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Research Article

Constructing Uniform Approximate Analytical Solutions for the Blasius Problem

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We propose a simple constructive method which assures uniform accuracy of the approximate analytical solutions for the Blasius problem on the semi-infinite interval $[0, \infty)$. The method is based on a weight function having an S-shape to reflect a series solution near the origin $x = 0$ and a reference solution far from the origin. Numerical results show the efficiency of the proposed method.

1. Introduction

For the Blasius problem

$$Nf(x) := f'''(x) + \alpha f(x) f''(x) = 0, \quad 0 \le x < \infty, \tag{1}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = \beta, \tag{2}$$

we recall the well-known properties [1–3] of the so-called Blasius function $f(x)$ as follows:

(i) $f''(0) = \kappa = \sqrt{\alpha\beta^3}\kappa_0$ with $\kappa_0 = 0.4695999883\cdots$

(ii) $\lim_{x\to\infty}\{f(x) - \beta x\} = \sqrt{(\beta/\alpha)}B_0$ with $B_0 = -1.2167806216\cdots$.

Though the Blasius problem looks simple, search for an approximate analytical solution is known to be quite difficult. Until now, in the literature [4–22], lots of analytical methods have been proposed. Recently, in approximation of the solutions of nonlinear differential equations in unbounded domain, several efficient spectral methods [23–27] have been proposed. These methods reduce solving the nonlinear equation to solving a system of nonlinear algebraic equations.

In this paper, we introduce a weight function $w_L(k; x)$ in (8) whose values cluster to 0 for $x < L/2$ and to 1 for $x > L/2$ when k is large enough. Then, employing a series approximate solutions $S_n f(x)$ for the Blasius function $f(x)$ near the origin $x = 0$ and a reference solution $Rf(x)$ away from the origin, we propose a weighted averaging method (11) based on the function $w_L(k; x)$. The presented analytical solution $f_{n,L}(k; x)$, a smooth function on interval $[0, L]$, highly reflects the near origin solution $S_n f(x)$ for $x < L/2$ and the faraway solution $Rf(x)$ for $x > L/2$. Furthermore, the solution $f_{n,L}(k; x)$ can be continuously extended to the semi-infinite interval $[0, \infty)$. For practical performance, a procedure to choose appropriate parameters (n, L, k) in $f_{n,L}(k; x)$ is included. In addition, to improve the accuracy of $f_{n,L}(k; x)$, we propose a corrected approximation formula including an auxiliary term which properly reflects the behavior of the deviation $f_{n,L}(k; x) - f(x)$. Results of numerical experiments, compared with the aforementioned existing method [27], illustrate availability of the proposed method.

2. Series Solutions and a Reference Solution

For simplicity we consider the case of $\alpha = 1/2$ and $\beta = 1$. The power series of the Blasius stream function $f(x)$ for this case is known as

$$Sf(x) = \sum_{j=0}^{\infty}\left(-\frac{1}{2}\right)^j \frac{a_j\kappa^{j+1}}{(3j+2)!}x^{3j+2}, \tag{3}$$

where $\kappa = \kappa_0/\sqrt{2}$ and the coefficients a_j are computed from the recurrence [1]

$$a_j = \begin{cases} 1, & j = 0, 1 \\ \sum_{r=0}^{j-1} \binom{3j-1}{3r} a_r a_{j-r-1}, & j \geq 2. \end{cases} \tag{4}$$

In fact, the series becomes

$$Sf(x) = \frac{\kappa}{2}x^2 - \frac{\kappa^2}{240}x^5 + \frac{11}{161280}\kappa^3 x^8 - \frac{73}{63866880}\kappa^4 x^{11} + \cdots . \tag{5}$$

This series, however, converges for $|x| < \rho = 5.6900380545$. In this paper, we will use a partial sum

$$S_n f(x) = \sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j \frac{a_j \kappa^{j+1}}{(3j+2)!} x^{3j+2}, \tag{6}$$

with an integer $n \geq 0$, for an approximate solution to the Blasius function $f(x)$ near the origin.

On the other hand, for a reference solution approximating $f(x)$ far from the origin, we consider the following linear function:

$$Rf(x) = \beta x + \sqrt{\frac{\beta}{\alpha}} B_0 = x + \sqrt{2} B_0, \tag{7}$$

based on the property (ii) in the previous section.

