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Research Article

Summation Formulas Involving Binomial Coefficients, Harmonic Numbers, and Generalized Harmonic Numbers

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A variety of identities involving harmonic numbers and generalized harmonic numbers have been investigated since the distant past and involved in a wide range of diverse fields such as analysis of algorithms in computer science, various branches of number theory, elementary particle physics, and theoretical physics. Here we show how one can obtain further interesting and (almost) serendipitous identities about certain finite or infinite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers by simply applying the usual differential operator to well-known Gauss's summation formula for ${}_2F_1(1)$.

1. Introduction and Preliminaries

The generalized harmonic numbers $H_n^{(s)}$ of order s which are defined by (cf. [1]; see also [2, 3], [4, page 156], and [5, Section 3.5])

$$H_n^{(s)} := \sum_{j=1}^{n} \frac{1}{j^s} \quad (n \in \mathbb{N};\ s \in \mathbb{C}), \tag{1}$$

$$H_n := H_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}) \tag{2}$$

are the harmonic numbers. Here $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the set of complex numbers, respectively, and we assume that

$$H_0 := 0, \qquad H_0^{(s)} := 0 \quad (s \in \mathbb{C} \setminus \{0\}), \qquad H_0^{(0)} := 1. \tag{3}$$

The generalized harmonic functions $H_n^{(s)}(z)$ are defined by (see [2, 6]; see also [7, 8])

$$H_n^{(s)}(z) := \sum_{j=1}^{n} \frac{1}{(j+z)^s} \quad (n \in \mathbb{N};\ s \in \mathbb{C} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1, -2, -3, \ldots\}), \tag{4}$$

so that, obviously,

$$H_n^{(s)}(0) = H_n^{(s)}. \tag{5}$$

Equation (1) can be written in the following form:

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1) \quad (\Re(s) > 1;\ n \in \mathbb{N}), \tag{6}$$

by recalling the well-known (easily derivable) relationship between the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ (see [4, equation 2.3(9)]):

$$\zeta(s) = \zeta(s, n+1) + \sum_{k=1}^{n} k^{-s} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{7}$$

The polygamma functions $\psi^{(n)}(s)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(s) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(s) = \frac{d^n}{ds^n} \psi(s) \quad (n \in \mathbb{N}_0;\ s \in \mathbb{C} \setminus \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}), \tag{8}$$

where $\Gamma(s)$ is the familiar gamma function and the psi-function ψ is defined by

$$\psi(s) := \frac{d}{ds} \log \Gamma(s), \quad \psi^{(0)}(s) = \psi(s) \quad (s \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{9}$$

A well-known (and potentially useful) relationship between the polygamma functions $\psi^{(n)}(s)$ and the generalized zeta function $\zeta(s,a)$ is given by

$$\psi^{(n)}(s) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+s)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, s) \qquad \left(n \in \mathbb{N};\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{10}$$

It is also easy to have the following expression (cf. [4, equation 1.2(54)]):

$$\psi^{(m)}(s+n) - \psi^{(m)}(s) = (-1)^m m! H_n^{(m+1)}(s-1) \qquad (m, n \in \mathbb{N}_0), \tag{11}$$

which immediately gives $H_n^{(s)}$ another expression for $H_n^{(s)}$ as follows (cf. [9, equation (20)]):

$$H_n^{(m)} = \frac{(-1)^{m-1}}{(m-1)!}\left[\psi^{(m-1)}(n+1) - \psi^{(m-1)}(1)\right] \qquad (m \in \mathbb{N};\ n \in \mathbb{N}_0). \tag{12}$$

The following identity was discovered by Euler in 1775 and has a long history (see, e.g., [10, page 252 *et seq.*]):

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3). \tag{13}$$

Identity (13) is a special case of the following more general sum due to Euler:

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1) \qquad (n \in \mathbb{N} \setminus \{1\}) \tag{14}$$

or, equivalently,

$$2\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^n} = n\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1) \qquad (n \in \mathbb{N} \setminus \{1\}), \tag{15}$$

where (*and in what follows*) an empty sum is understood to be nil.

Many different techniques have been used, in the vast mathematical literature, in order to evaluate harmonic sums of the types (13) and (15). For example, D. Borwein and J. M. Borwein [11] established the following interesting sums by applying Parseval's identity to a Fourier series and contour integrals to a generating function:

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n+1}\right)^2 = \frac{11}{4}\zeta(4), \tag{16}$$

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4), \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \tag{18}$$

where, in light of Euler sum (18), nonlinear harmonic sums (16) and (17) are substantially the same, since it is easily verified that

$$\sum_{k=1}^{\infty} \left(\frac{H_k}{k}\right)^2 = \sum_{k=1}^{\infty} \left(\frac{H_k}{k+1}\right)^2 + \frac{3}{2}\zeta(4). \tag{19}$$

