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Research Article

Cauchy Problems for Evolutionary Pseudodifferential Equations over p-Adic Field

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We study a class of evolutionary pseudodifferential equations of the second order in t, $(\partial^2 u(t,x)/\partial t^2 + 2a^2 T^{\alpha/2}(\partial u(t,x)/\partial t) + b^2 T^{\alpha} u(t,x) + c^2 u(t,x) = q(t,x))$, where $t \in (0,z]$ and T^{α} is pseudodifferential operator in $x \in Q_p$, which defined by Weiyi Su in 1992. We obtained the exact solutions to the equations which belong to mixed classes of real and p-adic functions.

1. Introduction

In recent years p-adic analysis has received a lot of attention due to its applications in mathematical physics; see, for example, [1–9] and references therein. The definition of pseudodifferential operator is very important in the theory of PDE on p-adic field. In 1960s, Gibbs defined logic derivative over dyadic field. Then, Vladimirov et al. [8] generalized logic derivative over p-adic field, and we called the operator referred to as Vladimirov pseudodifferential operator. Chuong et al. have done a lot of work on PDE over p-adic field using Vladimirov operator; see, for example, [9–12]. However, as a kind of operation, Vladimirov pseudodifferential operator is not closed in the test function space $S(\mathbb{Q}_p)$. This makes the definition of Vladimirov operator difficult to be applied to distribution space $S'(\mathbb{Q}_p)$. In 1992, Su [13] redefined derivative and integral operator T^{α} over p-adic field. The definition makes the operator closed in $S(\mathbb{Q}_p)$ and can be extended to its dual space $S'(\mathbb{Q}_p)$. In 2011, Su [14] has applied the differential operator to study two-dimensional wave equations with fractal boundaries.

In this paper, we consider the exact solutions to the pseudodifferential equations of the second order in t over p-adic field $\mathbb{Q}_p$ of the type

$$\frac{\partial^2 u(t,x)}{\partial t^2} + 2a^2 T^{\alpha/2}\frac{\partial u(t,x)}{\partial t} + b^2 T^{\alpha} u(t,x) + c^2 u(t,x) = q(t,x), \quad (1)$$

with initial conditions

$$u(0,x) = f(x), \qquad u'_t(0,x) = g(x), \quad (2)$$

where $a, b, c \in \mathbb{R}$ and $a^4 - b^2 < 0$, $t \in [0,z]$, using pseudodifferential operator T^{α} $(\alpha > 0)$ which was introduced by Su in [14, 15]. Here, q, f, g are functions given by

$$q(t,x) : [0,z] \times \mathbb{Q}_p \longrightarrow \mathbb{C},$$
$$f(x) : \mathbb{Q}_p \longrightarrow \mathbb{C}, \quad (3)$$
$$g(x) : \mathbb{Q}_p \longrightarrow \mathbb{C},$$

and unknown function is

$$u(t,x) : [0,z] \times \mathbb{Q}_p \longrightarrow \mathbb{C}. \quad (4)$$

We will give the existence of the solution $u(t,x)$ to (1) and (2) with the form

$$u(t,x) = \sum_{N,j,I} u_{NjI}(t)\,\psi_{NjI}(x), \tag{5}$$

under some assumptions of f, g, q where $\psi_{NjI}(x)$ is an orthonormal base of eigenfunctions of the operator T^{α} in $L^2(\mathbb{Q}_p)$, which is constructed by Qiu and Su in [17].

2. Preliminaries

We will use the notations and results from Taibleson's book [16]. Let $\mathbb{Q}_p$ be the p-adic field, in which p is a prime number. It is a nondiscrete, locally compact, totally disconnected and complete topological field endowed with nonarchimedean norm $|\cdot| : \mathbb{Q}_p \to \mathbb{R}^+$ satisfying

(i) $|x| \geq 0$; $|x| = 0 \Leftrightarrow x = 0$;

(ii) $|xy| = |x||y|$;

(iii) $|x+y| \leq \max\{|x|, |y|\}$

for $x, y \in \mathbb{Q}_p$, so that it is also ultrametric.