Figure 1(a) illustrates graphs of the series approximate solutions $S_n f(x)$ with $n = 0, 1, 2, 3, 4$ on the interval $[0, 8]$. Therein, the dotted line indicates the numerical solution for the Blasius function $f(x)$. It is observed that $S_n f(x)$ overshoots $f(x)$ when n is even and undershoots when n is odd. In addition, Figure 1(b) shows the graph of the reference solution $Rf(x)$ which undershoots $f(x)$. To illustrate motivation of the main idea proposed in the next section, graphs of the differences $S_n f(x) - f(x)$ with $n = 0, 2, 4$ and $Rf(x) - f(x)$ are included in Figure 2, where $f(x)$ is replaced by the numerical solution.

3. Uniform Approximate Analytical Solutions

For some $k > 1$ and $L > 0$ we introduce a weight function $w_L(k; x)$ defined as

$$w_L(k; x) = \frac{x^k}{x^k + (L-x)^k}, \quad 0 \leq x \leq L. \tag{8}$$

It should be noted that $0 \leq w_L(k; x) \leq 1$ and it is strictly increasing on the interval $[0, L]$ with $w_L(k; L/2) = 1/2$ for any k. In addition, for a large k it follows that

$$w_L(k; x) = \begin{cases} O\left(\left(\frac{x}{L}\right)^k\right), & \text{for } 0 \leq x < \frac{L}{2} \\ 1 + O\left(\left(\frac{L}{x} - 1\right)^k\right), & \text{for } \frac{L}{2} < x \leq L. \end{cases} \tag{9}$$

This implies that the value of $w_L(k; x)$ goes close to 0 for $x < L/2$ and to 1 for $x > L/2$ as k increases. Figure 3 shows the graphs of $w_L(k; x)$ with $L = 10$ and $k = 2, 4, 8$, for example.

Moreover, we can find that the inverse function of $w_L(k; x) = y$ takes a form of

$$w_L^{-1}(k; y) = L \cdot \frac{y^{1/k}}{y^{1/k} + (1-y)^{1/k}} = L \cdot w_1\left(\frac{1}{k}; y\right), \quad 0 \leq y \leq 1. \tag{10}$$

In order to improve the accuracy of the approximate solutions for the Blasius function, we propose a weighted average of the series solution $S_n f(x)$ and the reference solution $Rf(x)$ as

$$f_{n,L}(k; x) = \{1 - w_L(k; x)\} S_n f(x) + w_L(k; x) Rf(x), \quad x \in [0, L]. \tag{11}$$

Therein, for given n and L, we may take the optimal value of k, denoted by k^*, which minimizes the L_2-norm of the residual function $Nf_{n,L}(k; x)$ defined as

$$\|Nf_{n,L}(k; x)\|_2^2 = \int_0^L \left\{ f'''_{n,L}(k; x) + \frac{1}{2} f_{n,L}(k; x) f''_{n,L}(k; x) \right\}^2 dx. \tag{12}$$

From the property (9) of the weight function $w_L(k; x)$, it follows that for k large enough

$$f_{n,L}(k; x) \sim \begin{cases} S_n f(x), & \text{for } 0 \leq x < \frac{L}{2} \\ Rf(x), & \text{for } \frac{L}{2} < x \leq L \end{cases} \tag{13}$$

with

$$f_{n,L}\left(k; \frac{L}{2}\right) = \frac{\{S_n f(L/2) + Rf(L/2)\}}{2}. \tag{14}$$

This implies that the point $x = L/2$ is a threshold between the near origin series solution $S_n f(x)$ and the faraway reference solution $Rf(x)$.