Euler started this line of investigation in the course of his correspondence with Goldbach beginning in 1742 and he was the first to consider the linear harmonic sums:

$$\mathcal{S}_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}. \tag{20}$$

Euler, whose investigations were completed by Nielsen in 1906 (see Nielsen [12]), showed that the linear harmonic sums in (20) can be evaluated in terms of zeta values in the following special cases: $p = 1$, $p = q$, $p + q$ odd, and $p + q$ even, but with the pair (p, q) being restricted to a finite set of the so-called *exceptional* configurations $\{(2,4), (4,2)\}$ (see Flajolet and Salvy [13]). Of these special cases, in the ones with $p \neq q$, if $\mathcal{S}_{p,q}$ is known, then $\mathcal{S}_{q,p}$ can be found by means of the symmetry relation:

$$\mathcal{S}_{p,q} + \mathcal{S}_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q), \tag{21}$$

and *vice versa* (see also [1, page 140, Proposition 6]. Some typical examples are

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{7}{4}\zeta(4), \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7). \tag{23}$$

Rather extensive numerical search for linear relations between linear Euler sums and polynomials in zeta values (see Bailey et al. [14]; see also Flajolet and Salvy [13]) strongly suggests that Euler found all the possible evaluations of linear harmonic sums.

The nonlinear harmonic sums involve products of at least two harmonic numbers. Let $P = (p_1, \ldots, p_k)$ be a partition of an integer p into k summands, so that $p = p_1 + \cdots + p_k$ and $p_1 \leqq p_2 \leqq \cdots \leqq p_k$. The Euler sum of index P, q is defined by

$$\mathcal{S}_{P,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_k)}}{n^q}, \tag{24}$$

where the quantity $q + p_1 + \cdots + p_k$ is called the *weight* and the quantity k is the *degree*. A few basic nonlinear sums were recently evaluated by de Doelder [15] by invoking their relations with the Eulerian beta integrals or with polylogarithm functions. A detailed numerical search was conducted by Bailey et al. [14] who showed the existence of many surprising evaluations like

$$\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^4} = \frac{231}{16}\zeta(7) - \frac{51}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5). \quad (25)$$

Flajolet and Salvy [13] clearly and extensively analyzed most of the hitherto known evaluations for Euler sums and multiple zeta functions (see also Hoffman [16] and Zagier [17]). For a remarkably clear and insightful exposition of several important results and conjectures concerning multiple polylogarithms and the multiple zeta functions (including *especially* a broad survey of recent works on multiple zeta series and Euler sums of arbitrary degree), the interested reader should refer also to a survey-cum-expository paper by Bowman and Bradley [18], which contains a fairly comprehensive bibliography of as many as 83 *further* references on the subject.

Shen [19] investigated the connections between the Stirling numbers $s(n, k)$ of the first kind and the Riemann zeta function $\zeta(n)$ by means of the Gauss summation formula (28) for the hypergeometric series. In the course of his analysis, Shen [19] proved some known identities like (13), (18), and (22). In fact, by employing the univariate series expansion of classical hypergeometric formulas, Shen [19] and Choi and Srivastava [20, 21] investigated the evaluation of infinite series related to generalized harmonic numbers. On the other hand, more summation formulas have been systematically derived by Chu [22] and Chu and de Donno [23] who developed fully this approach to the multivariate case. We chose to recall two identities:

$$\sum_{n=1}^{\infty} \frac{H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}}{n^2} = 6\zeta(5),$$
$$\sum_{n=1}^{\infty} \left(\frac{H_n^{(2)}}{n^3} + \frac{3H_n}{n^4} \right) = \frac{9}{2}\zeta(5). \quad (26)$$

Many formulas of finite series involving binomial coefficients, the Stirling numbers of the first and second kinds, harmonic numbers, and generalized harmonic numbers have also been investigated in diverse ways (see, e.g., [2, 23–32]).

Here we show how one can obtain further interesting and (almost) serendipitous identities about certain finite or infinite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers by simply applying the usual differential operator to well-known Gauss's summation formula for ${}_2F_1(1)$. For example, see the identities in Corollary 5:

$$\sum_{n=1}^{\infty} \frac{(H_n)^3 - H_n H_n^{(2)}}{(n+1)(n+2)} = 4(1 + \zeta(2) + \zeta(3));$$
$$\sum_{n=1}^{\infty} \frac{(H_n)^4 - 3(H_n)^2 H_n^{(2)} + 2H_n H_n^{(3)}}{(n+1)(n+2)} = 12(1 + \zeta(2) + \zeta(3) + \zeta(4)). \quad (27)$$

Relevant connections between some of the identities presented here with those in earlier works are also pointed out.

