Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 438716, 17 pages
http://dx.doi.org/10.1155/2014/438716

Research Article

Wave Front Sets with respect to the Iterates of an Operator with Constant Coefficients

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Received 28 November 2013; Accepted 5 February 2014; Published 8 May 2014

Academic Editor: Luigi Rodino

We introduce the wave front set $\mathrm{WF}_*^P(u)$ with respect to the iterates of a hypoelliptic linear partial differential operator with constant coefficients of a classical distribution $u \in \mathscr{D}'(\Omega)$ in an open set Ω in the setting of ultradifferentiable classes of Braun, Meise, and Taylor. We state a version of the microlocal regularity theorem of Hörmander for this new type of wave front set and give some examples and applications of the former result.

1. Introduction

In the 1960s Komatsu characterized in [1] analytic functions f in terms of the behaviour not of the derivatives $D^\alpha f$, but of successive iterates $P(D)^j f$ of a partial differential elliptic operator $P(D)$ with constant coefficients, proving that a C^∞ function f is real analytic in Ω if and only if for every compact set $K \subset\subset \Omega$ there is a constant $C > 0$ such that

$$\left\| P(D)^j f \right\|_{2,K} \le C^{j+1} (j!)^m, \tag{1}$$

where m is the order of the operator and $\|\cdot\|_{2,K}$ is the L^2 norm on K.

This result was generalized for elliptic operators with variable analytic coefficients by Kotake and Narasimhan [2, Theorem 1]. Later, this result was extended to the setting of Gevrey functions by Newberger and Zielezny [3] and completely characterized by Métivier [4] (see also [5]). Spaces of Gevrey type given by the iterates of a differential operator are called *generalized Gevrey classes* and were used by Langenbruch [6–9] for different purposes. We mention modern contributions like [10–13] also. More recently, Juan-Huguet [14] extended the results of Komatsu [1], Newberger and Zielezny [3], and Métivier [4] to the setting of nonquasianalytic classes in the sense of Braun et al. [15]. In [14], Juan-Huguet introduced the generalized spaces of ultradifferentiable functions $\mathscr{E}_*^P(\Omega)$ on an open subset Ω of $\mathbb{R}^n$ for a fixed linear partial differential operator P with constant coefficients and proved that these spaces are complete if and only if P is hypoelliptic. Moreover, Juan-Huguet showed that, in this case, the spaces are nuclear. Later, the same author in [16] established a Paley-Wiener theorem for the classes $\mathscr{E}_*^P(\Omega)$ again under the hypothesis of the hypoellipticity of P.

The microlocal version of the problem of iterates was considered by Bolley et al. [17] to extend the microlocal regularity theorem of Hörmander [18, Theorem 5.4]. Bolley and Camus [19] generalized the microlocal version of the problem of iterates in [17] for some classes of hypoelliptic operators with analytic coefficients. We mention [20, 21] for investigations of the same problem for anisotropic and multianisotropic Gevrey classes. On the other hand, a version of the microlocal regularity theorem of Hörmander in the setting of [15] can be found in [22, 23] by one of the authors, which continues the study begun in [24].

Here, we continue in a natural way the previous work in [14] and study the microlocal version of the problem of iterates for generalized ultradifferentiable classes in the sense of Braun et al. [15]. We begin in Section 2 with some notation and preliminaries. In Section 3, we fix a hypoelliptic linear partial differential operator with constant coefficients P and introduce the wave front set $\mathrm{WF}_*^P(u)$ with respect to the iterates of P of a distribution $u \in \mathscr{D}'(\Omega)$ (Definition 7). To do this, we describe carefully the singular support in this setting (Proposition 6). We also prove that the new wave font set gives a more precise information for the study of the propagation of singularities than previous ones in Proposition 9, Theorem 13, and Example 15 (improving the previous works [22, 23] by one of the authors for operators with constant coefficients). More precisely, we clarify in Theorem 13 the necessity of the hypoellipticity of P with a new version of the microlocal regularity theorem of Hörmander for an operator with constant coefficients. In Section 4, we prove that the product of a function in a suitable Gevrey class and a function in $\mathscr{E}_*^P(\Omega)$ is still in $\mathscr{E}_*^P(\Omega)$ (Proposition 17). This fact is used to give a more involved example, inspired in [25, Theorem 8.1.4], in which we construct a classical distribution with prescribed wave front set (Theorem 18). Finally, we mention that, as far as we know, this is the first time that a result like Proposition 17 is discussed.

2. Notation and Preliminaries

Let us recall from [15] the definitions of weight functions ω and of the spaces of ultradifferentiable functions of Beurling and Roumieu type.

Definition 1. A nonquasianalytic weight function is a continuous increasing function $\omega : [0,+\infty[\to [0,+\infty[$ with the following properties:

(α) $\exists L > 0$ s.t. $\omega(2t) \leq L(\omega(t)+1)$ $\forall t \geq 0$,

(β) $\int_1^{+\infty} (\omega(t)/t^2)dt < +\infty$,

(γ) $\log(t) = o(\omega(t))$ as $t \to +\infty$,

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

Normally, we will denote φ_ω simply by φ.

For a weight function ω, we define $\overline{\omega} : \mathbb{C}^n \to [0,+\infty[$ by $\overline{\omega}(z) := \omega(|z|)$ and again we denote this function by ω.

The *Young conjugate* $\varphi^* : [0,+\infty[\to [0,+\infty[$ is defined by

$$\varphi^*(s) := \sup_{t\geq 0} \{st - \varphi(t)\}. \tag{2}$$

There is no loss of generality to assume that ω vanishes on $[0,1]$. Then φ^* has only nonnegative values, it is convex, $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$, and $\varphi^{**} = \varphi$.

Example 2. The following functions are, after a change in some interval $[0,M]$, examples of weight functions:

(i) $\omega(t) = t^d$ for $0 < d < 1$.

(ii) $\omega(t) = (\log(1+t))^s$, $s > 1$.

(iii) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.

(iv) $\omega(t) = \exp(\beta(\log(1+t))^\alpha)$, $0 < \alpha < 1$.

In what follows, Ω denotes an arbitrary subset of $\mathbb{R}^n$ and $K \subset\subset \Omega$ means that K is a compact subset in Ω.

Definition 3. Let ω be a weight function.

(a) For a compact subset K in $\mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$, we define the seminorm

$$p_{K,\lambda}(f) := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^n} \left|f^{(\alpha)}(x)\right| \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right), \tag{3}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and set

$$\mathscr{E}_\omega^\lambda(K) := \{f \in C^\infty(K) : p_{K,\lambda}(f) < \infty\}, \tag{4}$$

which is a Banach space endowed with the $p_{K,\lambda}(\cdot)$-topology.

(b) For an open subset Ω in $\mathbb{R}^n$, the class of *ω-ultradifferentiable functions of Beurling type* is defined by

$$\mathscr{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : p_{K,\lambda}(f) < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0\}. \tag{5}$$

The topology of this space is

$$\mathscr{E}_{(\omega)}(\Omega) = \underset{\overleftarrow{K\subset\subset\Omega}}{\operatorname{proj}}\ \underset{\overleftarrow{\lambda>0}}{\operatorname{proj}} \mathscr{E}_\omega^\lambda(K), \tag{6}$$

and one can show that $\mathscr{E}_{(\omega)}(\Omega)$ is a Fréchet space.

