Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 510924, 4 pages
http://dx.doi.org/10.1155/2014/510924

Research Article

A Remark on the Regularity Criterion for the 3D Boussinesq Equations Involving the Pressure Gradient

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Received 10 November 2013; Accepted 28 December 2013; Published 21 January 2014

Academic Editor: Giovanni P. Galdi

We consider the three-dimensional Boussinesq equations and obtain a regularity criterion involving the pressure gradient in the Morrey-Companato space $M_{p,q}$. This extends and improves the result of Gala (Gala 2013) for the Navier-Stokes equations.

1. Introduction

This paper concerns itself with the following three-dimensional (3D) Boussinesq equations:

$$\begin{gathered} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \Delta\mathbf{u} + \nabla\pi = \theta\mathbf{e}_3, \quad \text{in } \mathbb{R}^3\times(0,T), \\ \theta_t + (\mathbf{u}\cdot\nabla)\,\theta - \Delta\theta = 0, \quad \text{in } \mathbb{R}^3\times(0,T), \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in } \mathbb{R}^3\times(0,T), \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \quad \text{on } \mathbb{R}^3, \end{gathered} \tag{1}$$

where $T > 0$ is given time, $\mathbf{u} = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the fluid velocity, $\pi = \pi(x,t)$ is a scalar pressure, and $\theta = \theta(x,t)$ is the temperature, while $\mathbf{u}_0$ and θ_0 are the prescribed initial velocity field and temperature, respectively.

When $\theta = 0$, (1) reduces to the incompressible Navier-Stokes equations. The regularity of its weak solutions and the existence of global strong solutions are challenging open problems; see [1–3]. Starting with [4, 5], there have been a lot of literature devoted to finding sufficient conditions to ensure the smoothness of the solutions; see [6–15] and the references cited therein. Since the convective terms are similar in the Navier-Stokes equations and Boussinesq equations, the authors also consider the regularity conditions for (1); see [16–20] and so forth.

In [6], Gala uses intricate decomposition technique to obtain the following regularity criterion for the Navier-Stokes equations:

$$\nabla\pi \in L^{2/(3-r)}\left(0,T;\dot{X}_r\right) \quad \text{with } 0 \le r \le 1. \tag{2}$$

Here, $\dot{X}_r$ is the point-wise multiplier space from $\dot{H}^r$ to L^2, which is strictly larger than $L^{3/r}(\mathbb{R}^3)$ (see [6, Lemma 1.2]).

In this paper, we will extend and improve the regularity condition (2) to the Boussineq equations (1).

Before stating the precise result, let us recall the weak formulation of (1).

Definition 1. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, $\theta_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$. A measurable pair $(\mathbf{u},\theta)$ is said to be a weak solution of (1) in $(0,T)$, provided that

(1) $(\mathbf{u},\theta) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$, $\theta \in L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^3))$;

(2) $(1)_{1,2,3}$ are satisfied in the sense of distributions;

(3) the energy inequality

$$\begin{aligned} &\|(\mathbf{u},\theta)\|_{L^2}^2 + 2\int_0^t \|\nabla(\mathbf{u},\theta)\|_{L^2}^2\,\mathrm{d}s \\ &\quad\le \left\|(\mathbf{u}_0,\theta_0)\right\|_{L^2}^2 \\ &\qquad + 2\int_0^t\int_{\mathbb{R}^3} \theta u_3\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3}$$

for all $0 \le t \le T$.

Now, our main result reads the following.

Theorem 2. *Let* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ *with* $\nabla\cdot\mathbf{u}_0 = 0$ *in the sense of distributions,* $\theta_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$. *Supposing that* $(\mathbf{u},\theta)$ *is a weak solution of* (1) *in* $[0,T)$, *and the pressure gradient* $\nabla\pi$ *satisfies*

$$\nabla\pi \in L^{2/(3-r)}\left(0,T;\dot{M}_{2,3/r}\right) \quad \text{with } 0 < r \le 1, \tag{4}$$

then the solution $(\mathbf{u},\theta) \in C^\infty((0,T)\times\mathbb{R}^3)$.

Here, $\dot{M}_{p,q}$ is the Morrey-Campanato space, which will be introduced in Section 2. And Section 3 is devoted to the proof of Theorem 2.

Remark 3. Noticing that $\dot{X}_r \subset \dot{M}_{2,3/r}$ for $0 < r < 1$ (see (10)), we indeed improve the result of [6] for the Navier-Stokes equations.

2. Preliminaries

In this section, we will introduce the definition of Morrey-Campanato space $\dot{M}_{p,q}$ and recall its fundamental properties. The space plays an important role in studying the regularity of solutions to partial differential equations (see [21–23], e.g.).