2. Infinite Series Involving Binomial Coefficients, Harmonic Numbers, and Generalized Harmonic Numbers

We begin by recalling well-known Gauss's summation formula for ${}_2F_1(1)$:

$${}_2F_1(a, b; c; 1) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \left(\Re(c-a-b) > 0;\ c \notin \mathbb{Z}_0^-\right), \quad (28)$$

where $(\alpha)_n$ denotes the Pochhammer symbol defined (for $\alpha \in \mathbb{C}$) by

$$(\alpha)_n := \begin{cases} 1, & (n = 0), \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & (n \in \mathbb{N}). \end{cases} \quad (29)$$

For convenient reference, without proof, we collect a set of easily derivable formulas necessary to provide further interesting identities about certain finite or infinite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers asserted as in the following lemma.

Lemma 1. *Each of the following identities holds true:*

$$\frac{d}{d\alpha}(\alpha)_n = (\alpha)_n H_n^{(1)}(\alpha - 1) \quad \left(n \in \mathbb{N}_0;\ \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\right);$$
$$\frac{d}{d\alpha}\frac{1}{(\alpha)_n} = -\frac{H_n^{(1)}(\alpha-1)}{(\alpha)_n} \quad \left(n \in \mathbb{N}_0;\ \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\right);$$
$$\frac{d^2}{d\alpha^2}(\alpha)_n = (\alpha)_n \left[\left\{H_n^{(1)}(\alpha-1)\right\}^2 - H_n^{(2)}(\alpha-1)\right] \quad \left(n \in \mathbb{N}_0;\ \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-\right);$$

$$\frac{d^2}{d\alpha^2}\frac{1}{(\alpha)_n} = \frac{1}{(\alpha)_n}\left[\left\{H_n^{(1)}(\alpha-1)\right\}^2 + H_n^{(2)}(\alpha-1)\right]$$
$$\left(n\in\mathbb{N}_0;\ \alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-\right);$$
$$\frac{d^\ell}{dz^\ell}H_n^{(s)}(z) = (-1)^\ell (s)_\ell H_n^{(s+\ell)}(z)$$
$$\left(n\in\mathbb{N};\ \ell\in\mathbb{N}_0;\ s\in\mathbb{C}\setminus\mathbb{Z}^-\right);$$
$$\frac{d}{da}\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right\} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\psi(c-a)-\psi(c-a-b)\right);$$
$$\frac{d}{dc}\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right\} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\cdot\left(\psi(c)+\psi(c-a-b) -\psi(c-a)-\psi(c-b)\right). \quad (30)$$

Now we are ready to present certain general identities of infinite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers as in the following theorem.

Theorem 2. *Each of the following summation formulas holds true:*

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n^{(1)}(a-1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\psi(c-a)-\psi(c-a-b)\right); \quad (31)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(a-1)\right\}^2 - H_n^{(2)}(a-1)\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\cdot\left[\left(\psi(c-a)-\psi(c-a-b)\right)^2 - \left(\psi'(c-a)-\psi'(c-a-b)\right)\right]; \quad (32)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(a-1)\right\}^3 - 3H_n^{(1)}(a-1)H_n^{(2)}(a-1) + 2H_n^{(3)}(a-1)\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\left(\psi(c-a)-\psi(c-a-b)\right)^3 - 3\left(\psi(c-a)-\psi(c-a-b)\right) \times\left(\psi'(c-a)-\psi'(c-a-b)\right) + \left(\psi^{(2)}(c-a)-\psi^{(2)}(c-a-b)\right)\right]; \quad (33)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(a-1)\right\}^4 - 6\left\{H_n^{(1)}(a-1)\right\}^2 H_n^{(2)}(a-1) + 8H_n^{(1)}(a-1)H_n^{(3)}(a-1) + 3\left\{H_n^{(2)}(a-1)\right\}^2 - 6H_n^{(4)}(a-1)\right]$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\left(\psi(c-a)-\psi(c-a-b)\right)^4 - 6\left(\psi(c-a)-\psi(c-a-b)\right)^2 \times\left(\psi'(c-a)-\psi'(c-a-b)\right) + 4\left(\psi(c-a)-\psi(c-a-b)\right) \times\left(\psi^{(2)}(c-a)-\psi^{(2)}(c-a-b)\right) + 3\left(\psi'(c-a)-\psi'(c-a-b)\right)^2 - \left(\psi^{(3)}(c-a)-\psi^{(3)}(c-a-b)\right)\right]; \quad (34)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n^{(1)}(a-1)H_n^{(1)}(b-1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\cdot\left[\left(\psi(c-a)-\psi(c-a-b)\right) \times\left(\psi(c-b)-\psi(c-a-b)\right) +\psi'(c-a-b)\right]; \quad (35)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[H_n^{(1)}(a-1)\left\{H_n^{(1)}(b-1)\right\}^2 - H_n^{(1)}(a-1)H_n^{(2)}(b-1)\right]$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\left(\psi(c-a)-\psi(c-a-b)\right) \times\left(\psi(c-b)-\psi(c-a-b)\right)^2 + 2\left(\psi(c-a)-\psi(c-a-b)\right) \times\psi'(c-a-b) - \psi^{(2)}(c-a-b) - \left(\psi(c-a)-\psi(c-a-b)\right) \times\left(\psi'(c-b)-\psi'(c-a-b)\right)\right]; \quad (36)$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[H_n^{(1)}(a-1)\left\{H_n^{(1)}(b-1)\right\}^3 - 3H_n^{(1)}(a-1)H_n^{(1)}(b-1)H_n^{(2)}(b-1) +2H_n^{(1)}(a-1)H_n^{(3)}(b-1)\right]$$