(c) For a compact subset K in $\mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$, set

$$\mathscr{E}_{\{\omega\}}(K) = \{f \in C^\infty(K) : \text{there exists } m \in \mathbb{N} \text{ such that } p_{K,1/m}(f) < \infty\}. \tag{7}$$

This space is the strong dual of a nuclear Fréchet space (i.e., a (DFN) space) if it is endowed with its natural inductive limit topology; that is,

$$\mathscr{E}_{\{\omega\}}(K) = \underset{\overrightarrow{m\in\mathbb{N}}}{\operatorname{ind}} \mathscr{E}_\omega^{1/m}(K). \tag{8}$$

(d) For an open subset Ω in $\mathbb{R}^n$, the class of *ω-ultradifferentiable functions of Roumieu type* is defined by

$$\mathscr{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega\ \exists\lambda > 0 \text{ such that } p_{K,\lambda}(f) < \infty\}. \tag{9}$$

Its topology is the following:

$$\mathscr{E}_{\{\omega\}}(\Omega) = \underset{\overleftarrow{K\subset\subset\Omega}}{\operatorname{proj}} \mathscr{E}_{\{\omega\}}(K); \tag{10}$$

that is, it is endowed with the topology of the projective limit of the spaces $\mathscr{E}_{\{\omega\}}(K)$ when K runs the compact subsets of Ω. This is a complete PLS-space, that is, a complete space which is a projective limit of LB-spaces (i.e., a countable inductive limit of Banach spaces) with compact linking maps in the (LB) steps. Moreover, $\mathscr{E}_{\{\omega\}}(\Omega)$ is also a nuclear and reflexive locally convex space. In particular, $\mathscr{E}_{\{\omega\}}(\Omega)$ is an ultrabornological (hence barrelled and bornological) space.

The elements of $\mathscr{E}_{(\omega)}(\Omega)$ (resp., $\mathscr{E}_{\{\omega\}}(\Omega)$) are called ultradifferentiable functions of Beurling type (resp., Roumieu type) in Ω.

In the case that $\omega(t) := t^d$ $(0 < d < 1)$, the corresponding Roumieu class is the Gevrey class with exponent $1/d$. In the limit case $d = 1$, not included in our setting, the corresponding Roumieu class $\mathscr{E}_{\{\omega\}}(\Omega)$ is the space of real analytic functions on Ω, whereas the Beurling class $\mathscr{E}_{(\omega)}(\mathbb{R}^n)$ gives the entire functions.

If a statement holds in the Beurling and the Roumieu case, then we will use the notation $\mathscr{E}_*(\Omega)$. It means that in all cases, $*$ can be replaced either by (ω) or $\{\omega\}$.

For a compact set K in $\mathbb{R}^n$, define

$$\mathscr{D}_*(K) := \{f \in \mathscr{E}_*(\mathbb{R}^n) : \operatorname{supp} f \subset K\}, \tag{11}$$

endowed with the induced topology. For an open set Ω in $\mathbb{R}^n$, define

$$\mathscr{D}_*(\Omega) := \underset{\overrightarrow{K \subset\subset \Omega}}{\operatorname{ind}} \mathscr{D}_*(K). \tag{12}$$

Following [14], we consider smooth functions in an open set Ω such that there exists $C > 0$ verifying for each $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\left\|P^j(D) f\right\|_{2,K} \leq C \exp\left(\lambda \varphi^*\left(\frac{jm}{\lambda}\right)\right), \tag{13}$$

where K is a compact subset in Ω, $\|\cdot\|_{2,K}$ denotes the L^2-norm on K, and $P^j(D)$ is the jth iterate of the partial differential operator $P(D)$ of order m; that is,

$$P^j(D) = P(D) \underbrace{\circ \cdots \circ}_{j} P(D). \tag{14}$$

If $j = 0$, then $P^0(D) f = f$.

Given a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ with degree m, $P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$, the partial differential operator $P(D)$ is the following: $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where $D = (1/i)\partial$.

The spaces of ultradifferentiable functions with respect to the successive iterates of P are defined as follows.

Let ω be a weight function. Given a polynomial P, an open set Ω of $\mathbb{R}^n$, a compact subset $K \subset\subset \Omega$, and $\lambda > 0$, we define the seminorm

$$\|f\|_{K,\lambda} := \sup_{j \in \mathbb{N}_0} \left\|P^j(D) f\right\|_{2,K} \exp\left(-\lambda \varphi^*\left(\frac{jm}{\lambda}\right)\right) \tag{15}$$

and set

$$\mathscr{E}^\lambda_{P,\omega}(K) = \{f \in C^\infty(K) : \|f\|_{K,\lambda} < +\infty\}. \tag{16}$$

It is a Banach space endowed with the $\|\cdot\|_{K,\lambda}$-norm (it can be proved by the same arguments used for the standard class $\mathscr{E}^\lambda_\omega(K)$ in the sense of Braun et al.; see [15]).

The space of *ultradifferentiable functions of Beurling type with respect to the iterates of* P is

$$\mathscr{E}^P_{(\omega)}(\Omega) = \{f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < +\infty \text{ for each } K \subset\subset \Omega, \lambda > 0\}, \tag{17}$$

endowed with the topology given by

$$\mathscr{E}^P_{(\omega)}(\Omega) := \underset{\overleftarrow{K \subset\subset \Omega}}{\operatorname{proj}} \underset{\overleftarrow{\lambda > 0}}{\operatorname{proj}} \mathscr{E}^\lambda_{P,\omega}(K). \tag{18}$$

If $\{K_n\}_{n \in \mathbb{N}}$ is a compact exhaustion of Ω, we have

$$\mathscr{E}^P_{(\omega)}(\Omega) = \underset{\overleftarrow{n \in \mathbb{N}}}{\operatorname{proj}} \underset{\overleftarrow{k \in \mathbb{N}}}{\operatorname{proj}} \mathscr{E}^k_{P,\omega}(K_n) = \underset{\overleftarrow{n \in \mathbb{N}}}{\operatorname{proj}} \mathscr{E}^n_{P,\omega}(K_n). \tag{19}$$

This metrizable locally convex topology is defined by the fundamental system of seminorms $\{\|\cdot\|_{K_n,n}\}_{n \in \mathbb{N}}$.

The space of *ultradifferentiable functions of Roumieu type with respect to the iterates of* P is defined by

$$\mathscr{E}^P_{\{\omega\}}(\Omega) = \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega\ \exists \lambda > 0 \text{ such that } \|f\|_{K,\lambda} < +\infty\}. \tag{20}$$

Its topology is defined by

$$\mathscr{E}^P_{\{\omega\}}(\Omega) := \underset{\overleftarrow{K \subset\subset \Omega}}{\operatorname{proj}} \underset{\overrightarrow{\lambda > 0}}{\operatorname{ind}} \mathscr{E}^\lambda_{P,\omega}(K). \tag{21}$$

As in the Gevrey case, we call these classes *generalized nonquasianalytic classes.* We observe that in comparison with the spaces of generalized nonquasianalytic classes as defined in [14] we add here m as a factor inside φ^* in (15), where m is the order of the operator P, which does not change the properties of the classes and will simplify the notation in the following.

