where

$$G(t, s)=G_1(t, s)+\frac{\xi t^{n-1}}{1-\xi \eta^{n-p-1}} G_2(\eta, s), \tag{10}$$

$$\begin{aligned}
& G_1(t, s) \\
& =\frac{1}{(n-1)!}\begin{cases}t^{n-1}(1-s)^{n-p-1}-(t-s)^{n-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{n-1}(1-s)^{n-p-1}, & 0 \leqslant t \leqslant s \leqslant 1,\end{cases}
\end{aligned} \tag{11}$$

$$\begin{aligned}
& G_2(\eta, s) \\
& =\frac{1}{(n-1)!}\begin{cases}\eta^{n-p-1}(1-s)^{n-p-1}-(\eta-s)^{n-p-1}, & 0 \leqslant s \leqslant \eta, \\ \eta^{n-p-1}(1-s)^{n-p-1}, & \eta \leqslant s \leqslant 1.\end{cases}
\end{aligned} \tag{12}$$

Proof. One can reduce equation $u^{(n)}(t)+h(t)=0$ to an equivalent integral equation

$$u(t)=-\frac{1}{(n-1)!} \int_0^t(t-s)^{n-1} h(s) d s+\sum_{i=0}^{n-1} A_i t^i \tag{13}$$

for some $A_i \in \mathbb{R}$ $(i=0,1,2, \ldots, n-1)$. By $u^{(i)}(0)=0$ $(i=0,1, \ldots, n-2)$, it follows $A_i=0$ $(i=0,1,2, \ldots, n-2)$. Thus,

$$u(t)=-\frac{1}{(n-1)!} \int_0^t(t-s)^{n-1} h(s) d s+A_{n-1} t^{n-1}. \tag{14}$$

Now we solve for A_{n-1} by $u^{(p)}(1)=\xi u^{(p)}(\eta)$. It follows from

$$\begin{aligned}
u^{(p)}(t)= & -\frac{1}{(n-p-1)!} \int_0^t(t-s)^{n-p-1} h(s) d s \\
& +\frac{(n-1)!}{(n-p-1)!} A_{n-1} t^{n-p-1}
\end{aligned} \tag{15}$$

that

$$\begin{aligned}
-&\frac{1}{(n-p-1)!} \int_0^1(1-s)^{n-p-1} h(s) d s+\frac{(n-1)!}{(n-p-1)!} A_{n-1} \\
& =-\frac{\xi}{(n-p-1)!} \int_0^\eta(\eta-s)^{n-p-1} h(s) d s \\
& \quad +\frac{(n-1)!}{(n-p-1)!} A_{n-1} \xi \eta^{n-p-1},
\end{aligned} \tag{16}$$

from which we get

$$\begin{aligned}
A_{n-1}= & \frac{1}{(n-1)!(1-\xi \eta^{n-p-1})} \\
& \times\left(\int_0^1(1-s)^{n-p-1} h(s) d s-\xi \int_0^\eta(\eta-s)^{n-p-1} h(s) d s\right) \\
= & \frac{1}{(n-1)!} \int_0^1(1-s)^{n-p-1} h(s) d s \\
& +\frac{\xi \eta^{n-p-1}}{(n-1)!(1-\xi \eta^{n-p-1})} \int_0^1(1-s)^{n-p-1} h(s) d s \\
& -\frac{\xi}{(n-1)!(1-\xi \eta^{n-p-1})} \int_0^\eta(\eta-s)^{n-p-1} h(s) d s \\
= & \frac{1}{(n-1)!} \int_0^1(1-s)^{n-p-1} h(s) d s \\
& +\frac{\xi}{1-\xi \eta^{n-p-1}} \int_0^1 G_2(\eta, s) h(s) d s.
\end{aligned} \tag{17}$$

Therefore, substituting A_{n-1} into (14), one has

$$\begin{aligned}
u(t) &= -\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} h(s)\,ds \\
&\quad + \frac{t^{n-1}}{(n-1)!}\int_0^1 (1-s)^{n-p-1} h(s)\,ds \\
&\quad + \frac{\xi t^{n-1}}{1-\xi\eta^{n-p-1}}\int_0^1 G_2(\eta,s) h(s)\,ds \\
&= \int_0^1 G_1(t,s) h(s)\,ds + \frac{\xi t^{n-1}}{1-\xi\eta^{n-p-1}}\int_0^1 G_2(t,s) h(s)\,ds \\
&= \int_0^1 G(t,s) h(s)\,ds,
\end{aligned} \tag{18}$$

where $G(t,s)$ is defined by (10). This completes the proof. □

Lemma 6. *Suppose* $0 < \xi\eta^{n-p-1} < 1$. *Then Green's function* $G(t,s)$ *defined by* (10) *has the following properties:*