Define $\mathfrak{D}$ as the ring of integers in $\mathbb{Q}_p$; $\mathfrak{D} = \{x \in \mathbb{Q}_p : |x| \leq 1\}$. It is the unique maximal compact subring in $\mathbb{Q}_p$ with the Haar measures $|\mathfrak{D}| = 1$. Define $B = \{x \in \mathbb{Q}_p : |x| \leq p^{-1}\}$ as the prime ideal. There exists a prime element β of $\mathbb{Q}_p$ with $|\beta| = p^{-1}$ such that $B = \beta\mathfrak{D}$. It is the unique maximal ideal in $\mathfrak{D}$. Define the fractional ideal in $\mathbb{Q}_p$ as $B_k = \{x \in \mathbb{Q}_p : |x| \leq p^{-k}\}$ with the Haar measures $|B_k| = p^{-k}$, $k \in \mathbb{Z}$.

For $x \in \mathbb{Q}_p$, it has a unique expression $x = x_t\beta^t + x_{t+1}\beta^{t+1} + \cdots$, $t \in \mathbb{Z}$ with $|x| = p^{-t}$. For each $l \in \mathbb{Z}$, we choose elements $z_{l,i} \in \mathbb{Q}_p$, $i \in \mathbb{Z}^+$, so that the subsets $B_{l,i} = z_{l,i} + B_l \subset \mathbb{Q}_p$ satisfy $B_{l,i} \cap B_{l,j} = \emptyset$ if $i \neq j$ and $\cup_{i=0}^{\infty} B_{l,i} = \mathbb{Q}_p$.

Define indicative function of Haar measurable subset $E \subset \mathbb{Q}_p$ as

$$\Phi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \in E^c; \end{cases} \tag{6}$$

then, the Haar measure of E is $|E| = \int_E dx = \int_{\mathbb{Q}_p} \Phi_E(x)dx$ where dx denote the Haar measure on $\mathbb{Q}_p$ normalized by the condition $\int_{\mathfrak{D}} dx = 1$.

Define translation operator $\tau_h : f \to \tau_h f$, $h \in \mathbb{Q}_p$ as $\tau_h f(x) = f(x-h)$, $x \in \mathbb{Q}_p$. Then, the test function space $S = S(\mathbb{Q}_p)$ is defined as

$$S(\mathbb{Q}_p) = \Big\{ \varphi : \mathbb{Q}_p \longrightarrow \mathbb{C}, \varphi(x) = \sum_{j=1}^{n} c_j \tau_{h_j} \Phi_{B_{k_j}(x)}, \quad c_j \in \mathbb{C}, h_j \in \mathbb{Q}_{\mathbb{p}}, k_j \in \mathbb{Z}, 1 \leq j \leq n \Big\}, \tag{7}$$

where the element $\varphi(x)$ is called test function.

For the test function space S, we give the following topology: for $\varphi \in S(\mathbb{Q}_{\mathbb{p}})$, there exists unique integers (k,l) such that the function φ is constant on the coset of B_k, with supports in the ball B_l; $\lim_{n\to+\infty}\varphi_n(x) = 0$ converges uniformly for $x \in \mathbb{Q}_{\mathbb{p}}$. Then, S is complete topological linear spaces.

Denote by $S' = S'(\mathbb{Q}_p)$ the distribution space of test function space S. S' is a complete topological linear space under the dual topology.

Let $\chi(x)$ be a fixed nontrivial character of $\mathbb{Q}_p$ which is trivial on $\mathfrak{D}$. For the p-adic field, χ can be constructed by the base value [17] as

$$\chi(\beta^{-j}) = \begin{cases} \exp\left(\dfrac{2\pi i}{p^j}\right), & \text{for } j \in \mathbb{N}, \\ 1, & \text{otherwise}. \end{cases} \tag{8}$$