The inclusion map $\mathscr{E}_*(\Omega) \hookrightarrow \mathscr{E}^P_*(\Omega)$ is continuous (see [14, Theorem 4.1]). The space $\mathscr{E}^P_*(\Omega)$ is complete if and only if P is hypoelliptic (see [14, Theorem 3.3]). Moreover, under a mild condition on ω introduced by Bonet et al. [26], $\mathscr{E}^P_*(\Omega)$ coincides with the class of ultradifferentiable functions $\mathscr{E}_*(\Omega)$ if and only if P is elliptic (see [14, Theorem 4.12]).

Denoting by

$$\widehat{f}(\xi) := \int e^{-i\langle x,\xi\rangle} f(x)\, dx \tag{22}$$

the classical Fourier transform of $f \in \mathscr{E}'(\Omega)$, we recall from [22, Proposition 3.3] the following characterization of the $*$-singular support in the sense of Braun et al. [15].

Proposition 4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in \mathscr{D}'(\Omega)$, and $x_0 \in \Omega$.*

(a) *Then u is a $\mathscr{E}_{\{\omega\}}$-function in some neighborhood of x_0 if and only if for some neighborhood U of x_0 there exists a bounded sequence $u_N \in \mathscr{E}'(\Omega)$ which is equal to u in U and satisfies, for some $C > 0$ and $k \in \mathbb{N}$, the estimates*

$$|\xi|^N |\widehat{u}_N(\xi)| \leq C e^{(1/k)\varphi^*(Nk)}, \quad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n. \tag{23}$$

(b) *Then u is a $\mathscr{E}_{(\omega)}$-function in some neighborhood of x_0 if and only if for some neighborhood U of x_0 there exists a bounded sequence $u_N \in \mathscr{E}'(\Omega)$ which is equal to u in U and such that for every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ satisfying*

$$|\xi|^N |\widehat{u}_N(\xi)| \leq C_k e^{k\varphi^*(N/k)}, \quad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n. \tag{24}$$

This led, in [22, Definition 3.4], to the following definition of wave front set $\mathrm{WF}_*(u)$ in the sense of Braun et al. [15].

Definition 5. Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. The $\{\omega\}$-wave front set $\mathrm{WF}_{\{\omega\}}(u)$, resp., (ω)-wave front set $\mathrm{WF}_{(\omega)}(u)$, of u is the complement in $\Omega \times (\mathbb{R}^n \setminus 0)$ of the set of points (x_0, ξ_0) such that there exist an open neighborhood U of x_0 in Ω, a conic neighborhood Γ of ξ_0, and a bounded sequence $u_N \in \mathscr{E}'(\Omega)$ (the set of classical distributions with compact support in Ω) equal to u in U such that there are $k \in \mathbb{N}$ and $C > 0$ with the property

$$|\xi|^N |\widehat{u}_N(\xi)| \le Ce^{(1/k)\varphi^*(kN)}, \quad N = 1, 2, \ldots, \ \xi \in \Gamma \tag{25}$$

Resp., which satisfies that for every $k \in \mathbb{N}$ there is $C_k > 0$ with the property

$$|\xi|^N |\widehat{u}_N(\xi)| \le C_k e^{k\varphi^*(N/k)}, \quad N = 1, 2, \ldots, \ \xi \in \Gamma. \tag{26}$$

3. Wave Front Sets with respect to the Iterates of an Operator

Now, we assume that A is a bounded open set in $\mathbb{R}^n$ and we use the following notation:

$$A_s := \{x \in A : d(x, \partial A) > s\}, \tag{27}$$

where $d(x, \partial A)$ is the distance of x to the boundary of A. Given a linear partial differential operator $P(D)$, we denote by $P^{(\alpha)}(D)$ the operator corresponding to the polynomial $P^{(\alpha)}(\xi)$. If $P(D)$ is hypoelliptic, by [27, Theorem 4.1] and the argument used in the proof of [3, Theorem 1], there are constants $C > 0$ and $\gamma > 0$ such that for every $s \ge 0$ and $t > 0$ we have

$$\left\|P^{(\alpha)}(D) f\right\|_{2,A_{s+t}} \le C\left(t^{|\alpha|}\|P(D) f\|_{2,A_s} + t^{|\alpha|-\gamma}\|f\|_{2,A_s}\right), \quad f \in C^\infty(A). \tag{28}$$

We observe also that if $P(D)$ has constant coefficients, its formal adjoint is $P(-D)$ and, if $P(D)$ is hypoelliptic, $P(-D)$ is also hypoelliptic (because of the behavior of the associated polynomial $P(-\xi)$). Moreover, any power $P(D)^\ell$ or $P(-D)^\ell$, with $\ell \in \mathbb{N}$, of $P(D)$ or $P(-D)$, is also hypoelliptic.

We now want to generalize the notion of $*$-singular support of Proposition 4, using the iterates of a hypoelliptic linear partial differential operator P with constant coefficients. The idea is to substitute the sequence u_N which satisfies an estimate of the form (23) or (24) by the sequence $f_N = P(D)^N u$ whose Fourier transform satisfies the following estimates (29) or (30).

Proposition 6. *Let $P(D)$ be a linear partial differential operator of order m with constant coefficients which is hypoelliptic. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, $x_0 \in \Omega$ and consider the following three conditions:*

(i) $f^N = P(D)^N u$,

(ii) *(Roumieu)* $\exists k \in \mathbb{N}$, $\forall M \in \mathbb{R}$, $\exists C_M > 0$, $\forall N \in \mathbb{N}$, *and* $\xi \in \mathbb{R}^n$, *we have*

$$\left|\widehat{f}_N(\xi)\right| \le C_M e^{(1/k)\varphi^*(kNm)}(1 + |\xi|)^M, \tag{29}$$

(iii) *(Beurling)* $\forall k \in \mathbb{N}$ *and* $M \in \mathbb{R}$, $\exists C_{k,M} > 0$, $\forall N \in \mathbb{N}$, *and* $\xi \in \mathbb{R}^n$, *we have*

$$\left|\widehat{f}_N(\xi)\right| \le C_{k,M} e^{k\varphi^*(Nm/k)}(1 + |\xi|)^M. \tag{30}$$

Then, the distribution $u \in \mathscr{E}^P_{\{\omega\}}(U)$ ($u \in \mathscr{E}^P_{(\omega)}(U)$), where U is some neighborhood of x_0, if and only if there exist a neighborhood V of x_0 and a sequence $\{f_N\}$ in $\mathscr{E}'(\Omega)$ that satisfies (i) and (ii) in V (that satisfies (i) and (iii) in V).

Proof.

Sufficiency (Roumieu case). Let $u \in \mathscr{E}^P_{\{\omega\}}(U)$ with $U = B_{3r}(x_0)$, the ball in $\mathbb{R}^n$ of center x_0 and radius $3r$, $r > 0$. We choose $\chi \in \mathscr{D}(\Omega)$ such that $\chi = 1$ in $B_r(x_0)$ and $\chi = 0$ in $(B_{2r}(x_0))^c$. We set $f_N = \chi P(D)^N u$. Then, $f_N \in \mathscr{E}'(\Omega)$ and $f_N = P(D)^N u$ in $B_r(x_0)$.