(a) $\partial^j G(t,s)/\partial t^j$ *is continuous on* $[0,1]\times[0,1]$, $j = 0,1,2,\ldots,n-2$;

(b) $\partial G(t,s)/\partial t \geqslant 0$, *for all* $t,s\in[0,1]$;

(c) $t^{n-1}G(1,s) \leqslant G(t,s) \leqslant G(1,s)$, *for all* $t,s\in[0,1]$;

(d) $\int_0^1 G(1,s)ds = (p(1-\xi\eta^{n-p-1}) + n\xi(1-\eta)\eta^{n-p-1}) \times (n!(n-p)(1-\xi\eta^{n-p-1}))^{-1} =: A^{-1}$;

(e) $\int_\eta^1 G(1,s)ds = (n(1-\eta)^{n-p} - (n-p)(1-\eta)^n(1-\xi\eta^{n-p-1})) \times (n!(n-p)(1-\xi\eta^{n-p-1}))^{-1} =: C^{-1}$;

(f) $\int_\eta^1 G(\eta,s)ds = (\eta^{n-1}(1-\eta)^{n-p}) \times ((n-1)!(n-p)(1-\xi\eta^{n-p-1}))^{-1} =: B^{-1}$.

Proof. (a) This conclusion holds obviously, so we omit the proof here.

(b) First we prove

$$\frac{\partial G_1(t,s)}{\partial t} \geqslant 0, \quad t,s\in[0,1]. \tag{19}$$

In fact, If $t \leqslant s$, obviously, $\partial G_1(t,s)/\partial t \geqslant 0$. If $s \leqslant t$, from (11) we know

$$\begin{aligned}
\frac{\partial G_1(t,s)}{\partial t} &= \frac{1}{(n-2)!}\left[t^{n-2}(1-s)^{n-p-1} - (t-s)^{n-2}\right] \\
&\geqslant \frac{1}{(n-2)!}\left[t^{n-2}(1-s)^{n-2} - (t-s)^{n-2}\right] \\
&= \frac{1}{(n-2)!}\left[(t-ts)^{n-2} - (t-s)^{n-2}\right] \\
&\geqslant 0.
\end{aligned} \tag{20}$$

Now we prove

$$G_2(\eta,s) \geqslant 0, \quad s\in[0,1]. \tag{21}$$

In fact, if $s \geqslant \eta$, obviously, (21) holds. If $s \leqslant \eta$, one has

$$\begin{aligned}
&\eta^{n-p-1}(1-s)^{n-p-1} - (\eta-s)^{n-p-1} \\
&\quad = (\eta-\eta s)^{n-p-1} - (\eta-s)^{n-p-1} \geqslant 0,
\end{aligned} \tag{22}$$

which implies that (21) is also true. Therefore, By (10), (20), and (21), we find

$$\frac{\partial G(t,s)}{\partial t} = \frac{\partial G_1(t,s)}{\partial t} + \frac{(n-1)\xi t^{n-2}}{1-\xi\eta^{n-p-1}} G_2(\eta,s) \geqslant 0. \tag{23}$$

The proof of (b) is completed.

(c) From (b) we know that $\partial G(t,s)/\partial t \geqslant 0$ for any $t,\ s\in[0,1]$. Thus $G(t,s)$ is increasing in t, so

$$G(t,s) \leqslant G(1,s), \quad \text{for } (t,s)\in[0,1]\times[0,1]. \tag{24}$$

On the other hand, if $s \leqslant t$, then from (11), we have

$$\begin{aligned}
G_1(t,s) &= \frac{1}{(n-1)!}\left[t^{n-1}(1-s)^{n-p-1} - (t-s)^{n-1}\right] \\
&= \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&\quad + \frac{1}{(n-1)!}\left[(t-ts)^{n-1} - (t-s)^{n-1}\right] \\
&\geqslant \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&= t^{n-1}G_1(1,s).
\end{aligned} \tag{25}$$

If $s \geqslant t$, then from (11), we have

$$\begin{aligned}
G_1(t,s) &= \frac{1}{(n-1)!}t^{n-1}(1-s)^{n-p-1} \\
&= \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&\quad + \frac{1}{(n-1)!}(t-ts)^{n-1} \\
&\geqslant \frac{t^{n-1}}{(n-1)!}\left[(1-s)^{n-p-1} - (1-s)^{n-1}\right] \\
&= t^{n-1}G_1(1,s).
\end{aligned} \tag{26}$$