Then for $x = x_t\beta_t + x_{t+1}\beta_{t+1} + \cdots$, $\chi(x) = \exp(2\pi i \sum_{j=t}^{-1} x_j p^j)$ and for $\lambda = \lambda_\tau\beta_\tau + \lambda_{\tau+1}\beta_{\tau+1} + \cdots$

$$\begin{aligned} \chi_\lambda(x) &= \chi(\lambda x) \\ &= \exp\left(2\pi i \sum_{k=0}^{-(t+\tau+1)} \left(\sum_{j=0}^{k} x_{t+\tau-j}\lambda_{t+\tau}\right) p^{t+\tau+k}\right). \end{aligned} \tag{9}$$

For $\varphi \in S(\mathbb{Q}_p)$, we define its Fourier transform $\varphi^{\wedge}$ by

$$\varphi^{\wedge}(\xi) = \int_{\mathbb{Q}_p} \varphi(x)\,\overline{\chi_\xi}(x)\,dx, \quad \xi \in \mathbb{Q}_p \tag{10}$$

and inverse Fourier transform $\varphi^{\vee}$ by

$$\varphi^{\vee}(x) = \int_{\mathbb{Q}_p} \varphi(\xi)\,\chi_x(\xi)\,d\xi, \quad x \in \mathbb{Q}_p. \tag{11}$$

In 1992, Su [13] has given definitions of the derivative for the p-adic local fields $\mathbb{Q}_p$, including derivatives of the fractional orders and real orders.

Definition 1. Let $\langle\xi\rangle = \max\{1, |\xi|\}$, $\alpha \geq 0$ if for $\varphi \in S(\mathbb{Q}_p)$, the integral

$$T^{\alpha}\varphi(x) = \left(\langle\xi\rangle^{\alpha}\varphi^{\wedge}(\xi)\right)^{\vee}(x) \tag{12}$$

exists at $x \in \mathbb{Q}_p$, where $\chi(x)$ is a fixed nontrivial character of $\mathbb{Q}_p$. Then it is called a pointwise derivative of order α of φ at x.

Note that the defined domain of T^{α} in the definition can be extended to the space $S'(\mathbb{Q}_p)$, where $S'(\mathbb{Q}_p)$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p)$.

Let $D(T^{\alpha})$ be the domain of T^{α} defined as

$$D(T^{\alpha}) = \left\{\varphi \in L^2 : \langle\xi\rangle^{\alpha}\varphi^{\wedge} \in L^2\right\}. \tag{13}$$

We have the following.

Lemma 2. *Consider*

$$D(T^{\alpha}) \subset D(T^{\alpha/2}), \tag{14}$$

with $\alpha > 0$.

Proof. Let $\varphi \in D(T^{\alpha})$, then

$$\int_{K_p} \left|\varphi^{\wedge}(\xi)\right|^2 d\xi < \infty, \qquad \int_{K_p} \langle\xi\rangle^{2\alpha} \left|\varphi^{\wedge}(\xi)\right|^2 d\xi < \infty, \tag{15}$$

with $\langle\xi\rangle^{\alpha} \leq ((1 + \langle\xi\rangle^{2\alpha})/2)$; thus

$$\int_{K_p} \langle\xi\rangle^{\alpha} \left|\varphi^{\wedge}(\xi)\right|^2 d\xi \leq \int_{K_p} \frac{1 + \langle\xi\rangle^{2\alpha}}{2} \left|\varphi^{\wedge}(\xi)\right|^2 d\xi < \infty, \tag{16}$$

and we have

$$T^{\alpha/2}\varphi \in L^2, \qquad \left\|T^{\alpha/2}\varphi\right\|^2 = \int_{K_p} \langle\xi\rangle^{\alpha} \left|\varphi^{\wedge}(\xi)\right|^2 d\xi. \tag{17}$$

Then $\varphi \in D(T^{\alpha/2})$. □

Lemma 3 (see [17]). *T^{α} is a positive definite self-adjoint operator on $D(T^{\alpha})$; $\{\psi_{NjI}\}$ is an orthonormal base of L^2 consisting of eigenfunctions of the operator T^{α}, defined as follows:*

$$\psi_{NjI}(x) = p^{-N/2} \chi_j\left(p^{N-1}x\right) \Phi_{B^0}\left(p^N x - z_I\right), \qquad N \in \mathbb{Z}, \quad I = z_I + B^0 \in \frac{K_p}{B^0}, \tag{18}$$

where $\Phi_{B^0}(x)$ is a characteristic function of a unit ball.