Now, fix $\ell \in \mathbb{N}$. From the hypoellipticity of $P(D)$, there are constants $D, d > 0$ such that, for $|\xi|$ large enough, $|P(\xi)| \ge D|\xi|^d$. Then, from the definition of f_N we obtain, for $|\xi|$ large enough,

$$\begin{aligned} D^\ell |\xi|^{d\ell} \left|\widehat{f}_N(\xi)\right| &\le \left|P(\xi)\right|^\ell \cdot \left|\widehat{f}_N(\xi)\right| \\ &= \left|P(\xi)\right|^\ell \left|\int_{\mathbb{R}^n} \chi(x) P(D)^N u(x) e^{-i\langle x,\xi\rangle} dx\right| \\ &= \left|\int_{\mathbb{R}^n} \chi(x) P(D)^N u(x) P(-D)^\ell \left(e^{-i\langle x,\xi\rangle}\right) dx\right|. \end{aligned} \tag{31}$$

We integrate by parts in the integral above, which will be equal to

$$\left|\int_{\mathbb{R}^n} P(D)^\ell \left(\chi(x) \cdot P(D)^N u(x)\right) e^{-i\langle x,\xi\rangle} dx\right|. \tag{32}$$

From the generalized Leibniz rule, we can write (here m is the order of $P(D)$)

$$\begin{aligned} &P(D)^\ell \left(\chi(x) \cdot P(D)^N u(x)\right) \\ &\quad = \sum_{|\alpha| \le m\ell} \frac{1}{\alpha!} D^\alpha \chi(x) \cdot \left(P^\ell\right)^{(\alpha)}(D) \left(P(D)^N u(x)\right). \end{aligned} \tag{33}$$

Since $P(D)^\ell$ is hypoelliptic and $P(D)^N u$ is a C^∞-function in the bounded set $B_{3r}(x_0)$, we can apply formula (28) to the operator $P(D)^\ell$ with $t = \varepsilon$, for $0 < \varepsilon < r$, $A_{s+t} = B_{2r}(x_0)$, and $f = P(D)^N u$ (and $A_s = B_{2r+\varepsilon}(x_0)$) to obtain constants $C_\ell, \gamma > 0$ (which do not depend on N) such that

$$\begin{aligned}\left\|\left(P^{\ell}\right)^{(\alpha)}(D)\left(P(D)^N u\right)\right\|_{2,B_{2r}(x_0)} &\le C_\ell\left(\varepsilon^{|\alpha|}\left\|P(D)^{N+\ell}u\right\|_{2,B_{2r+\varepsilon}(x_0)}\right.\\ &\quad\left.+\varepsilon^{|\alpha|-\gamma}\left\|P(D)^N u\right\|_{2,B_{2r+\varepsilon}(x_0)}\right).\end{aligned} \tag{34}$$

Now, as $u \in \mathscr{E}^P_{\{\omega\}}(U)$, there are constants $k \in \mathbb{N}$ and $C > 0$ such that (we use the convexity of φ^*)

$$\begin{aligned}\left\|P(D)^{N+\ell}u\right\|_{2,B_{2r+\varepsilon}} &\le Ce^{(1/k)\varphi^*(km(N+\ell))}\\ &\le Ce^{(1/2k)\varphi^*(2kmN)+(1/2k)\varphi^*(2km\ell)}, \quad \ell, N \in \mathbb{N}.\end{aligned} \tag{35}$$

Therefore, we can estimate, by Hölder's inequality, the Fourier transform $\widehat{f}_N(\xi)$ for $|\xi|$ big enough in the following way (at the end, we use the fact that $\varphi^*(x)/x$ is an increasing function):

$$\begin{aligned}D^{\ell}|\xi|^{d\ell}\left|\widehat{f}_N(\xi)\right| &\le C_\ell\sum_{|\alpha|\le m\ell}\frac{1}{\alpha!}\left\|D^\alpha\chi\right\|_{2,B_{2r}(x_0)}\\ &\quad\cdot\left(\varepsilon^{|\alpha|}\left\|P(D)^{N+\ell}u\right\|_{2,B_{2r+\varepsilon}(x_0)}\right.\\ &\quad\left.+\varepsilon^{|\alpha|-\gamma}\left\|P(D)^N u\right\|_{2,B_{2r+\varepsilon}(x_0)}\right)\\ &\le D_{m,\ell}\left(e^{(1/k)\varphi^*(km(N+\ell))}+e^{(1/k)\varphi^*(kmN)}\right)\\ &\le E_{m,\ell}e^{(1/2k)\varphi^*(2kmN)}.\end{aligned} \tag{36}$$

On the other hand, if $|\xi|$ is bounded, we put $D_r = \|\chi\|_{2,B_{2r}(x_0)}$ and, by Hölder's inequality, we have

$$\begin{aligned}\left|\widehat{f}_N(\xi)\right| &\le \left|\int_{\mathbb{R}^n}\chi(x)\,P(D)^N u(x)\,e^{-i\langle x,\xi\rangle}dx\right|\\ &\le D_r\left\|P(D)^N u\right\|_{2,B_{2r}} \le CD_r e^{(1/2k)\varphi^*(2kNm)}.\end{aligned} \tag{37}$$

From the last estimates, we can conclude that $\exists k \in \mathbb{N}$, $\forall M \in \mathbb{R}$, $\exists C_M > 0$, $\forall N \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$,

$$\left|\widehat{f}_N(\xi)\right| \le C_M e^{(1/k)\varphi^*(kNm)}(1+|\xi|)^M, \tag{38}$$

which finishes this implication.

The *Beurling case* is similar.

Necessity (Roumieu case). Let $\{f_N\}_{N\in\mathbb{N}} \subset \mathscr{E}'(\Omega)$ satisfying (i) in some neighborhood U of x_0 and (ii). We fix a compact set $K \subset\subset U$ and take $M > (n+1)/2$. Now, by (ii), there is $k \in \mathbb{N}$ and a constant $C > 0$ that depends on n and $P(D)$ such that, by Parseval's formula,

$$\begin{aligned}\left\|P(D)^N u\right\|_{L_2(K)} &= \left\|f_N\right\|_{L_2(K)} \le \left\|f_N\right\|_{L_2(\mathbb{R}^n)}\\ &= \frac{1}{(2\pi)^n}\left(\int_{\mathbb{R}^n}(1+|\xi|)^{-2M}\right.\\ &\quad\left.\times(1+|\xi|)^{2M}\left|\widehat{f}_N(\xi)\right|^2 d\xi\right)^{1/2}\\ &\le Ce^{(1/k)\varphi^*(kNm)}\left(\int_{\mathbb{R}^n}(1+|\xi|)^{-2M}d\xi\right)^{1/2}.\end{aligned} \tag{39}$$

In a similar way, using the Fourier transform, we can see that the distributions $D^\alpha u$ satisfy analogous estimates for each multi-index α on K. By the hypoellipticity of $P(D)$ we conclude that $u \in C^\infty(U)$, and this finishes the proof in the Roumieu case.