$$
\begin{aligned}
&= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \Big[\left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi(c-b)-\psi(c-a-b)\right)^3 \\
&\quad + 3\left(\psi(c-b)-\psi(c-a-b)\right)^2 \\
&\quad \times \psi'(c-a-b) + \psi^{(3)}(c-a-b \\
&\quad - 3\left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi(c-b)-\psi(c-a-b)\right) \\
&\quad \cdot \left(\psi'(c-b)-\psi'(c-a-b)\right) \\
&\quad - 3\left(\psi(c-b)-\psi(c-a-b)\right) \\
&\quad \times \psi^{(2)}(c-a-b) - 3\psi'(c-a-b) \\
&\quad \times \left(\psi'(c-b)-\psi'(c-a-b)\right) \\
&\quad + \left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi^{(2)}(c-b)-\psi^{(2)}(c-a-b)\right)\Big];
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} \Big[H_n^{(1)}(a-1) \left\{H_n^{(1)}(b-1)\right\}^4 \\
&\quad - 6H_n^{(1)}(a-1)\left\{H_n^{(1)}(b-1)\right\}^2 H_n^{(2)}(b-1) \\
&\quad + 8H_n^{(1)}(a-1)\,H_n^{(1)}(b-1)\,H_n^{(3)}(b-1) \\
&\quad + 3H_n^{(1)}(a-1)\left\{H_n^{(2)}(b-1)\right\}^2 \\
&\quad - 6H_n^{(1)}(a-1)\,H_n^{(4)}(b-1) \Big] \\
&= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \Big[\left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi(c-b)-\psi(c-a-b)\right)^4 \\
&\quad + 4\left(\psi(c-b)-\psi(c-a-b)\right)^3 \\
&\quad \times \psi'(c-a-b) - \psi^{(4)}(c-a-b) \\
&\quad - 6\left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi(c-b)-\psi(c-a-b)\right)^2 \\
&\quad \cdot \left(\psi'(c-b)-\psi'(c-a-b)\right) \\
&\quad - 6\left(\psi(c-b)-\psi(c-a-b)\right)^2 \\
&\quad \times \psi^{(2)}(c-a-b) \\
&\quad - 12\left(\psi(c-b)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi'(c-b)-\psi'(c-a-b)\right) \\
&\quad + 4\left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi(c-b)-\psi(c-a-b)\right) \\
&\quad \cdot \left(\psi^{(2)}(c-b)-\psi^{(2)}(c-a-b)\right) \\
&\quad + 4\ \left(\psi(c-b)-\psi(c-a-b)\right) \\
&\quad \times \psi^{(3)}(c-a-b) \\
&\quad + 3\ \left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi'(c-b)-\psi'(c-a-b)\right)^2 \\
&\quad + 6\ \left(\psi'(c-b)-\psi'(c-a-b)\right) \\
&\quad \times \psi^{(2)}(c-a-b) + 4\ \psi'(c-a-b \\
&\quad \times \left(\psi^{(2)}(c-b)-\psi^{(2)}(c-a-b)\right) \\
&\quad - \left(\psi(c-a)-\psi(c-a-b)\right) \\
&\quad \times \left(\psi^{(3)}(c-b)-\psi^{(3)}(c-a-b)\right)\Big].
\end{aligned}
\tag{38}
$$

Proof. Differentiating each side of (28) with respect to the variable a and using some suitable identities in Section 1 and Lemma 1, we obtain (31). Differentiating each side of (31) with respect to the variable a, we get (32). Similarly we prove (33) and (34). Differentiating each side of (31) with respect to the variable b and using some suitable identities in Section 1 and Lemma 1, we obtain (35). Similarly we prove (36), (37), and (38). □

Setting $c = 2$ and $a = b = 1$ in (31) to (34) and using some suitable identities in Section 1, we obtain some interesting identities involving harmonic numbers and generalized harmonic numbers given in the following corollary.

Corollary 3. *Each of the following identities holds true:*

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+2)} = 1; \\
&\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \left[(H_n)^2 - H_n^{(2)}\right] = 2 = 1\cdot 2; \\
&\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \left[(H_n)^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right] \\
&\quad = 6 = 2\cdot 3; \\
&\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \Big[(H_n)^4 - 6(H_n)^2 H_n^{(2)} \\
&\quad + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 - 6H_n^{(4)}\Big] \\
&\quad = 24 = 6\cdot 4.
\end{aligned}
\tag{39}
$$

Differentiating *only* the left-hand side of (34) with respect to the variable a twice and setting $c = 3$ and $a = b = 1$ in each expression, in view of the identities (39), we may guess two identities asserted by the following conjecture.