Thus from (25) and (26), we obtain that

$$G_1(t,s) \geqslant t^{n-1}G_1(1,s). \tag{27}$$

From (10) and (27), one has

$$\begin{aligned}
G(t,s) &= G_1(t,s) + \frac{\xi t^{n-1}}{1-\xi\eta^{n-p-1}} G_2(\eta,s) \\
&\geqslant t^{n-1}G_1(1,s) + \frac{\xi t^{n-1}}{1-\xi\eta^{n-p-1}} G_2(\eta,s) \\
&= t^{n-1}G(1,s).
\end{aligned} \tag{28}$$

The equations (d), (e), and (f) can be verified by direct calculations. Then the proof is completed. □

Define the cone P by

$$P = \{u \in C[0,1] : u(t) \geqslant 0, \quad u(t) \text{ is increasing on } [0,1] \text{ and } u(t) \geqslant t^{n-1}\|u\|, \quad t \in [0,1]\}. \tag{29}$$

Then P is a normal cone of E. Define the operator $T : P \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \qquad t \in [0,1]. \tag{30}$$

Thus, to solve BVP (1), we only need to find a fixed point of the operator T in P.

Lemma 7. *The operator defined in* (30) *is completely continuous and satisfies* $T(P) \subseteq P$.

Proof. By Lemma 6(b) and (c), $T(P) \subseteq P$. T is completely continuous by an application of Arzela-Ascoli theorem. □

Finally, let us define two continuous functionals α and γ on the cone P by

$$\begin{aligned} \alpha(u) &:= \min_{t\in[\eta,1]} u(t) = u(\eta), \\ \gamma(u) &:= \max_{t\in[0,1]} u(t) = u(1) = \|u\|. \end{aligned} \tag{31}$$

It is clear that $\alpha(u) \leqslant \gamma(u)$ for all $u \in P$.

4. Main Results

Consider firstly the existence of positive solution for BVP (1).

Theorem 8. *Suppose that there exists positive numbers r, R with $r < \eta^{n-1}R$ such that the following conditions are satisfied:*

(A1) *$f(t,x) \geqslant Br$, for all $(t,x) \in [\eta,1] \times [r,R]$,*

(A2) *$f(t,x) \leqslant AR$, for all $(t,x) \in [0,1] \times [0,R]$.*

Then BVP (1) *admits a positive solution u^* such that*

$$r \leqslant \min_{t\in[\eta,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \tag{32}$$

Proof. Let

$$\Omega_1 = \{u : \alpha(u) < r\}, \qquad \Omega_2 = \{u : \gamma(u) < R\}, \tag{33}$$

we have that $0 \in \Omega_1$, Ω_1, and Ω_2 being bounded open subsets of E. Let $u \in \overline{\Omega}_1$, then we have

$$r \geqslant \alpha(u) = \min_{t\in[\eta,1]} u(t) \geqslant \eta^{n-1}\|u\| = \eta^{n-1}\gamma(u), \tag{34}$$

which implies that $R > r/\eta^{n-1} \geqslant \gamma(u)$; that is, $u \in \Omega_2$, thus $\overline{\Omega}_1 \subseteq \Omega_2$.

Claim 1. If $u \in P \cap \partial\Omega_1$, then $\alpha(Tu) \geqslant \alpha(u)$.

To see this let $u \in P \cap \partial\Omega_1$, then $R = \gamma(u) \geqslant u(s) \geqslant \alpha(u) = r$, $s \in [\eta,1]$. Thus it follows from (A1), Lemma 6(f), and (30), one has

$$\begin{aligned} \alpha(Tu) = (Tu)(\eta) &= \int_0^1 G(\eta,s) f(s,u(s))\, ds \\ &\geqslant \int_\eta^1 G(\eta,s) f(s,u(s))\, ds \\ &\geqslant Br \int_\eta^1 G(\eta,s)\, ds = r = \alpha(u). \end{aligned} \tag{35}$$

Claim 2. If $u \in P \cap \partial\Omega_2$, then $\gamma(Tu) \leqslant \gamma(u)$.