As above, in the *Beurling case* we can argue in a similar way. □

In the rest of the paper, it is assumed that the operator $P(D)$ is hypoelliptic, but not elliptic. Hypoellipticity is not only useful for Proposition 6, but also because it gives some good properties of the space $\mathscr{E}^P_*(\Omega)$, such as completeness (cf. [14]). On the contrary, the elliptic case is not really interesting here since $\mathscr{E}^P_*(\Omega) = \mathscr{E}_*(\Omega)$ if and only if P is elliptic, as we have already mentioned at the end of Section 2.

Proposition 6 leads us to define the wave front set with respect to the iterates of an operator.

Definition 7. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, and $P(D)$ a linear partial differential hypoelliptic operator of order m with constant coefficients. We say that a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the $\{\omega\}$-wave front set with respect to the iterates of P, $\mathrm{WF}^P_{\{\omega\}}(u)$ ((ω)-wave front set with respect to the iterates of P, $\mathrm{WF}^P_{(\omega)}(u)$), if there are a neighborhood U of x_0, an open conic neighborhood Γ of ξ_0, and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathscr{E}'(\Omega)$ such that (i) and (ii) of the following conditions hold ((i) and (iii) of the following conditions hold):

(i) For every $N \in \mathbb{N}$, $f_N = P(D)^N u$ in U.

(ii) *Roumieu*:

(a) there are constants $k \in \mathbb{N}$, $M > 0$, and $C > 0$, such that

$$\begin{aligned}\left|\widehat{f}_N(\xi)\right| &\le C^N\left(e^{(1/Nmk)\varphi^*(Nmk)}+|\xi|\right)^{Nm}(1+|\xi|)^M,\\ &\qquad N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n;\end{aligned} \tag{40}$$

(b) there is a constant $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}_0$, there is $C_\ell > 0$ with the property

$$\left|\widehat{f}_N(\xi)\right| \le C_\ell e^{(1/k)\varphi^*(kNm)}(1+|\xi|)^{-\ell}, \quad N\in\mathbb{N},\ \xi\in\Gamma. \tag{41}$$

(iii) *Beurling*:

(a) there are $M, C > 0$ such that for all $k \in \mathbb{N}$, there is $C_k > 0$ such that

$$\left|\widehat{f}_N(\xi)\right| \le C_k C^N \left(e^{(k/Nm)\varphi^*(Nm/k)} + |\xi|\right)^{Nm} (1+|\xi|)^M, \quad N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n; \tag{42}$$

(b) for all $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there is $C_{k,\ell} > 0$ such that

$$\left|\widehat{f}_N(\xi)\right| \le C_{k,\ell} e^{k\varphi^*(Nm/k)} (1+|\xi|)^{-\ell}, \quad N \in \mathbb{N}, \quad \xi \in \Gamma. \tag{43}$$

If we compare the last definition with Definition 5 we can deduce, as Proposition 9 will show, that the new wave front set gives more precise information about the propagation of singularities of a distribution than the $*$-wave front set of a classical distribution ($* = \{\omega\}$ or (ω)). We first recall the following result that we state as a lemma (see [19, Proposition 1.8]).

Lemma 8. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, and $P(D)$ a linear partial differential operator with analytic coefficients in Ω of order m. Let $\chi_N \in \mathscr{D}(\Omega)$ such that*

$$\left|D^\alpha \chi_N\right| \le C(CN)^{|\alpha|}, \quad |\alpha| \le N, \tag{44}$$

where $C > 0$ does not depend on $N = 0, 1, 2, \ldots$. Then the sequence $f_N = \chi_{pmN} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of N satisfies

$$\left|\widehat{f}_N(\xi)\right| \le \widetilde{C}^N (mN + |\xi|)^{mN} (1+|\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots, \tag{45}$$

for some constants $\widetilde{C} > 0$ and $M > 0$.

Proposition 9. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, ω a weight function, and $P(D)$ a hypoelliptic linear partial differential operator of order m with constant coefficients. Then, the following inclusions hold:*

$$WF^P_{\{\omega\}} u \subset WF_{\{\omega\}} u, \qquad WF^P_{(\omega)} u \subset WF_{(\omega)} u. \tag{46}$$

Proof.

Roumieu Case. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\{\omega\}} u$. From Definition 5, there exist a neighborhood U of x_0, an open conic neighborhood F of ξ_0, and a bounded sequence $\{u_N\}_{N \in \mathbb{N}} \subset \mathscr{E}'(\Omega)$ such that $u_N = u$ in U for every $N \in \mathbb{N}$ and for some constants $C > 0$, $k \in \mathbb{N}$

$$|\xi|^N \left|\widehat{u}_N(\xi)\right| \le C e^{(1/k)\varphi^*(kN)}, \quad \xi \in F, \ N \in \mathbb{N}. \tag{47}$$

By [18, Lemma 2.2], we can find a sequence $\chi_N \in \mathscr{D}(U)$ such that $\chi_N = 1$ in a neighborhood V of x_0 and

$$\left|D^{\alpha+\beta} \chi_N\right| \le C_\alpha (C_\alpha N)^{|\beta|}, \quad \beta \in \mathbb{N}_0^n, \ |\beta| \le N. \tag{48}$$

We select $p \in \mathbb{N}$ as in Lemma 8 (or bigger if necessary) and set $f_N = \chi_{Nmp} P(D)^N u$. We first observe that, as $u = u_N$ in U for all $N \in \mathbb{N}$ and $\chi_N \in \mathscr{D}(U)$, we have $f_N = \chi_{Nmp} P(D)^N u_s$ for all $s \in \mathbb{N}$. We want to prove (i), (ii)(a), and (ii)(b) in Definition 7. By the choice of χ_N, condition (i) is fulfilled in the neighborhood V. To see (ii)(a), we observe that from Lemma 8 there is $\widetilde{C} > 0$ such that

$$\left|\widehat{f}_N(\xi)\right| \le \widetilde{C}^N (mN + |\xi|)^{mN} (1+|\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots, \tag{49}$$

for some constant $M > 0$. Since the weight function ω satisfies $\omega(t) = o(t)$ as t tends to infinity, from [22, Remark 2.4(b)], for every $k \in \mathbb{N}$ there is $C_k > 0$ such that

$$Nm \le (C_k)^{1/Nm} e^{(k/Nm)\varphi^*(Nm/k)}, \quad N \in \mathbb{N}. \tag{50}$$

In particular, for $k = 1$, we obtain

$$\left|\widehat{f}_N(\xi)\right| \le C_1 \widetilde{C}^N \left(e^{(1/Nm)\varphi^*(Nm)} + |\xi|\right)^{mN} (1+|\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots, \tag{51}$$

which proves (ii)(a).

We prove now (ii)(b). We fix $\ell \in \mathbb{N}$ and set, for $f_N = \chi_{Nmp} P(D)^N u_{Nm+\ell}$,

$$\begin{aligned}(1+|\xi|)^\ell \left|\widehat{f}_N(\xi)\right| &\le (1+|\xi|)^\ell \int \left|\widehat{\chi}_{Nmp}(\eta)\right| \left|P(\xi-\eta)\right|^N \times \left|\widehat{u}_{Nm+\ell}(\xi-\eta)\right| d\eta \\ &=: J_1(\xi) + J_2(\xi),\end{aligned} \tag{52}$$

where $J_1(\xi)$ is the integral when $|\eta| \le c|\xi|$, for $c > 0$ to be chosen, and $J_2(\xi)$ is the integral when $|\eta| \ge c|\xi|$, both considered with the factor $(1+|\xi|)^\ell$. In $J_2(\xi)$, we have

$$|\xi - \eta| \le |\xi| + |\eta| \le \left(1 + c^{-1}\right)|\eta|. \tag{53}$$

Since u_N is a bounded sequence in $\mathscr{E}'(\Omega)$, there is $M > 0$ such that $|\widehat{u}_N(\xi)| \le C_1(1+|\xi|)^M$ for all $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.