Conjecture 4. *Each of the following identities may hold true:*

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\Big[(H_n)^5 - 10(H_n)^3H_n^{(2)} + 20(H_n)^2H_n^{(3)} + 15H_n\big(H_n^{(2)}\big)^2 - 30H_nH_n^{(4)} - 20H_n^{(2)}H_n^{(3)} + 24H_n^{(5)}\Big] = 120 = 24\cdot 5;$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\Big[(H_n)^6 - 15(H_n)^4H_n^{(2)} + 40(H_n)^3H_n^{(3)} + 45(H_n)^2\big(H_n^{(2)}\big)^2 - 90(H_n)^2H_n^{(4)} - 120H_nH_n^{(2)}H_n^{(3)} + 144H_nH_n^{(5)} - 15\big(H_n^{(2)}\big)^3 + 90H_n^{(2)}H_n^{(4)} + 40\big(H_n^{(3)}\big)^2 - 120H_n^{(5)}\Big] = 720 = 120\cdot 6. \tag{40}$$

Setting $c = 3$ and $a = b = 1$ in (35) to (44) and using some suitable identities in Section 1 and Lemma 1, we obtain a set of very interesting identities in the following corollary.

Corollary 5. *Each of the following identities holds true:*

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{(n+1)(n+2)} = 2(1+\zeta(2)); \tag{41}$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^3 - H_nH_n^{(2)}}{(n+1)(n+2)} = 4(1+\zeta(2)+\zeta(3)); \tag{42}$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^4 - 3(H_n)^2H_n^{(2)} + 2H_nH_n^{(3)}}{(n+1)(n+2)} = 12(1+\zeta(2)+\zeta(3)+\zeta(4)); \tag{43}$$

$$\sum_{n=1}^{\infty}\Big(\Big((H_n)^5 - 6(H_n)^3H_n^{(2)} + 8(H_n)^2H_n^{(3)} + 3H_n(H_n^{(2)})^2 - 6H_nH_n^{(4)}\Big)\times((n+1)(n+2))^{-1}\Big) = 48(1+\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)). \tag{44}$$

Differentiating *only* the left-hand side of (38) with respect to the variable b twice and setting $c = 3$ and $a = b = 1$ in each expression, in view of the identities (41) to (44), we may guess two identities asserted by the following conjecture.

Conjecture 6. *Each of the following identities may hold true:*

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\Big[(H_n)^6 - 10(H_n)^4H_n^{(2)} + 20(H_n)^3H_n^{(3)} + 15(H_n)^2(H_n^{(2)})^2 - 30(H_n)^2H_n^{(4)} - 20H_nH_n^{(2)}H_n^{(3)} + 24H_nH_n^{(5)}\Big] = 240(1+\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)); \tag{45}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\Big[(H_n)^7 - 15(H_n)^5H_n^{(2)} + 40(H_n)^4H_n^{(3)} + 45(H_n)^3(H_n)^2 - 90(H_n)^3H_n^{(4)} - 120(H_n)^2H_n^{(2)}H_n^{(3)} + 144(H_n)^2H_n^{(5)} - 15H_n\big(H_n^{(2)}\big)^3 + 90H_nH_n^{(2)}H_n^{(4)} + 40H_n(H_n^{(3)})^2 - 120H_nH_n^{(6)}\Big] = 1440(1+\zeta(2)+\zeta(3)+\zeta(4)+\zeta(5)+\zeta(6)+\zeta(7)). \tag{46}$$

Remark 7. It is interesting to observe that the number of terms in the numerator of each of the left-hand sides of (41) to (46) is equal to the number of partitions of $k - 1$ ($k = 2, 3, 4, 5, 6, 7$), respectively. For example, for (46), all the partitions of $7 - 1 = 6$ are as follows:

$$\begin{aligned}7 &= 1+1+1+1+1+1+1\\ &= 1+1+1+1+1+2 = 1+1+1+2+2\\ &= 1+2+2+2 = 1+1+1+1+3\\ &= 1+1+2+3 = 1+3+3\\ &= 1+1+1+4 = 1+1+5 = 1+2+4,\end{aligned} \tag{47}$$

where k denotes $H_n^{(k)}$ ($k = 1, 2, 3, 4, 5$) and + is translated into a multiplication of its corresponding $H_n^{(k)}$. The coefficient of each of the right-hand sides of (41) to (46) has the following rule:

$$\begin{gathered}2,\quad 4 = 2\cdot 2,\quad 12 = 4\cdot 3,\quad 48 = 12\cdot 4,\\ 240 = 48\cdot 5,\quad 1440 = 240\cdot 6,\ldots.\end{gathered} \tag{48}$$

The remaining thing is to find a rule that dominates the coefficients of each term of the numerator of the left-hand sides of (41) to (46).