Definition 4. For $1 < p \le q \le +\infty$, the Morrey-Campanato space $\dot{M}_{p,q}$ is defined as

$$\dot{M}_{p,q} = \Bigg\{ f \in L^p_{\text{loc}}\left(\mathbb{R}^3\right); \|f\|_{\dot{M}_{p,q}} = \sup_{x\in\mathbb{R}^3, R>0} \frac{1}{R^{(3/p)-(3/q)}} \left(\int_{B(x,R)} |f(y)|^p \, \mathrm{d}y\right)^{1/p} < +\infty \Bigg\}, \tag{5}$$

where $B(x,R) \subset \mathbb{R}^3$ is the ball with center x and radius R.

One sees readily that $\dot{M}_{p,q}$ is a Banach space under the norm $\|\cdot\|_{\dot{M}_{p,q}}$ and contains the classical Lebesgue space as a subspace:

$$L^q = \dot{M}_{q,q} \subset \dot{M}_{p,q}. \tag{6}$$

Moreover, the following scaling property holds:

$$\|f(\lambda\cdot)\|_{\dot{M}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{M}_{p,q}}, \quad \text{for } \lambda > 0. \tag{7}$$

Due to the following characterization in [24].

Lemma 5. *For* $0 \le r < 3/2$, *the space* $\dot{Z}_r$ *is defined as the space of all functions* $f \in L^2_{\text{loc}}(\mathbb{R}^3)$ *such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \le 1} \|fg\|_{L^2} < +\infty. \tag{8}$$

Then $f \in \dot{M}_{2,3/r}$ *if and only if* $f \in \dot{Z}_r$ *with equivalent norm.*

And with the fact that

$$L^2 \cap \dot{H}^r \subset \dot{B}^r_{2,1} \subset \dot{H}^r \quad \text{for } 0 < r < 1, \tag{9}$$

we have

$$\dot{X}_r \subset \dot{M}_{2,3/r}. \tag{10}$$

Here $\dot{B}^r_{2,1}$ is the Besov space, which is intermediate between L^2 and $\dot{H}^1$ (see [25]):

$$\|f\|_{\dot{B}^r_{2,1}} \le C\|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}, \quad \text{for } 0 < r < 1. \tag{11}$$

3. Proof of Theorem 2

In this section, we will prove Theorem 2.

Due to the Serrin type regularity criterion

$$\mathbf{u} \in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right) \quad \text{with } \frac{2}{p} + \frac{3}{q} = 1,\ 3 < q \le +\infty \tag{12}$$

in [19], we need only to prove

$$\mathbf{u} \in L^\infty\left(0,T;L^4\left(\mathbb{R}^4\right)\right) \subset L^8\left(0,T;L^4\left(\mathbb{R}^3\right)\right). \tag{13}$$

We just do a priori estimates, with the justification being from passage to limits for the Galerkin approximated solutions.

Taking the inner product of $(1)_2$ with 2θ in $L^2(\mathbb{R}^3)$, we find

$$\frac{d}{dt}\|\theta\|^2_{L^2} + 2\|\nabla\theta\|^2_{L^2} = 0. \tag{14}$$

Thus,

$$\|\theta\|_{L^2} \le \|\theta_0\|_{L^2}. \tag{15}$$

One can also take the inner product of $(1)_2$ with $p\theta^{p-1}$ $(1 \le p < \infty)$ in $L^2(\mathbb{R}^3)$ to derive the estimate of θ in L^p-norm and invoke the maximum principle to bound the L^∞-norm of θ, as stated in Definition 1.

Taking the divergence of $(1)_1$, we get

$$-\Delta\pi = \sum_{i,j=1}^{3} \partial_i\partial_j\left(u_iu_j\right) - \partial_3\theta. \tag{16}$$

Consequently,

$$\|\nabla\pi\|_{L^2} \le C\||\mathbf{u}|\cdot|\nabla\mathbf{u}|\|_{L^2} + \|\theta\|_{L^2} \le C\left(\||\mathbf{u}|\cdot|\nabla\mathbf{u}|\|_{L^2} + 1\right). \tag{17}$$

Taking the inner product of $(1)_1$ with $4|\mathbf{u}|^2\mathbf{u}$ in $L^2(\mathbb{R}^3)$, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\||\mathbf{u}|^2\right\|^2_{L^2} + 4\int_{\mathbb{R}^3}|\mathbf{u}|^2|\nabla\mathbf{u}|^2\mathrm{d}x + 2\int_{\mathbb{R}^3}\left|\nabla|\mathbf{u}|^2\right|\mathrm{d}x \\ &\quad \le 4\int_{\mathbb{R}^3}|\nabla\pi|\cdot|\mathbf{u}|^3\mathrm{d}x + 4\int_{\mathbb{R}^3}|\theta|\cdot|\mathbf{u}|^3\mathrm{d}x \\ &\quad \equiv I_1 + I_2. \end{aligned} \tag{18}$$