From (48), we can differentiate χ_{Nmp} up to the order Nm to obtain constants $C_2 > 0, C_\ell$ that depend on n, ℓ, and M such that (see [22, Lemma 3.5])

$$\left|\widehat{\chi}_{Nmp}(\eta)\right| \le C_\ell C_2^{Nm+1} \times \frac{e^{(1/k)\varphi^*(Nkm)}}{\left(|\eta| + e^{(1/Nkm)\varphi^*(Nkm)}\right)^{Nm}} (1+|\eta|)^{-n-1-M-\ell}, \quad \eta \in \mathbb{R}^n. \tag{54}$$

As $P(D)$ has order m, we also have $|P(\xi)|^N \le C(1+|\xi|)^{Nm}$ for some constant $C > 0$ and each $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.

Moreover, in $J_2(\xi)$, $(1+|\xi|)^\ell \le (1+c^{-1})^\ell (1+|\eta|)^\ell$ and

$$(1+|\xi-\eta|)^{Nm+M} \le \left(1+c^{-1}\right)^{Nm+M}(1+|\eta|)^{Nm+M}. \quad (55)$$

Therefore, from (54), we obtain

$$\begin{aligned}|J_2(\xi)| &\le DC_\ell \left(1+c^{-1}\right)^{M+Nm+\ell} \\ &\quad \times \int_{|\eta|\ge c|\xi|} (1+|\eta|)^{Nm+\ell}(1+|\eta|)^M \left|\widehat{\chi}_{Nmp}(\eta)\right| d\eta \\ &\le D' C_\ell C_2^{Nm+1}(1+c^{-1})^{M+Nm+\ell} e^{(1/k)\varphi^*(Nmk)}\end{aligned} \quad (56)$$

for some $D, D' > 0$.

On the other hand, if we consider the estimate $(1+|\xi|)^\ell \le (1+|\xi-\eta|)^\ell (1+|\eta|)^\ell$, we obtain

$$\begin{aligned}|J_1(\xi)| \le &\left(\int (1+|\eta|)^\ell \left|\widehat{\chi}_{Nmp}(\eta)\right| d\eta\right) \\ &\cdot \sup_{|\eta|\le c|\xi|} \left|\widehat{u}_{Nm+\ell}(\xi-\eta)\right| \\ &\cdot (1+|\xi-\eta|)^\ell \cdot |P(\xi-\eta)|^N.\end{aligned} \quad (57)$$

We observe that the integral is less than or equal to $C_\ell A^N$ for some constant $C_\ell > 0$ that depends on ℓ and the support of χ_{Nmp} and some constant $A > 0$. Now, we write $\zeta = \xi - \eta$. If Γ is a conic neighborhood of ξ_0 such that $\Gamma \subset F$, we can select $0 < c < 1$ such that for $\xi \in \Gamma$ and $|\xi - \zeta| \le c|\xi|$, we have $\zeta \in F$. Consequently, we obtain, by assumption on $\widehat{u}_{Nm+\ell}$ (see (47)), and by the estimate $|P(\zeta)|^N \le C^N (1+|\zeta|)^{Nm}$ for some positive constant C, for $\xi \in \Gamma$,

$$\begin{aligned}|J_1(\xi)| &\le C_\ell A^N \cdot \sup_{|\xi-\zeta|\le c|\xi|} \left|\widehat{u}_{Nm+\ell}(\zeta)\right| \cdot (1+|\zeta|)^\ell \cdot |P(\zeta)|^N \\ &\le \widetilde{C}_\ell \widetilde{C}^{N+1} e^{(1/k)\varphi^*(Nkm+k\ell)}\end{aligned} \quad (58)$$

for some $\widetilde{C} > 0$. We conclude, using the convexity of φ^*, that there are constants $D_\ell > 0$ and $E > 0$ such that

$$\begin{aligned}(1+|\xi|)^\ell \left|\widehat{f}_N(\xi)\right| &\le |J_1(\xi)| + |J_2(\xi)| \\ &\le D_\ell E^{N+1} e^{(1/2k)\varphi^*(2kNm)}, \quad \xi \in \Gamma.\end{aligned} \quad (59)$$

Beurling Case. Let us assume now that $(x_0, \xi_0) \notin \mathrm{WF}_{(\omega)} u$. From Definition 5, there exist a neighborhood U of x_0, an open conic neighborhood F of ξ_0, and a bounded sequence $\{u_N\}_{N\in\mathbb{N}} \subset \mathscr{E}'(\Omega)$ such that $u_N = u$ in U for every $N \in \mathbb{N}$ and for every $k \in \mathbb{N}$ there is $C_k > 0$, such that

$$|\xi|^N \left|\widehat{u}_N(\xi)\right| \le C_k e^{k\varphi^*(N/k)}, \quad \xi \in F, \ N \in \mathbb{N}. \quad (60)$$

We take χ_N and f_N as in the Roumieu case. From (50), for any $k \in \mathbb{N}$, there is $D_k > 0$ satisfying

$$\left|\widehat{f}_N(\xi)\right| \le D_k \widetilde{C}^N \left(e^{(k/Nm)\varphi^*(Nm/k)} + |\xi|\right)^{mN} (1+|\xi|)^M, \quad \xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots, \quad (61)$$

which proves (iii)(a).

To prove (iii)(b), fix $\ell \in \mathbb{N}$ and consider now the estimate (use (48) and (50))

$$\begin{aligned}\left|\widehat{\chi}_{Nmp}(\eta)\right| &\le C_\ell C^{Nm+1} \frac{C_k e^{k\varphi^*(Nm/k)}}{\left(|\eta| + e^{(k/Nm)\varphi^*(Nm/k)}\right)^{Nm}} \\ &\quad \times (1+|\eta|)^{-n-1-M-\ell}, \quad \eta \in \mathbb{R}^n.\end{aligned} \quad (62)$$

Here,

$$\begin{aligned}(1+|\xi|)^\ell \left|\widehat{f}_N(\xi)\right| &\le (1+|\xi|)^\ell \\ &\quad \times \int \left|\widehat{\chi}_{Nmp}(\eta)\right| |P(\xi-\eta)|^N \\ &\quad \times \left|\widehat{u}_{Nm+\ell}(\xi-\eta)\right| d\eta \\ &=: J_1(\xi) + J_2(\xi),\end{aligned} \quad (63)$$

where $J_1(\xi)$ is the integral when $|\eta| \le c|\xi|$, for $c > 0$ to be chosen, and $J_2(\xi)$ is the integral when $|\eta| \ge c|\xi|$. In this case, we use (60) and obtain a constant $C_\ell > 0$ which depends on ℓ (and M, n) and a constant $E > 0$ with the property that for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that for any $\xi \in \Gamma$ and $N \in \mathbb{N}$,

$$(1+|\xi|)^\ell \left|\widehat{f}_N(\xi)\right| \le C_\ell E^{N+1} C_k e^{k\varphi^*(Nm/k)}, \quad \xi \in \Gamma, \ N \in \mathbb{N}. \quad (64)$$