And

$$T^{\alpha}\psi_{NjI}(x) = \begin{cases} p^{(1-N)\alpha}\psi_{NjI}(x), & N < 1, \\ \psi_{NjI}(x), & N \geq 1. \end{cases} \tag{19}$$

3. Main Results

We will solve the following pseudodifferential equation over p-adic field by using the orthonormal base $\{\psi_{NjI}\}$ constructed in Lemma 3.

First, we consider the case of homogeneous equation.

Theorem 4. *Let*

$$\begin{gathered} \frac{\partial^2 u(t,x)}{\partial t^2} + 2a^2 T^{\alpha/2} \frac{\partial u(t,x)}{\partial t} + b^2 T^{\alpha} u(t,x) + c^2 u(t,x) = 0, \\ u(0,x) = f(x), \\ u_t'(0,x) = g(x), \end{gathered} \tag{20}$$

where $a, b, c \in \mathbb{R}$, $a^4 - b^2 < 0$, $\alpha > 0$, $f \in D(T^{\alpha})$, $g \in D(T^{\alpha/2})$.

Then one has a formal solution

$$u(t,x) = \sum_{N,j,I} u_{NjI}(t)\psi_{NjI}(x), \quad t \in [0,z], \ x \in \mathbb{Q}_p, \tag{21}$$

and $u(t,x) \in V = C([0,z], D(T^{\alpha})) \cap C^1([0,z], D(T^{\alpha/2})) \cap C^2([0,z], L^2(\mathbb{Q}_p))$.

Proof. Consider the following.

Step 1. We will write $\sum$ instead of $\sum_{N,j,I}$ in the following proof.

Let $u(t,x) = \sum u_{NjI}(t)\psi_{NjI}(x)$ be the exact form of problem (20); it is a lacunary series. Then

$$\begin{aligned} \frac{\partial^2 u(t,x)}{\partial t^2} &= \sum \frac{d^2 u_{NjI}(t)}{dt^2}\psi_{NjI}(x), \\ T_x^{\alpha} u(t,x) &= \sum_{N<1} p^{\alpha(1-N)} u_{NjI}(t)\psi_{NjI}(x) \\ &\quad + \sum_{N\geq 1} u_{NjI}(t)\psi_{NjI}(x). \end{aligned} \tag{22}$$

From

$$\frac{\partial^2 u(t,x)}{\partial t^2} + 2a^2 T^{\alpha/2} \frac{\partial u(t,x)}{\partial t} + b^2 T^{\alpha} u(t,x) + c^2 u(t,x) = 0, \tag{23}$$

we get

$$\begin{aligned} &\sum_{N<1} \left\{u_{NjI}''(t) + 2a^2 p^{(\alpha/2)(1-N)} u_{NjI}'(t) + b^2 p^{\alpha(1-N)} u_{NjI}(t) + c^2 u_{NjI}(t)\right\}\psi_{NjI}(x) \\ &\quad + \sum_{N\geq 1} \left\{u_{NjI}''(t) + 2a^2 u_{NjI}'(t) + \left(b^2 + c^2\right) u_{NjI}(t)\right\}\psi_{NjI}(x) = 0. \end{aligned} \tag{24}$$

Due to the orthogonality of $\{\psi_{NjI}(x)\}$, we have

$$\begin{aligned} &u_{NjI}''(t) + 2a^2 p^{(\alpha/2)(1-N)} u_{NjI}'(t) + \left(b^2 p^{\alpha(1-N)} + c^2\right) u_{NjI}(t) = 0, \quad N < 1, \\ &u_{NjI}''(t) + 2a^2 u_{NjI}'(t) + \left(b^2 + c^2\right) u_{NjI}(t) = 0, \quad N \geq 1. \end{aligned} \tag{25}$$

Then we obtain an ODE of order 2 on $\mathbb{R}$. And the characteristic equation is

$$\begin{gathered} \lambda^2 + 2a^2 p^{(\alpha/2)(1-N)}\lambda + b^2 p^{\alpha(1-N)} + c^2 = 0, \quad N < 1, \\ \lambda^2 + 2a^2\lambda + b^2 + c^2 = 0, \quad N \geq 1. \end{gathered} \tag{26}$$