Differentiating each side of (28) with respect to the variable c successively and using some suitable identities in Section 1 and Lemma 1, we obtain a set of infinite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers different from those in Theorem 2 as in the following theorem.

Theorem 8. *Each of the following summation formulas holds true:*

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n^{(1)}(c-1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b));$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(c-1)\right\}^2+H_n^{(2)}(c-1)\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times\left[(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b))^2 -\left(\psi'(c-a)+\psi'(c-b)-\psi'(c)-\psi'(c-a-b)\right)\right];$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(c-1)\right\}^3+3H_n^{(1)}(c-1) \times H_n^{(2)}(c-1)+2H_n^{(3)}(c-1)\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times\left[(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b))^3 - 3(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b)) \cdot\left(\psi'(c-a)+\psi'(c-b)-\psi'(c)-\psi'(c-a-b)\right) +\left(\psi^{(2)}(c-a)+\psi^{(2)}(c-b) -\psi^{(2)}(c)-\psi^{(2)}(c-a-b)\right)\right];$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\left[\left\{H_n^{(1)}(c-1)\right\}^4+6\left\{H_n^{(1)}(c-1)\right\}^2 \times H_n^{(2)}(c-1)+8H_n^{(1)}(c-1)H_n^{(3)}(c-1) +3\left\{H_n^{(2)}(c-1)\right\}^2+6H_n^{(4)}(c-1)\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \times\left[(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b))^4 - 6(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b))^2 \cdot\left(\psi'(c-a)+\psi'(c-b)-\psi'(c)-\psi'(c-a-b)\right) +4(\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b)) \cdot\left(\psi^{(2)}(c-a)+\psi^{(2)}(c-b)-\psi^{(2)}(c) -\psi^{(2)}(c-a-b)\right) +3\left(\psi'(c-a)+\psi'(c-b)-\psi'(c) -\psi'(c-a-b)\right)^2 -\left(\psi^{(3)}(c-a)+\psi^{(3)}(c-b)-\psi^{(3)}(c) -\psi^{(3)}(c-a-b)\right)\right]. \tag{49}$$

Setting $c = 1$ and $a = b = -1/2$ in (49) and using some suitable identities in Section 1 and special values of ψ-function (see, e.g., [4, Section 1.2] and [5, Section 1.3]), we obtain a set of interesting infinite series involving binomial coefficients and harmonic numbers given in the following corollary.

Corollary 9. *Each of the following identities holds true:*

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}}H_n = \frac{4}{\pi}(3-4\log 2);$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}}\left[(H_n)^2+H_n^{(2)}\right] = \frac{4}{\pi}\left[(3-4\log 2)^2+7-4\zeta(2)\right];$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}}\left[(H_n)^3+3H_nH_n^{(2)}+2H_n^{(3)}\right] = \frac{4}{\pi}\left[(3-4\log 2)^3-3(3-4\log 2)(4\zeta(2)-7) -24\zeta(3)+30\right];$$

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{(2n-1)^2 2^{4n}}\left[(H_n)^4+6(H_n)^2H_n^{(2)} +8H_nH_n^{(3)}+3(H_n^{(2)})^2+6H_n^{(4)}\right] = \frac{4}{\pi}\left[(3-4\log 2)^4-6(3-4\log 2)^2(4\zeta(2)-7) +24(3-4\log 2)(5-4\zeta(3))+3(4\zeta(2)-7)^2 +186-168\zeta(4)\right]. \tag{50}$$

3. Finite Series Involving Binomial Coefficients, Harmonic Numbers, and Generalized Harmonic Numbers

Setting $b=-N\in\mathbb{N}$ in some chosen formulas in Theorems 2 and 8 and using some suitable identities in Section 1 and the following known and easily derivable formula:

$$(-N)_n=\begin{cases}\dfrac{(-1)^n N!}{(N-n)!}, & (0\le n\le N;\ N\in\mathbb{N}),\\ 0, & (n>N),\end{cases}$$
$$=\begin{cases}(-1)^n n!\dbinom{N}{n}, & (0\le n\le N;\ N\in\mathbb{N}),\\ 0, & (n>N),\end{cases} \tag{51}$$

we obtain a set of finite series involving binomial coefficients, harmonic numbers, and generalized harmonic numbers given in the following theorem.