This concludes the Beurling case. □

Corollary 10. *Let $u \in \mathscr{D}'(\Omega)$, and let K be a compact subset of Ω and F a closed cone in $\mathbb{R}^n$. Let ω be a weight function. Suppose that $\{\chi_N\} \subset \mathscr{D}(K)$ is like in (48). Then, we have the following:*

(a) *If $WF^P_{\{\omega\}}(u) \cap (K \times F) = \emptyset$, then the sequence $g_N = \chi_{Nmp} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of N, satisfies that there is $k \in \mathbb{N}$ such that for every $\ell \in \mathbb{N}$, there is $C_\ell > 0$ with*

$$\left|\widehat{g_N}(\xi)\right| \le C_\ell e^{(1/k)\varphi^*(kNm)} (1+|\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}. \quad (65)$$

(b) *If $WF^P_{(\omega)}(u) \cap (K \times F) = \emptyset$, then the sequence $g_N = \chi_{Nmp} P(D)^N u$, for $p \in \mathbb{N}$ large enough independent of N, satisfies that for every $k, \ell \in \mathbb{N}$ there is $C_{k,\ell} > 0$ with*

$$\left|\widehat{g_N}(\xi)\right| \le C_{k,\ell} e^{k\varphi^*(Nm/k)} (1+|\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}. \quad (66)$$

Proof. We make a sketch of proof of (a) only. Let $x_0 \in K$, $\xi_0 \in F\setminus\{0\}$ and choose U and Γ, with Γ a conic subset of F and f_N according to Definition 7. If the support of χ_N is in U, we have $\chi_{Nmp}P(D)^N u = \chi_{Nmp} f_N$. Now, the proof is like (ii)(b) of Proposition 9 for the set Γ and f_N instead of $P(D)^N u_{Nm+\ell}$. To obtain a uniform estimate in F, we can proceed as in [22, Lemma 3.5] at the end of the proof of (a). See also the proof of [25, Lemma 8.4.4]. □

The singular support of a classical distribution $u \in \mathscr{D}'(\Omega)$ with respect to the class $\mathscr{E}_*^P$ is the complement in Ω of the biggest open set U, where $u|_U \in \mathscr{E}_*^P(U)$. As a consequence of Propositions 6 and 9 and Corollary 10, we obtain the following result.

Corollary 11. *The projection in Ω of $WF_*^P(u)$ is the singular support with respect to the class $\mathscr{E}_*^P(\Omega)$ if $u \in \mathscr{D}'(\Omega)$.*

Proof. Follow the lines of the proofs of [22, Theorem 3.6] and [25, Theorem 8.4.5]. □

Remark 12. We observe that from the definition it is obvious that if P is hypoelliptic, then for $* = (\omega)$ or $\{\omega\}$

$$\mathrm{WF}_*^P(u) = \mathrm{WF}_*^P(Pu). \tag{67}$$

Then, by Proposition 9, the following inclusions hold:

$$\mathrm{WF}_*^P(u) = \mathrm{WF}_*^P(Pu) \subset \mathrm{WF}_*(Pu) \subset \mathrm{WF}_*(u). \tag{68}$$

Now, we can state an improvement of [22, Theorem 4.8] for operators with constant coefficients.

Theorem 13. *Let $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, be a hypoelliptic linear partial differential operator with constant coefficients and order m and let Ω be an open subset of $\mathbb{R}^n$. Let P_m be the principal part of P and $\Sigma = \{(x,\xi) \in \Omega\times\mathbb{R}^n \setminus \{0\} : P_m(\xi) = 0\}$ the characteristic set of $P(D)$. Then, for any distribution $u \in \mathscr{D}'(\Omega)$*

$$WF_*(u) \subset WF_*^P(u) \cup \Sigma. \tag{69}$$

Proof. Let $(x_0,\xi_0) \notin \mathrm{WF}_*^P(u)$ such that $P_m(\xi_0) \ne 0$. Then, there are a neighborhood U of x_0, a conic neighborhood Γ of ξ_0, and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathscr{E}'(\Omega)$ that verify (i), (ii)(a)-(ii)(b) in the Roumieu case, and (iii)(a)-(iii)(b) in the Beurling case of Definition 7. We take $F \subset \Gamma$ such that $P_m(\xi) \ne 0$ for $\xi \in F$. We take a compact neighborhood $K \subset U$ of x_0 and consider a sequence $\{\chi_N\}_{N\in\mathbb{N}} \subset \mathscr{D}(U)$ satisfying (48) such that $\chi_N \equiv 1$ on K.

We set now $u_N = \chi_{3m^2N} u$. We want to estimate

$$\begin{aligned}\widehat{u}_N(\xi) &= \left\langle u, \chi_{3m^2N} e^{-i\langle x,\xi\rangle}\right\rangle \\ &= \int u(x)\,\chi_{3m^2N}(x)\,e^{-i\langle x,\xi\rangle}\,dx.\end{aligned} \tag{70}$$

To estimate $|\widehat{u}_N(\xi)|$ in F, we will solve in an approximate way the following equation:

$${}^tP(D)^N v(x) = \chi_{3m^2N}(x)\,e^{-i\langle x,\xi\rangle}. \tag{71}$$

As in [17], we put $v(x) = e^{-i\langle x,\xi\rangle} w(x,\xi)/P_m(\xi)^N$. For $(x,\xi) \in K\times F$, we have

$$\begin{aligned}&{}^tP(D)\left(e^{-i\langle x,\xi\rangle} P_m^{-1}(\xi)\,w\right)\\ &\quad= \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha D_x^\alpha \left(e^{-i\langle x,\xi\rangle} P_m^{-1}(\xi)\,w\right)\\ &\quad= \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha e^{-i\langle x,\xi\rangle} P_m^{-1}(\xi)\\ &\qquad\times \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (-\xi)^\beta D_x^{\alpha-\beta} w\\ &\quad=: e^{-i\langle x,\xi\rangle}(I-R)\,w,\end{aligned} \tag{72}$$

where $R = R_1 + \cdots + R_m$, with $R_j = R_j(\xi, D)$ a differential operator of order $\le j$ which depends on the parameter ξ such that $R_j|\xi|^j$ is homogeneous of order 0. Recursively, it is easy to compute then

$${}^tP(D)^N\left(e^{-i\langle x,\xi\rangle} P_m^{-N}(\xi)\,w\right) = e^{-i\langle x,\xi\rangle}(I-R)^N w. \tag{73}$$

Therefore, we want to give an approximate solution of

$$e^{-i\langle x,\xi\rangle}(I-R)^N w = \chi_{3m^2N}(x)\,e^{-i\langle x,\xi\rangle}. \tag{74}$$

A formal solution of (74) is given by the series:

$$w = (I-R)^{-N}\chi_{3m^2N} = \sum_{j=0}^{+\infty} \binom{-N}{j} (-1)^j R^j \chi_{3m^2N}. \tag{75}$$