To see this let $u \in P \cap \partial\Omega_2$, then $u(s) \leqslant \gamma(u) = R$, $s \in [0,1]$. Thus condition (A2) and Lemma 6(d) yield

$$\begin{aligned} \gamma(Tu) = (Tu)(1) &= \int_0^1 G(1,s) f(s,u(s))\, ds \\ &\leqslant AR \int_0^1 G(1,s)\, ds = R = \gamma(u). \end{aligned} \tag{36}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore the hypothesis (K1) of Theorem 4 is satisfied, and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$; that is, BVP (1) has at least one positive solution $u^*(t) \in P$ such that

$$r \leqslant \min_{t\in[\eta,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \tag{37}$$

This completes the proof. □

Theorem 9. *Suppose that there exist positive numbers r, R with $r < R$ such that the following conditions are satisfied:*

(A3) *$f(t,x) \leqslant Ar$, for $(t,x) \in [0,1] \times [0,r]$,*

(A4) *$f(t,x) \geqslant BR$, for $(t,x) \in [\eta,1] \times [R, R/\eta^{n-1}]$.*

Then BVP (1) *has at least one positive solution u^* satisfying*

$$r \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[\eta,1]} u^*(t) \leqslant R. \tag{38}$$

Proof. For all $u \in P$ we have $\alpha(u) \leqslant \gamma(u)$. Thus if we let

$$\Omega_3 = \{u : \gamma(u) < r\}, \qquad \Omega_4 = \{u : \alpha(u) < R\}, \tag{39}$$

we have that $0 \in \Omega_3$ and $\overline{\Omega}_3 \subseteq \Omega_4$, with Ω_3 and Ω_4 being bounded open subsets of E.

Claim 1. If $u \in P \cap \partial\Omega_3$, then $\gamma(Tu) \leqslant \gamma(u)$.

To see this let $u \in P \cap \partial\Omega_3$, then $u(s) \leqslant \gamma(u) = r$, $s \in [0,1]$. Thus condition (A3) and Lemma 6(d) yield

$$\begin{aligned} \gamma(Tu) = (Tu)(1) &= \int_0^1 G(1,s) f(s,u(s))\, ds \\ &\leqslant Ar \int_0^1 G(1,s)\, ds = r = \gamma(u). \end{aligned} \tag{40}$$

Claim 2. If $u \in P \cap \partial\Omega_4$, then $\alpha(Tu) \geqslant \alpha(u)$.

To see this let $u \in P \cap \partial\Omega_4$, then $R/\eta^{n-1} = \alpha(u)/\eta^{n-1} \geqslant \gamma(u) \geqslant u(s) \geqslant \alpha(u) = R,\ s \in [\eta, 1]$. Thus it follows from (A4) and Lemma 6(f), one has

$$\begin{aligned}\alpha(Tu) = (Tu)(\eta) &= \int_0^1 G(\eta, s) f(s, u(s))\, ds \\ &\geqslant \int_\eta^1 G(\eta, s) f(s, u(s))\, ds \\ &\geqslant BR \int_\eta^1 G(\eta, s)\, ds \\ &= R = \alpha(u).\end{aligned} \tag{41}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore the hypothesis (K2) of Theorem 4 is satisfied, and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$; that is, BVP (1) has at least one positive solution $u^* \in P$ such that

$$r \leqslant \max_{t \in [0,1]} u^*(t), \qquad \min_{t \in [\eta,1]} u^*(t) \leqslant R. \tag{42}$$

This completes the proof. □

Now we discuss nonexistence of positive solutions of BVP (1).

Theorem 10. *Suppose that $f \in C([0,1] \times [0,\infty), [0,\infty))$ satisfies the condition*

$$\sup_{(t,x) \in [0,1] \times (0,\infty)} \frac{f(t,x)}{x} < A. \tag{43}$$

Then BVP (1) does not admit positive solutions.

Proof. Assume to the contrary that $u = u(t)$ is a positive solution of BVP (1), then from Lemma 6(d) and (43) we get

$$\begin{aligned}u(1) &= \int_0^1 G(1, s) f(s, u(s))\, ds \\ &< A \int_0^1 G(1, s) u(s)\, ds \leqslant A \|u\| \int_0^1 G(1, s)\, ds = u(1).\end{aligned} \tag{44}$$

This is a contradiction and completes the proof. □

Theorem 11. *Suppose that $f \in C([0,1] \times [0,\infty), [0,\infty))$ satisfies the condition*

$$\inf_{(t,x) \in [\eta,1] \times (0,\infty)} \frac{f(t,x)}{x} > \frac{C}{\eta^{n-1}}, \tag{45}$$

then BVP (1) does not admit positive solutions.