Theorem 10. *Each of the following finite summation formulas holds true:*

$$\sum_{n=1}^{N}(-1)^{n+1}\frac{(a)_n}{(c)_n}\binom{N}{n}H_n^{(1)}(a-1)=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}H_N^{(1)}(c-a-1)\quad(N\in\mathbb{N}_0), \tag{52}$$

where the empty sum is (as usual) understood to be nil throughout this paper,

$$\sum_{n=1}^{N}(-1)^{n}\frac{(a)_n}{(c)_n}\binom{N}{n}\left[\left\{H_n^{(1)}(a-1)\right\}^2-H_n^{(2)}(a-1)\right]=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}\left[\left\{H_N^{(1)}(c-a-1)\right\}^2-H_N^{(2)}(c-a-1)\right]\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}(-1)^{n+1}\frac{(a)_n}{(c)_n}\binom{N}{n}\left[\left\{H_n^{(1)}(a-1)\right\}^3-3H_n^{(1)}(a-1)\times H_n^{(2)}(a-1)+2H_n^{(3)}(a-1)\right]=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}\left[\left\{H_N^{(1)}(c-a-1)\right\}^3-3H_N^{(1)}(c-a-1)\times H_N^{(2)}(c-a-1)+2H_N^{(3)}(c-a-1)\right]\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}(-1)^{n}\frac{(a)_n}{(c)_n}\binom{N}{n}\left[\left\{H_n^{(1)}(a-1)\right\}^4-6\left\{H_n^{(1)}(a-1)\right\}^2\times H_n^{(2)}(a-1)+8H_n^{(1)}(a-1)\times H_n^{(3)}(a-1)+3\left\{H_n^{(2)}(a-1)\right\}^2-6H_n^{(4)}(a-1)\right]$$
$$=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}\left[\left\{H_N^{(1)}(c-a-1)\right\}^4-6\left\{H_N^{(1)}(c-a-1)\right\}^2\times H_N^{(2)}(c-a-1)+8H_N^{(1)}(c-a-1)H_N^{(3)}(c-a-1)+3\left\{H_N^{(2)}(c-a-1)\right\}^2-6H_N^{(4)}(c-a-1)\right]\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}(-1)^{n}\frac{(a)_n}{(c)_n}\binom{N}{n}H_n^{(1)}(c-1)=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}\left[H_N^{(1)}(c-1)-H_N^{(1)}(c-a-1)\right]\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}(-1)^{n}\binom{N}{n}\frac{(a)_n}{(c)_n}\left[\left\{H_n^{(1)}(c-1)\right\}^2+H_n^{(2)}(c-1)\right]=\frac{\Gamma(c-a+N)\Gamma(c)}{\Gamma(c-a)\Gamma(c+N)}\left[\left\{H_N^{(1)}(c-1)-H_N^{(1)}(c-a-1)\right\}^2+H_N^{(2)}(c-1)-H_N^{(2)}(c-a-1)\right]\quad(N\in\mathbb{N}_0). \tag{53}$$

Setting $a=1$ and $c=2$ in (52) to (53) and using some suitable identities in Section 1, we obtain a set of interesting identities involving binomial coefficients, harmonic numbers, and generalized harmonic numbers given in the following corollary.

Corollary 11. *Each of the following identities holds true:*

$$\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n+1}\binom{N}{n}H_n=\frac{H_N}{N+1}\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}\frac{(-1)^{n}}{n+1}\binom{N}{n}\left[(H_n)^2-H_n^{(2)}\right]=\frac{(H_N)^2-H_N^{(2)}}{N+1}\quad(N\in\mathbb{N}_0);$$

$$\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n+1}\binom{N}{n}\left[(H_n)^3-3H_nH_n^{(2)}+2H_n^{(3)}\right]$$

$$= \frac{1}{N+1}\left[(H_N)^3 - 3H_N H_N^{(2)} + 2H_N^{(3)}\right] \quad (N \in \mathbb{N}_0);$$

$$\sum_{n=1}^{N} \frac{(-1)^n}{n+1}\binom{N}{n}\Big[(H_n)^4 - 6(H_n)^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 - 6H_n^{(4)}\Big]$$

$$= \frac{1}{N+1}\Big[(H_N)^4 - 6(H_N)^2 H_N^{(2)} + 8H_N H_N^{(3)} + 3(H_N^{(2)})^2 - 6H_N^{(4)}\Big] \quad (N \in \mathbb{N}_0);$$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n+1}\binom{N}{n} H_n^{(1)}(1) = \frac{N}{(N+1)^2} \quad (N \in \mathbb{N}_0);$$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n+1}\binom{N}{n}\Big[\big\{H_n^{(1)}(1)\big\}^2 + H_n^{(2)}(1)\Big] = \frac{2N}{(N+1)^3} \quad (N \in \mathbb{N}_0). \tag{54}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] V. S. Adamchik and H. M. Srivastava, "Some series of the zeta and related functions," *Analysis*, vol. 18, no. 2, pp. 131–144, 1998.

[2] J. Choi and H. M. Srivastava, "Some summation formulas involving harmonic numbers and generalized harmonic numbers," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2220–2234, 2011.

[3] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, A Foundation for Computer Science, Addison-Wesley, Reading, Mass, USA, 2nd edition, 1989.

[4] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, London, UK, 2001.

[5] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science, Amsterdam, The Netherlands, 2012.

[6] J. Choi, "Certain summation formulas involving harmonic numbers and generalized harmonic numbers," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 734–740, 2011.