With $a^4 - b^2 < 0$, we have

$$\Delta = \begin{cases} 4\left[\left(a^4 - b^2\right) p^{\alpha(1-N)} - c^2\right], & N < 1, \\ 4\left[\left(a^4 - b^2\right) - c^2\right], & N \geq 1. \end{cases} \tag{27}$$

The solution of the equation is

$$\begin{aligned} u_{NjI}(t) &= A_{NjI} e^{-a^2 p^{(\alpha/2)(1-N)} t}\cos(tA) + B_{NjI} e^{-a^2 p^{(\alpha/2)(1-N)} t}\sin(tA), \quad N < 1, \\ u_{NjI}(t) &= C_{NjI} e^{-a^2 t}\cos(tB) + D_{NjI} e^{-a^2 t}\sin(tB), \quad N \geq 1, \end{aligned} \tag{28}$$

where $A = \sqrt{(b^2 - a^4)p^{\alpha(1-N)} + c^2}$, $B = \sqrt{(b^2 - a^4) + c^2}$.

To determine the coefficients A_{NjI}, B_{NjI}, C_{NjI}, and D_{NjI}, we assume that $f \in D(T^{\alpha})$ can be expanded as lacunary series $f = \sum f_{NjI}\psi_{NjI}(x)$, where

$$f_{NjI} = \left\langle f(x), \psi_{NjI}(x)\right\rangle = \int_{\mathbb{Q}_p} f(x)\overline{\psi_{NjI}(x)}dx,$$
$$\sum \left|f_{NjI}\right|^2 < +\infty,$$
$$T^{\alpha} f(x) = \sum_{N<1} p^{\alpha(1-N)} f_{NjI}\psi_{NjI}(x) + \sum_{N\geq 1} f_{NjI}\psi_{NjI}(x),$$
$$\sum_{N<1} p^{2\alpha(1-N)}|f_{NjI}|^2 + \sum_{N\geq 1}|f_{NjI}|^2 < +\infty. \tag{29}$$

With the initial condition $u(0,x) = f(x)$ and then $u_{NjI}(0) = f_{NjI}$, we obtain

$$A_{NjI} = f_{NjI}, \quad N < 1,$$
$$C_{NjI} = f_{NjI}, \quad N \geq 1. \tag{30}$$

The same as with $g \in D(T^{\alpha/2})$, we get $g = \sum g_{NjI}\psi_{NjI}(x)$, where

$$g_{NjI} = \left\langle g(x), \psi_{NjI}(x)\right\rangle = \int_{\mathbb{Q}_p} g(x)\overline{\psi_{NjI}(x)}dx,$$
$$\sum \left|g_{NjI}\right|^2 < +\infty,$$
$$T^{\alpha} g(x) = \sum_{N<1} p^{\alpha(1-N)/2} g_{NjI}\psi_{NjI}(x) + \sum_{N\geq 1} g_{NjI}\psi_{NjI}(x),$$
$$\sum_{N<1} p^{\alpha(1-N)}\left|g_{NjI}\right|^2 + \sum_{N\geq 1}\left|g_{NjI}\right|^2 < +\infty. \tag{31}$$

With the initial condition $u'(0,x) = g(x)$ and then $u'_{NjI}(0) = g_{NjI}$, we obtain

$$B_{NjI} = \frac{a^2 p^{(\alpha/2)(1-N)}}{A} f_{NjI} + \frac{1}{A} g_{NjI}, \quad N < 1,$$
$$D_{NjI} = \frac{a^2}{B} f_{NjI} + \frac{1}{B} g_{NjI}, \quad N \geq 1. \tag{32}$$

Then the exact solution of the equation is

$$\begin{aligned} u(t,x) = &\sum_{N<1} e^{-a^2 p^{(\alpha/2)(1-N)} t} \\ &\times \left[f_{NjI}\cos(tA) + \left(\frac{a^2 p^{(\alpha/2)(1-N)}}{A} f_{NjI} + \frac{1}{A} g_{NjI}\right) \right. \\ &\left. \times \sin(tA) \right] \psi_{NjI}(x) \\ &+ \sum_{N\geq 1} e^{-a^2 t} \left[f_{NjI}\cos(tB) + \left(\frac{a^2}{B} f_{NjI} + \frac{1}{B} g_{NjI}\right)\right. \\ &\left. \times \sin(tB) \right] \psi_{NjI}(x). \end{aligned} \tag{33}$$

Step 2. We will prove that the solution we obtained in Step 1 satisfies the conditions in Theorem 4.