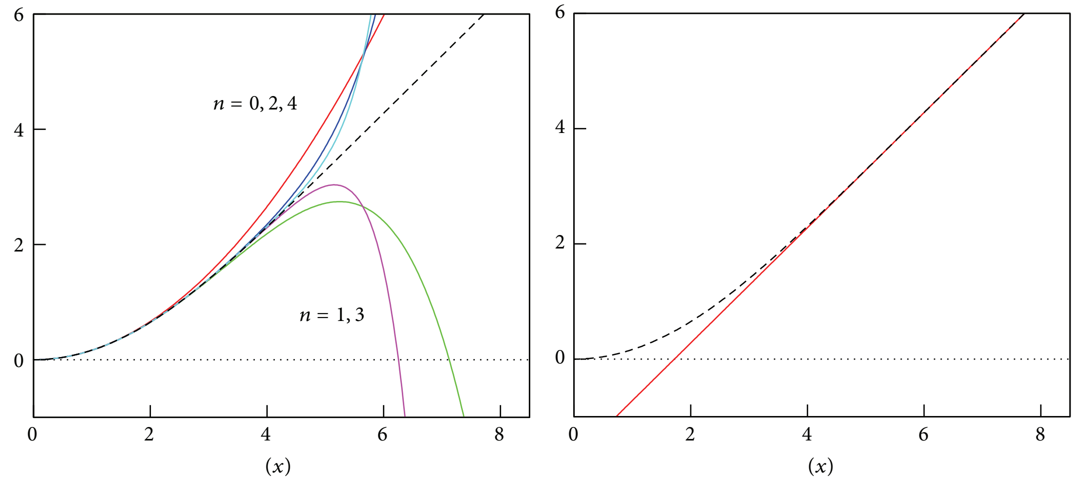
We now summarize the procedure to choose the parameters n, L, and k in the proposed solution $f_{n,L}(k; x)$ in (11) as follows.

(S1) Considering the undershoot of the reference solution $Rf(x)$, take an even integer $n \geq 0$ in the series solution $S_n f$ which overshoots the Blasius function $f(x)$ (see Figure 1).

(S2) Choose a length $L = 2d$ of the interval $[0, L]$ for some d satisfying

$$(S_n f(d) - f(d)) + (Rf(d) - f(d)) \approx 0 \tag{15}$$

or $S_n f(d) + Rf(d) \approx 2f(d)$.



(a) Series solutions $S_n f(x)$ (b) Reference solution $Rf(x)$

Figure 1: Approximations of the series solutions $S_n f(x)$ for each $n = 0, 1, 2, 3, 4$ in (a) and the reference solution $Rf(x)$ in (b).

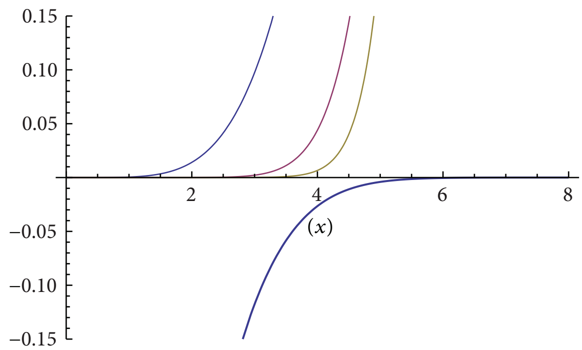


Figure 2: Differences $S_n f(x) - f(x)$ with $n = 0, 2, 4$ and $Rf(x) - f(x)$ indicated by the thin lines and the thick line, respectively.

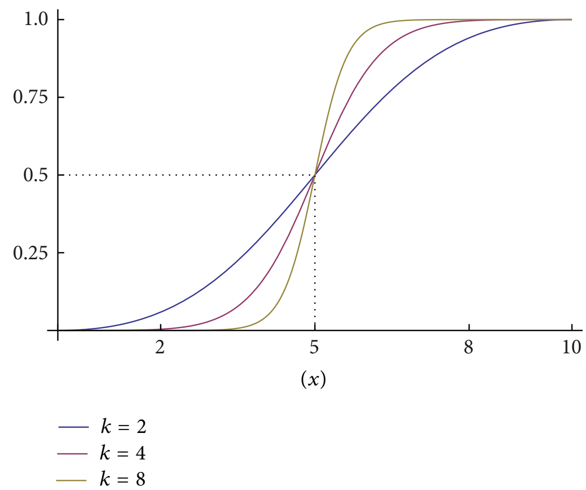


Figure 3: Behavior of the weight function $w_L(k; x)$ with $L = 10$ for each $k = 2, 4, 8$.

(S3) Find the optimal exponent $k = k^*$ of $w_L(k; x)$ which minimizes $\|Nf_{n,L}(k; x)\|_2$ defined in (12), that is, satisfies

$$\|Nf_{n,L}(k^*; x)\|_2 = \min_{k>1} \|Nf_{n,L}(k; x)\|_2. \tag{16}$$

As a result, we may expect that the presented approximate solution $f_{n,L}(k; x)$ with the parameters (n, L, k) determined by the procedure (S1)–(S3) will become a corrected approximate solution which improves accuracy of both the series solution $S_n f(x)$ and the reference solution $Rf(x)$ over the interval $[0, L]$.