For I_1, we estimate as

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} |\nabla\pi|^{1/2} \cdot |\nabla\pi|^{1/2} |\mathbf{u}| \cdot |\mathbf{u}|^2 \, dx \le \left\| |\nabla\pi|^{1/2} \right\|_{L^4} \cdot \left\| |\nabla\pi|^{1/2} |\mathbf{u}| \right\|_{L^4} \cdot \left\| |\mathbf{u}|^2 \right\|_{L^2} \quad \text{(by Hölder inequality)} \\
&= \|\nabla\pi\|_{L^2}^{1/2} \cdot \left\| |\nabla\pi| \cdot |\mathbf{u}|^2 \right\|_{L^2}^{1/2} \cdot \left\| |\mathbf{u}|^2 \right\|_{L^2} \le \|\nabla\pi\|_{L^2}^{1/2} \cdot \|\nabla\pi\|_{\dot{M}_{2,3/r}}^{1/2} \left\| |\mathbf{u}|^2 \right\|_{\dot{B}^r_{2,1}}^{1/2} \cdot \left\| |\mathbf{u}|^2 \right\|_{L^2} \quad \text{(by Lemma 5)} \\
&\le C\left(\| |\mathbf{u}| \cdot |\nabla\mathbf{u}| \|_{L^2} + 1 \right)^{1/2} \cdot \|\nabla\pi\|_{\dot{M}_{2,3/r}}^{1/2} \cdot \left\| |\mathbf{u}|^2 \right\|_{L^2}^{(1-r)/2} \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^{r/2} \cdot \left\| |\mathbf{u}|^2 \right\|_{L^2} \quad \text{(by (17) and (11))} \\
&\le C \| |\mathbf{u}| \cdot |\nabla\mathbf{u}| \|_{L^2}^{1/2} \cdot \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^{r/2} \cdot \|\nabla\pi\|_{\dot{M}_{2,3/r}}^{1/2} \left\| |\mathbf{u}|^2 \right\|_{L^2}^{(3-r)/2} + C \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^{r/2} \cdot 1 \cdot \|\nabla\pi\|_{\dot{M}_{2,3/r}}^{1/2} \left\| |\mathbf{u}|^2 \right\|_{L^2}^{(3-r)/2} \\
&\le 3 \| |\mathbf{u}| \cdot |\nabla\mathbf{u}| \|_{L^2}^2 + \frac{1}{2} \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^2 + C + C \|\nabla\pi\|_{\dot{M}_{2,3/r}}^{2/(3-r)} \left\| |\mathbf{u}|^2 \right\|_{L^2}^2 \\
&\quad \left(\begin{array}{c} \text{Young inequality } abc \le \varepsilon a^p + \delta b^q + C_{\varepsilon\delta} c^r, \ \dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1 \\ \text{with } p = 4, \ q = \dfrac{4}{r}, \ r = \dfrac{4}{3-r} \end{array} \right).
\end{aligned} \tag{19}$$

The term I_2 can be dominated as

$$\begin{aligned}
I_2 &\le 4 \|\theta\|_{L^2} \left\| |\mathbf{u}|^3 \right\|_{L^2} \\
&\le C \left\| |\mathbf{u}|^2 \right\|_{L^3}^{3/2} \quad \text{(by (15))} \\
&\le C \left\| |\mathbf{u}|^2 \right\|_{L^2}^{3/4} \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^{3/4} \quad \text{(by interpolation inequality)} \\
&\le C \left\| |\mathbf{u}|^2 \right\|_{L^2}^{6/5} + \frac{1}{2} \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^2 \\
&\le C + C \left\| |\mathbf{u}|^2 \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla |\mathbf{u}|^2 \right\|_{L^2}^2.
\end{aligned} \tag{20}$$

Plugging (19) and (20) into (18), we deduce that

$$\begin{aligned}
&\frac{d}{dt} \left\| |\mathbf{u}|^2 \right\|_{L^2}^2 + \int_{\mathbb{R}^3} |\mathbf{u}|^2 |\nabla\mathbf{u}|^2 dx + \int_{\mathbb{R}^3} \left| \nabla |\mathbf{u}|^2 \right| dx \\
&\quad \le C + C \left(\|\nabla\pi\|_{\dot{M}_{2,3/r}}^{2/(3-r)} + 1 \right) \left\| |\mathbf{u}|^2 \right\|_{L^2}^2.
\end{aligned} \tag{21}$$

Applying Gronwall inequality, we see that

$$\begin{aligned}
\|\mathbf{u}(t)\|_{L^4}^4 &\le \left(\|\mathbf{u}_0\|_{L^4}^4 + CT \right) \\
&\quad \cdot \exp\left\{ C \int_0^T \left(\|\nabla\pi\|_{\dot{M}_{2,3/r}}^{2/(3-r)} + 1 \right) ds \right\},
\end{aligned} \tag{22}$$

for every $t \in [0, T]$. Recalling (13), we complete the proof of Theorem 2.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Zujin Zhang was partially supported by the Youth Natural Science Foundation of Jiangxi Province (20132BAB211007), the Science Foundation of Jiangxi Provincial Department of Education (GJJ13658, GJJ13659), and the National Natural Science Foundation of China (11326138, 11361004, and 11326238).

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