(i) Consider that

$$0 < e^{-a^2 p^{(\alpha/2)(1-N)} t} \leq 1; \qquad 0 < e^{-a^2 t} \leq 1;$$
$$|\cos(tx)| \leq 1; \qquad |\sin(tx)| \leq 1. \tag{34}$$

Then the series of $u(t,x)$ converges uniformly in $L^2(\mathbb{Q}_p)$ where $t \in [0,z]$.

With the assumptions of $f \in D(T^{\alpha})$, $g \in D(T^{\alpha/2})$, the series

$$\begin{aligned} &T_x^{\alpha} u(t,x) \\ &= \sum_{N<1} e^{-a^2 p^{(\alpha/2)(1-N)} t} p^{\alpha(1-N)} \\ &\times \left[f_{NjI}\cos(tA) + \left(\frac{a^2 p^{(\alpha/2)(1-N)}}{A} f_{NjI} + \frac{1}{A} g_{NjI}\right)\sin(tA)\right] \\ &\times \psi_{NjI}(x) \\ &+ \sum_{N\geq 1} e^{-a^2 t}\left[f_{NjI}\cos(tB) + \left(\frac{a^2}{B} f_{NjI} + \frac{1}{B} g_{NjI}\right)\sin(tB)\right] \\ &\times \psi_{NjI}(x) \end{aligned} \tag{35}$$

is converging uniformly in $L^2(\mathbb{Q}_p)$.

(ii) We obtain

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} \\ &\quad = \sum_{N<1} e^{-a^2 p^{(\alpha/2)(1-N)} t} \\ &\qquad \times \left[g_{NjI}\cos(tA) \right. \\ &\qquad\quad - \frac{1}{A}\left(a^4 p^{\alpha(1-N)} f_{NjI} + a^2 p^{(\alpha/2)(1-N)} g_{NjI}\right) \\ &\qquad\quad \left. \times \sin(tA) - A f_{NjI}\sin(tA) \right] \psi_{NjI}(x) \\ &\quad + \sum_{N\geq 1} e^{-a^2 t} \\ &\qquad \times \left[g_{NjI}\cos(tB) - \frac{1}{B}\left(a^4 f_{NjI} + a^2 g_{NjI}\right)\sin(tB) \right. \\ &\qquad\quad \left. - B f_{NjI}\sin(tB) \right] \psi_{NjI}(x) \end{aligned} \tag{36}$$

which converges uniformly in $L^2(\mathbb{Q}_p)$, with $f \in D(T^\alpha)$, $g \in D(T^{\alpha/2})$, and

$$0 < e^{-a^2 p^{(\alpha/2)(1-N)}t} \le 1, \qquad 0 < e^{-a^2 t} \le 1. \tag{37}$$

Furthermore

$$\begin{aligned} T_x^\alpha \frac{\partial u(t,x)}{\partial t} &= \sum_{N<1} e^{-a^2 p^{(\alpha/2)(1-N)}t} p^{(\alpha/2)(1-N)} \\ &\quad \times \Big[g_{NjI} \cos(tA) - \frac{1}{A}\left(a^4 p^{\alpha(1-N)} f_{NjI} + a^2 p^{(\alpha/2)(1-N)} g_{NjI}\right) \\ &\qquad \times \sin(tA) - A f_{NjI} \sin(tA) \Big] \psi_{NjI}(x) \\ &\quad + \sum_{N\ge 1} e^{-a^2 t} \Big[g_{NjI}\cos(tB) - \frac{1}{B}\left(a^4 f_{NjI} + a^2 g_{NjI}\right) \\ &\qquad \times \sin(tB) - B f_{NjI}\sin(tB) \Big] \psi_{NjI}(x) \end{aligned} \tag{38}$$

converges uniformly in $L^2(\mathbb{Q}_p)$ where $t \in [0, z]$; then $u \in C^1([0, z]; D(T^{\alpha/2}))$.