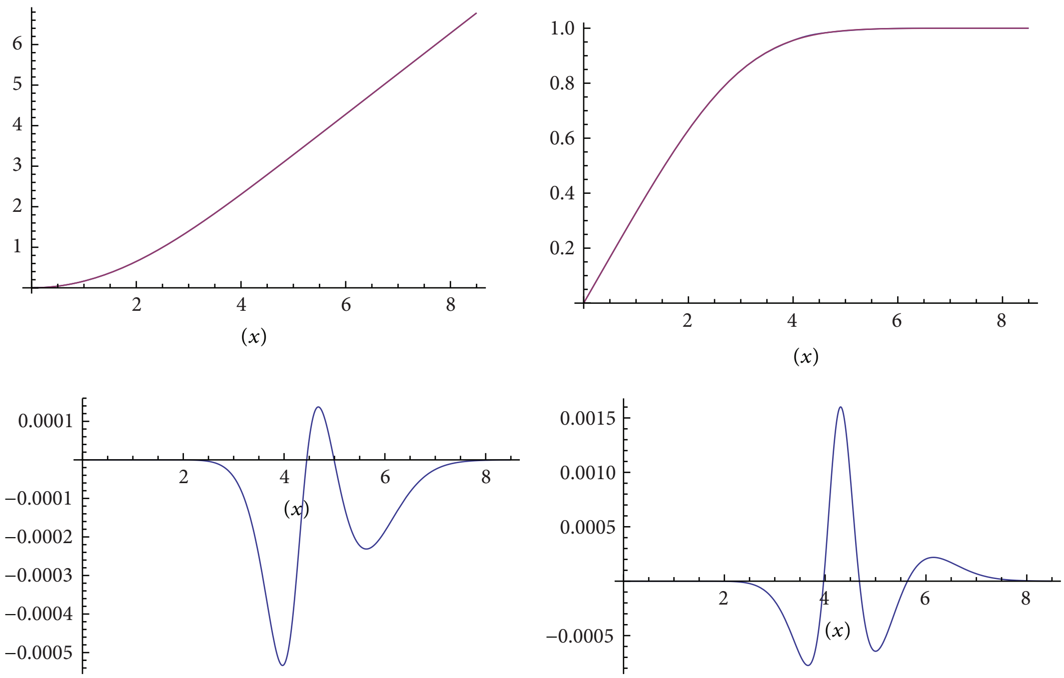
In addition, we may extend $f_{n,L}(k; x)$ to the semi-infinite interval $[0, \infty)$ continuously by setting $f_{n,L}(k; x) = Rf(x)$ for all $x \geq L$, which assures sufficient accuracy over the interval $[L, \infty)$ for $L > 6$ as can be observed in Figures 1(b) and 2.

For example, when we take $n = 0$, from Figure 2, we can find $d \approx 3$ and thus we may set $L = 2d = 6$. The optimal exponent is $k^* \approx 4.31$ which is obtained by the software, Mathematica V.9. By the similar way, we can choose the values of L and k^* for other cases of n. Table 1 includes the results for the some small values, $n = 0, 2, 4$, where k' indicates the nearest integer to the optimal exponent k^*.

Figure 4 illustrates the availability of the presented approximate solution $f_{n,L}(k; x)$ with $(n, L, k) = (4, 8.5, 11)$ given in Table 1. Additionally, numerical results for the L_2-norm errors of the approximate solution $f_{n,L}(k; x)$ and its derivatives are given in Table 2.

4. Further Improvement of the Approximate Solution

In a particular case of $(n, L, k) = (4, 8.5, 11.49)$, observing the behavior of the difference error $f_{n,L}(k; x) - f(x)$, we propose a



(a) $f_{n,L}(k;x)$ (b) $f'_{n,L}(k;x)$

Figure 4: Approximation of the weighted average $f_{n,L}(k;x)$ with the parameters $(n,L,k) = (4, 8.5, 11)$ and its error in (a) and those of the related velocity profile $f'_{n,L}(k;x)$ in (b).