Proof. Assume to the contrary that $u = u(t)$ is a positive solution of BVP (1), then by the definition of the cone K, we have $u(s) \geqslant s^{n-1}\|u\| \geqslant \eta^{n-1} u(1)$ for $s \in [\eta, 1]$. Thus from Lemma 6(e) and (45) we get

$$\begin{aligned}u(1) &= \int_0^1 G(1, s) f(s, u(s))\, ds \\ &\geqslant \int_\eta^1 G(1, s) f(s, u(s))\, ds \\ &> \frac{C}{\eta^{n-1}} \int_\eta^1 G(1, s) u(s)\, ds \geqslant u(1).\end{aligned} \tag{46}$$

This is a contradiction and completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors gratefully acknowledge the referee for his/her valuable comments and suggestions. This work was supported financially by the Natural Science Foundation of Zhejiang Province of China (Y12A01012), the National Natural Science Foundation of China (11371221, 11071141), the Specialized Research Foundation for the Doctoral Program of Higher Education of China (20123705110001), and the Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province.

References

[1] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, "Singular $(p, n-p)$ focal and (n, p) higher order boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 2, pp. 215–228, 2000.

[2] J. V. Baxley and C. R. Houmand, "Nonlinear higher order boundary value problems with multiple positive solutions," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 682–691, 2003.

[3] B. Yang, "Positive solutions for the (n, p) boundary value problem," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 31, pp. 1–13, 2009.

[4] P. W. Eloe and B. Ahmad, "Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 521–527, 2005.

[5] J. R. Graef and T. Moussaoui, "A class of nth-order BVPs with nonlocal conditions," *Computers & Mathematics with Applications*, vol. 58, no. 8, pp. 1662–1671, 2009.

[6] X. Hao, L. Liu, and Y. Wu, "Positive solutions for nonlinear nth-order singular nonlocal boundary value problems," *Boundary Value Problems*, vol. 2007, Article ID 74517, 10 pages, 2007.

[7] J. R. Graef, J. Henderson, and B. Yang, "Positive solutions of a nonlinear higher order boundary-value problem," *Electronic Journal of Differential Equations*, vol. 2007, no. 45, pp. 1–10, 2007.

[8] J. R. Graef, J. Henderson, P. J. Y. Wong, and B. Yang, "Three solutions of an nth order three-point focal type boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3386–3404, 2008.

[9] X. Zhang, M. Feng, and W. Ge, "Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 2, pp. 584–597, 2009.

[10] Z. Du, W. Liu, and X. Lin, "Multiple solutions to a three-point boundary value problem for higher-order ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1207–1218, 2007.

[11] Y. Gao, "Existence and uniqueness of solutions for nth-order nonlinear three-point boundary value problems," *Dynamics of Continuous, Discrete & Impulsive Systems A*, vol. 15, no. 2, pp. 243–250, 2008.

[12] J. R. Graef, C. Qian, and B. Yang, "A three point boundary value problem for nonlinear fourth order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 217–233, 2003.

[13] J. R. Graef and B. Yang, "Positive solutions to a multi-point higher order boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 409–421, 2006.

[14] Y. Ji and Y. Guo, "The existence of countably many positive solutions for nonlinear nth-order three-point boundary value problems," *Boundary Value Problems*, vol. 2009, Article ID 572512, 18 pages, 2009.

[15] I. Y. Karaca, "Positive solutions of an nth order three-point boundary value problem," *The Rocky Mountain Journal of Mathematics*, vol. 43, no. 1, pp. 205–224, 2013.

[16] X.-J. Liu, W.-H. Jiang, and Y.-P. Guo, "Multi-point boundary value problems for higher order differential equations," *Applied Mathematics E-Notes*, vol. 4, pp. 106–113, 2004.

[17] Y. Liu and W. Ge, "Positive solutions for $(n-1,1)$ three-point boundary value problems with coefficient that changes sign," *Journal of Mathematical Analysis and Applications*, vol. 282, no. 2, pp. 816–825, 2003.

[18] P. K. Palamides, "Multi point boundary-value problems at resonance for n-order differential equations: positive and monotone solutions," *Electronic Journal of Differential Equations*, vol. 2004, no. 25, pp. 1–14, 2004.

[19] X. Zhang, L. Liu, and H. Zou, "Positive solutions of fourth-order singular three point eigenvalue problems," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1359–1367, 2007.

[20] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Boston, Mass, USA, 1999.

[21] R. Avery, J. Henderson, and D. O'Regan, "Functional compression-expansion fixed point theorem," *Electronic Journal of Differential Equations*, vol. 2008, no. 22, pp. 1–12, 2008.

[22] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1988.