(iii) Similarly with the above case, the series

$$\frac{\partial^2 u(t,x)}{\partial^2 t} = -\left(2a^2 T^{\alpha/2} \frac{\partial u(t,x)}{\partial t} + b^2 T^\alpha u(t,x) + c^2 u(t,x)\right) \tag{39}$$

converges uniformly in $L^2(\mathbb{Q}_p)$ where $t \in [0, z]$.

Combining (i)–(iii) we obtain $u \in V$. □

Next, we will consider the case of nonhomogeneous equation.

Theorem 5. *Let*

$$\begin{aligned} &\frac{\partial^2 u(t,x)}{\partial t^2} + 2a^2 T^{\alpha/2} \frac{\partial u(t,x)}{\partial t} + b^2 T^\alpha u(t,x) + c^2 u(t,x) \\ &\quad = q(t,x), \\ &\qquad u(0,x) = f(x), \\ &\qquad \frac{\partial u(0,x)}{\partial t} = g(x), \end{aligned} \tag{40}$$

where $a, b, c \in \mathbb{R}$, $a^4 - b^2 < 0$, $\alpha > 0$, $f \in D(T^\alpha)$, $g \in D(T^{\alpha/2})$, $q(t,x) \in C([0,z], D(T^\alpha))$.

Then there exists an exact solution $u(t,x)$ *of equation* (40) *with the form*

$$u(t,x) = \sum_{N,j,I} u_{NjI}(t)\psi_{NjI}(x), \quad t \in [0,z], \; x \in \mathbb{Q}_p, \tag{41}$$

and $u(t,x) \in V$.

Proof. Consider the following.

Step 1. Similarly to the proof of Theorem 4, we expand $q(t,x)$ as lacunary series

$$q(t,x) = \sum q_{NjI}(t)\psi_{NjI}(x), \tag{42}$$

where

$$q_{NjI}(t) = \langle q(t,x), \psi_{NjI}(x)\rangle = \int_{\mathbb{Q}_p} q(t,x)\overline{\psi_{NjI}(x)}dx, \tag{43}$$

and we obtain

$$\begin{aligned} &\sum_{N<1} \{u''_{NjI}(t) + 2a^2 p^{(\alpha/2)(1-N)} u'_{NjI}(t) \\ &\qquad + b^2 p^{\alpha(1-N)} u_{NjI}(t) + c^2 u_{NjI}(t)\} \psi_{NjI}(x) \\ &\quad + \sum_{N\ge 1} \{u''_{NjI}(t) + 2a^2 u'_{NjI}(t) \\ &\qquad + (b^2 + c^2) u_{NjI}(t)\} \psi_{NjI}(x) \\ &\quad = \sum q_{NjI}(t)\psi_{NjI}(x). \end{aligned} \tag{44}$$

Due to the orthogonality of $\{\psi_{NjI}(x)\}$, we get

$$\begin{aligned} &u''_{NjI}(t) + 2a^2 p^{(\alpha/2)(1-N)} u'_{NjI}(t) + b^2 p^{\alpha(1-N)} u_{NjI}(t) \\ &\quad + c^2 u_{NjI}(t) = q_{NjI}(t), \quad N < 1, \\ &u''_{NjI}(t) + 2a^2 u'_{NjI}(t) + (b^2 + c^2) u_{NjI}(t) \\ &\quad = q_{NjI}(t), \quad N \ge 1. \end{aligned} \tag{45}$$