correction formula by adding an auxiliary term to the formula $f_{n,L}(k;x)$ as follows:

$$\tilde{f}_{n,L}(k;x) = f_{n,L}(k;x) + Ae^{-(x-c)^2}, \qquad (17)$$

where A is the maximum of the absolute error $|f_{n,L}(k;x) - f(x)|$ at the point $x = c$. Values of A and c are numerically evaluated as

$$A = 0.00064585, \qquad c = 5.0402. \qquad (18)$$

Numerical implementation for $\tilde{f}_{n,L}(k;x)$ results in the errors

$$\left\| f - \tilde{f}_{n,L} \right\|_2 = 3.9 \times 10^{-5}, \qquad \left\| f - \tilde{f}'_{n,L} \right\|_2 = 1.3 \times 10^{-4}, \qquad \left\| f - \tilde{f}''_{n,L} \right\|_2 = 2.0 \times 10^{-2}. \qquad (19)$$

Comparing the results with those in Table 2, one can find that the corrected approximation $\tilde{f}_{n,L}(k;x)$ and its first derivative $\tilde{f}'_{n,L}(k;x)$ reasonably improve the accuracy of $f_{n,L}(k;x)$ and $f'_{n,L}(k;x)$.

For comparison with the existing approximation method, we consider the modified generalized Laguerre function Tau method introduced in the literature [27] such as

$$f_N^{\text{par}}(x) = \exp\left(\frac{-x}{2l}\right) \sum_{j=0}^{N-1} a_j L_j^{\alpha}\left(\frac{x}{l}\right), \qquad (20)$$

based on the generalized Laguerre polynomials $L_j^{\alpha}(x)$ for $\alpha = 0.5, 0.8, 1, 1.3, 1.5$ and a scaling parameter $l > 0$. For the unknown coefficients a_j's the Tau method [28, 29] is used, which generates a nonlinear system of algebraic equations. Thus a Newton-like iterative method is required to determine the coefficients a_j's as a result.

To improve the accuracy, we introduced a correction method $\tilde{f}_{n,L}(k;x)$ in (17) which includes an additional term reflecting the deviation of $f_{n,L}(k;x)$ from the Blasius function $f(x)$. As a result we can observe that the presented method is available for approximation to $f(x)$ and $f'(x)$ while the approximation to the second derivative $f''(x)$ is not so effective.

Table 1: Values of n, L, and k^* obtained by (S1)–(S3).

n	Length (L)	Optimal exponent (k^*)	k'
0	6	4.31	4
2	8	7.87	8
4	8.5	11.49	11

Table 2: L_2-norm errors of $f_{n,L}(k;x)$ and its derivatives $f'_{n,L}(k;x)$ and $f''_{n,L}(k;x)$.

(n,L,k)	$\|f - f_{n,L}\|_2$	$\|f' - f'_{n,L}\|_2$	$\|f'' - f''_{n,L}\|_2$
$(0,6,4)$	1.4×10^{-2}	1.2×10^{-2}	3.1×10^{-2}
$(2,8,8)$	8.5×10^{-3}	1.3×10^{-2}	3.9×10^{-2}
$(4,8.5,11)$	4.6×10^{-4}	1.2×10^{-3}	2.0×10^{-2}
$(4,8.5,11.49)$	7.1×10^{-4}	7.1×10^{-4}	2.0×10^{-2}

Table 3 includes numerical results of the relative errors $Ef_{n,L}(k;x_j)$, $E\widetilde{f}_{n,L}(k;x_j)$ and $Ef_N^{\text{Par}}(x_j)$, with the parameters $(N,\alpha,l) = (21,1,1)$, for the Blasius function $f(x)$. Additionally, numerical results of $E_1 f_{n,L}(k;x_j)$ and $E_1\widetilde{f}_{n,L}(k;x_j)$, and $E_1 f_N^{\text{Par}}(x_j)$ for the first derivative $f'(x)$ are given in Table 4. In the tables, the relative errors are defined as

$$Eg(x_j) = \left|\frac{f(x_j) - g(x_j)}{f(x_j)}\right|,$$
$$E_1 g(x_j) = \left|\frac{f'(x_j) - g'(x_j)}{f'(x_j)}\right| \tag{21}$$

for an approximation $g(x)$ to the Blasius function $f(x)$. Therein, $f(x_j)$ and $f'(x_j)$ are replaced by the numerical solutions for a set of nodes $\{x_j\}_{j=1}^{9} = \{1,2,\ldots,9\}$. From Tables 3 and 4 we can see that the presented approximations $f_{n,L}(k;x_j)$ and $f'_{n,L}(k;x_j)$ are less accurate than $f_N^{\text{Par}}(x_j)$ and $f_N^{\text{Par}\prime}(x_j)$ on the region $4 \le x \le 7$, and vice versa outside the region. However, it is also noticed that the inferiority of the presented approximations is quite overcome by the corrected approximation $\widetilde{f}_{n,L}(k;x_j)$ and $\widetilde{f}'_{n,L}(k;x_j)$.