[7] T. M. Rassias and H. M. Srivastava, "Some classes of infinite series associated with the Riemann Zeta and Polygamma functions and generalized harmonic numbers," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 593–605, 2002.

[8] A. Sofo and H. M. Srivastava, "Identities for the harmonic numbers and binomial coefficients," *Ramanujan Journal*, vol. 25, no. 1, pp. 93–113, 2011.

[9] M. W. Coffey, "On some series representations of the Hurwitz zeta function," *Journal of Computational and Applied Mathematics*, vol. 216, no. 1, pp. 297–305, 2008.

[10] B. C. Berndt, *Ramanujan's Notebooks, Part I*, Springer, New York, NY, USA, 1985.

[11] D. Borwein and J. M. Borwein, "On an intriguing integral and some series related to $\zeta(4)$," *Proceedings of the American Mathematical Society*, vol. 123, no. 4, pp. 1191–1198, 1995.

[12] N. Nielsen, *Die Gammafunktion*, Chelsea, New York, NY, USA, 1965.

[13] P. Flajolet and B. Salvy, "Euler sums and contour integral representations," *Experimental Mathematics*, vol. 7, no. 1, pp. 15–35, 1998.

[14] D. H. Bailey, J. M. Borwein, and R. Girgensohn, "Experimental evaluation of Euler sums," *Experimental Mathematics*, vol. 3, no. 1, pp. 17–30, 1994.

[15] P. J. de Doelder, "On some series containing $\psi(x) - \psi(y\ >)$ and $(\psi(x) - \psi(y>))^2$ for certain values of x and y," *Journal of Computational and Applied Mathematics*, vol. 37, no. 1–3, pp. 125–141, 1991.

[16] M. E. Hoffman, "Multiple harmonic series," *Pacific Journal of Mathematics*, vol. 152, no. 2, pp. 275–290, 1992.

[17] D. Zagier, "Values of Zeta functions and their applications," in *Proceedings of the 1st European Congress of Mathematics, Volume II (Paris, France, 1992)*, A. Joseph, F. Mignot, F. Murat, B. Prum, and R. Rentschler, Eds., Progress in Mathematics 120, pp. 497–512, Birkhäauser, Basel, Switzerland, 1994.

[18] D. Bowman and D. M. Bradley, Multiple Polylogarithms : A brief survey in q-Series with Applications to Combinatorics , Number Theory, and Physics (Papers from the Conference held at the University of Illinois, Urbana, Ill, USA, October 2000) (B. C. Berndt and K. Ono, Editors), pp. 71–92, Contemporary Mathematics 291 , American Mathematical Society, Providence, RI, USA, 2001.

[19] L. Shen, "Remarks on some integrals and series involving the Stirling numbers and $\zeta(n)$," *Transactions of the American Mathematical Society*, vol. 347, no. 4, pp. 1391–1399, 1995.

[20] J. Choi and H. M. Srivastava, "Certain classes of infinite series," *Monatshefte für Mathematik*, vol. 127, no. 1, pp. 15–25, 1999.

[21] J. Choi and H. M. Srivastava, "Explicit evaluation of Euler and related sums," *Ramanujan Journal*, vol. 10, no. 1, pp. 51–70, 2005.

[22] W.-C. Chu, "Hypergeometric series and the Riemann zeta function," *Acta Arithmetica*, vol. 82, no. 2, pp. 103–118, 1997.

[23] W. Chu and L. de Donno, "Hypergeometric series and harmonic number identities," *Advances in Applied Mathematics*, vol. 34, no. 1, pp. 123–137, 2005.

[24] H. Alzer, D. Karayannakis, and H. M. Srivastava, "Series representations for some mathematical constants," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 1, pp. 145–162, 2006.

[25] J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer, Berlin, Germany, 1996.

[26] D. Cvijović, "The Dattoli-Srivastava conjectures concerning generating functions involving the harmonic numbers," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 4040–4043, 2010.

[27] G. Dattoli and H. M. Srivastava, "A note on harmonic numbers, umbral calculus and generating functions," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 686–693, 2008.

[28] D. H. Greene and D. E. Knuth, *Birkhauser*, Berlin, Germany, 3rd edition, 1990.

[29] P. Paule and C. Schneider, "Computer proofs of a new family of harmonic number identities," *Advances in Applied Mathematics*, vol. 31, no. 2, pp. 359–378, 2003.

[30] C. Schneider, "Solving parameterized linear difference equations in $\prod\Sigma$-fields," Technical Report 02-03, RISC-Linz, Johannes Kepler University, Linz, Austria, 2002, http://www.risc.uni-linz.ac.at/research/combinat/risc/publications.

[31] M. Z. Spivey, "Combinatorial sums and finite differences," *Discrete Mathematics*, vol. 307, no. 24, pp. 3130–3146, 2007.

[32] WolframMathWorld, http://mathworld.wolfram.com/HarmonicNumber.html.