It is clear that the exact solution of the equation is

$$u(t,x) = \sum_{N,j,I} u_{NjI}(t)\psi_{NjI}(x), \tag{46}$$

with

$$\begin{aligned} &u_{NjI}(t) \\ &\quad = e^{-a^2 p^{(\alpha/2)(1-N)}t} \\ &\qquad \times \Big[f_{NjI}\cos(tA) + \\ &\qquad\quad + \left(\frac{a^2 p^{(\alpha/2)(1-N)}}{A} f_{NjI} + \frac{1}{A} g_{NjI}\right)\sin(tA)\Big] \\ &\qquad + \frac{1}{A}\int_0^t e^{-a^2 p^{(\alpha/2)(1-N)}(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)A)\,d\tau, \\ &\qquad\qquad N < 1, \end{aligned}$$

$$u_{NjI}(t) = e^{-a^2 t} \times \left[f_{NjI}\cos(tB) + \left(\frac{a^2}{B} f_{NjI} + \frac{1}{B} g_{NjI}\right)\sin(tB)\right] + \frac{1}{B}\int_0^t e^{-a^2(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)B)\,d\tau, \quad N \geq 1. \tag{47}$$

Step 2. It will be proved that the solution satisfies the conditions of Theorem 5.

(i) With $q(t,x) \in C([0,T], D(T^\alpha))$ we obtain

$$q(t,x) = \sum q_{NjI}(t)\,\psi_{NjI}(x), \tag{48}$$

where $\sum |q_{NjI}(t)|^2$ converges in $t \in [0,z]$ uniformly. Then the series is bounded on $[0,z]$.

Furthermore we get

$$T_x^\alpha q(t,x) = \sum_{N<1} p^{\alpha(1-N)} q_{NjI}(t)\,\psi_{NjI}(x) + \sum_{N\geq 1} q_{NjI}(t)\,\psi_{NjI}(x), \tag{49}$$

where

$$\sum_{N<1} p^{2\alpha(1-N)} \left|q_{NjI}(t)\right|^2 + \sum_{N\geq 1}\left|q_{NjI}(t)\right|^2 \tag{50}$$

converges in $t \in [0,z]$ uniformly.

(ii) By using *Swartz* inequality, we obtain

$$\begin{aligned} &\left|\int_0^t e^{-a^2 p^{(\alpha/2)(1-N)}(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)A)\,d\tau\right|^2 \leq T^3 A^2 \int_0^T \left|q_{NjI}(\tau)\right|^2 d\tau, \\ &\left|\int_0^t e^{-a^2(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)B)\,d\tau\right|^2 \leq T^3 B^2 \int_0^T \left|q_{NjI}(\tau)\right|^2 d\tau, \end{aligned} \tag{51}$$

and we get

$$\sum_{N<1}\left|\frac{1}{A}\int_0^t e^{-a^2 p^{(\alpha/2)(1-N)}(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)A)\,d\tau\right|^2 \leq T^3 \int_0^t \sum_{N<1} |q_{NjI}(\tau)|^2 d\tau \leq T^3 \int_0^t M\,d\tau < \infty, \tag{52}$$

$$\sum_{N\geq 1}\left|\frac{1}{B}\int_0^t e^{-a^2(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)B)\,d\tau\right|^2 \text{leq} T^3 \times \int_0^t \sum_{N\geq 1}\left|q_{NjI}(\tau)\right|^2 d\tau \leq T^3 \int_0^t M\,d\tau < \infty, \tag{53}$$

with $\sum |q_{NjI}(\tau)|^2 \leq M$.

(iii) Consider

$$\sum_{N<1}\left|\frac{p^{\alpha(1-N)}}{A}\int_0^t e^{-a^2 p^{(\alpha/2)(1-N)}(t-\tau)} q_{NjI}(\tau)\sin((t-\tau)A)\,d\tau\right|^2 \leq T^3 \int_0^t \sum_{N<1}\left(p^{2\alpha(1-N)}\left|q_{NjI}(\tau)\right|^2\right) d\tau < \infty. \tag{54}$$

Combining (i)–(iii), we obtain $u(t,x) \in V$. □

4. Conclusion

In this work, a class of evolutionary pseudodifferential equations of the second order in t over p-adic field $\mathbb{Q}_p$ was investigated where T^α is a p-adic pseudodifferential operator defined by Su Weiyi. The exact solution to the equation was obtained and the uniform convergence of the series of the formal solution was constructed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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