Table 3: Relative errors for the Blasius function $f(x)$.

x_j	Existing method (in [27]) $Ef_N^{\text{Par}}(x_j)$	Presented methods $Ef_{n,L}(k;x_j)$	$E\widetilde{f}_{n,L}(k;x_j)$
1	8.1×10^{-6}	1.8×10^{-7}	1.8×10^{-7}
2	1.6×10^{-5}	3.9×10^{-7}	4.9×10^{-7}
3	1.2×10^{-5}	4.3×10^{-6}	2.9×10^{-6}
4	6.7×10^{-6}	9.3×10^{-5}	2.3×10^{-6}
5	5.4×10^{-6}	2.0×10^{-4}	7.3×10^{-8}
6	6.2×10^{-6}	6.2×10^{-5}	1.5×10^{-6}
7	4.4×10^{-6}	4.9×10^{-6}	2.2×10^{-6}
8	3.5×10^{-6}	1.6×10^{-7}	1.5×10^{-7}
9	3.6×10^{-6}	1.2×10^{-8}	1.2×10^{-8}

Table 4: Relative errors for the first derivative $f'(x)$.

x_j	Existing method (in [27]) $E_1 f_N^{\text{Par}}(x_j)$	Presented methods $E_1 f_{n,L}(k;x_j)$	$E_1\widetilde{f}_{n,L}(k;x_j)$
1	4.7×10^{-5}	1.2×10^{-7}	1.2×10^{-7}
2	1.3×10^{-5}	2.0×10^{-6}	2.6×10^{-6}
3	1.7×10^{-5}	5.3×10^{-5}	4.3×10^{-6}
4	7.4×10^{-6}	3.2×10^{-4}	1.5×10^{-4}
5	1.7×10^{-5}	6.5×10^{-5}	1.2×10^{-5}
6	1.5×10^{-5}	4.5×10^{-4}	4.3×10^{-5}
7	5.6×10^{-6}	7.3×10^{-5}	1.9×10^{-5}
8	3.8×10^{-6}	3.8×10^{-6}	3.2×10^{-6}
9	1.5×10^{-7}	1.5×10^{-7}	1.5×10^{-7}

5. Conclusions

For the Blasius problem on the semi-infinite interval we proposed a uniformly accurate approximation formula $f_{n,L}(k;x)$ in (11). The proposed method employs the weight function $w_L(k;x)$ in (8) to combine a near origin series solution and a faraway reference solution.

Comparing the presented solutions $f_{n,L}(k;x)$ and $\widetilde{f}_{n,L}(k;x)$ with the existing solution $f_N^{\text{Par}}(x_j)$, a solution from the generalized Laguerre spectral approach [27] based on Tau method, we summarize advantages of the presented method with discussions as follows.

(i) The presented solution $f_{n,L}(k;x)$ is composed of simple forms of known solutions, that is, a series solution $S_n f(x)$ and a reference solution $Rf(x) = x + \sqrt{2}B_0$, while the spectral method requires solving a nonlinear system of algebraic equations. This implies that the presented method will save number of evaluations in numerical implementation.

(ii) The corrected solution $\widetilde{f}_{n,L}(k;x)$ highly improves accuracy of $f_{n,L}(k;x)$ with a small number of terms $n = 4$, and numerical results show that it is comparable to the spectral solution $f_N^{\text{Par}}(x_j)$ with $N = 21$.

(iii) There is a room for further improvement of the present method, for example, by replacing the weight function $w_L(k,x)$ by some more appropriate one or employing other partial solutions instead of $S_n f(x)$ or $Rf(x)$.

To conclude, though the presented method is limitedly applicable to the Blasius problem unlike the spectral methods, we may expect to develop an extensive method for other nonlinear differential equations motivated by the advantages above.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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