For

$$w_N := \sum_{j=0}^{mN} \binom{-N}{j} (-1)^j R^j \chi_{3m^2N}, \tag{76}$$

we can write

$$\begin{aligned}(I-R)^N w_N &= \sum_{h=0}^{N} \binom{N}{h} (-1)^h R^h\\ &\quad\times \sum_{j=0}^{mN} \binom{-N}{j} (-1)^j R^j \chi_{3m^2N}\\ &= \sum_{h=0}^{N}\sum_{j=0}^{mN} \binom{N}{h}\binom{-N}{j} (-1)^{h+j} R^{h+j}\chi_{3m^2N}.\end{aligned} \tag{77}$$

We observe that the coefficient of $R^{h+j}\chi_{3m^2N} = R^k\chi_{3m^2N}$ with $h+j = k \le mN$ is given by

$$(-1)^k \sum_{h=0}^{k} \binom{N}{h}\binom{-N}{k-h} = 0, \quad k \ge 1, \tag{78}$$

by the Chu-Vandermonde identity. For $k \ge mN+1$, the term R^k does not appear anymore for $h = 0$. So, we do not have all the summands needed in the identity above and hence the coefficients of R^k are not zero. Therefore, (we write χ for χ_{3m^2N} for simplicity)

$$\begin{aligned}(I-R)^N w_N &= \chi + \sum_{h=1}^{N}\sum_{j=mN+1-h}^{mN}\binom{N}{h}\binom{-N}{j}(-1)^{h+j}R^{h+j}\chi \\ &= \chi - e_N\end{aligned} \tag{79}$$

for

$$e_N := \sum_{h=1}^{N}\sum_{j=mN+1-h}^{mN}\binom{N}{h}\binom{-N}{j}(-1)^{h+j+1}R^{h+j}\chi. \tag{80}$$

Then,

$$\begin{aligned}{}^tP(D)^N\left(e^{-i\langle x,\xi\rangle}P_m^{-N}w_N\right) &= e^{-i\langle x,\xi\rangle}(I-R)^N w_N \\ &= e^{-i\langle x,\xi\rangle}\left(\chi - e_N\right).\end{aligned} \tag{81}$$

If we apply these identities to u, we obtain

$$\begin{aligned}\widehat{u}_N(\xi) &= \int e^{-i\langle x,\xi\rangle}\chi_{3m^2N}u(x)\,dx \\ &= \int e^{-i\langle x,\xi\rangle}e_N(x,\xi)\,u(x)\,dx \\ &\quad + \int {}^tP(D)^N\left(e^{-i\langle x,\xi\rangle}P_m^{-N}w_N\right)\cdot u(x)\,dx \\ &= \int e^{-i\langle x,\xi\rangle}e_N(x,\xi)\,u(x)\,dx \\ &\quad + \int e^{-i\langle x,\xi\rangle}P_m^{-N}(\xi)\,w_N(x,\xi)\,P(D)^N u(x)\,dx \\ &=: H_1(\xi) + H_2(\xi),\end{aligned} \tag{82}$$

where the integrals denote action of distributions.

We suppose now that u has order $M > 0$ in a neigborhood of K. Since $H_1(\xi) = \langle u, e_N e^{-i\langle x,\xi\rangle}\rangle$, we have

$$\begin{aligned}\left|H_1(\xi)\right| &\le C\sum_{|\beta|\le M}\left|D_x^\beta\left(e_N(x,\xi)\,e^{-i\langle x,\xi\rangle}\right)\right| \\ &\le C\sum_{|\beta|\le M}\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\left|D_x^\alpha e_N(x,\xi)\right| \\ &\quad \cdot\left|D_x^{\beta-\alpha}e^{-i\langle x,\xi\rangle}\right| \\ &\le C'\sum_{|\alpha|\le M}\left(1+|\xi|\right)^{M-|\alpha|}\sup_x\left|D_x^\alpha e_N(x,\xi)\right|.\end{aligned} \tag{83}$$

In order to estimate this expression, first we estimate

$$\left|D_x^\alpha e_N\right| \le \left|\sum_{h=1}^{N}\sum_{j=mN+1-h}^{mN}\binom{N}{h}\binom{-N}{j}D_x^\alpha\left(R^{j+h}\chi_{3m^2N}\right)\right|. \tag{84}$$

The number of terms in e_N depends on

$$\begin{aligned}&\left|\sum_{j=mN+1-h}^{mN}\binom{-N}{j}\right| \\ &\quad\le \sum_{j=mN+1-h}^{mN}\binom{N+mN-1}{j} \le 2^{N+mN-1}.\end{aligned} \tag{85}$$

Now, since $\sum_{h=0}^{N}\binom{N}{h} = 2^N$ and in the sum of the expression of e_N, $mN < s = h + j \le mN + N$, we obtain (we recall that $R = R_1 + \cdots + R_m$)

$$\begin{aligned}\left|D_x^\alpha e_N\right| &\le 2^{(m+2)N}\sum_{s=mN+1}^{mN+N}\left|D_x^\alpha\left(R^s\chi_{3m^2N}\right)\right| \\ &\le C^N\sum_{s=mN+1}^{mN+N}\sum_{j_1+\cdots+j_m=s}\frac{s!}{j_1!\cdots j_m!} \\ &\qquad\times\left|D_x^\alpha\left(R_1^{j_1}\cdots R_m^{j_m}\chi_{3m^2N}\right)\right|.\end{aligned} \tag{86}$$

In the last expression, we obtain a sum of A^N terms, for some constant $A > 0$, of the form $R_{j_1}\cdots R_{j_k}$ which contain derivatives of order $mN + 1 + j_N$ and are homogeneous of degree $-mN - 1 - j_N$, where $0 \le j_N \le m^2N$. Then, if we take $|\xi| > N$, we get a new constant $B > 0$, such that

$$\begin{aligned}&\left|D_x^\alpha e_N\right| \\ &\quad\le A^N\sum_{p=0}^{m^2N}\left(3m^2N\right)^{Nm+1+p+|\alpha|}|\xi|^{-Nm-1-p} \\ &\quad\le B^{N+|\alpha|}N^{|\alpha|+N}|\xi|^{-N}.\end{aligned} \tag{87}$$

Therefore, we obtain a new constant $C > 0$ such that

$$\left|H_1(\xi)\right| \le C^N\left(1+|\xi|\right)^M N^{N+M}|\xi|^{-N}, \quad \forall|\xi| > N. \tag{88}$$

We study now

$$\begin{aligned}H_2(\xi) &= \int e^{-i\langle x,\xi\rangle}P_m^{-N}(\xi)\,w_N(x,\xi)\,P(D)^N u(x)\,dx \\ &= P_m^{-N}(\xi)\int e^{-i\langle x,\xi\rangle}w_N(x,\xi)\,f_N(x)\,dx \\ &= P_m^{-N}(\xi)\cdot\mathcal{F}\left(w_N f_N\right)(\xi) \\ &= P_m^{-N}(\xi)\cdot\frac{1}{(2\pi)^n} \\ &\qquad\times\int_{\mathbb{R}^n}\widehat{w_N}(\eta)\cdot\widehat{f_N}(\xi-\eta)\,d\eta := S_1(\xi) + S_2(\xi),\end{aligned} \tag{89}$$

where we have splitted $H_2(\xi)$ in the sum of $S_1(\xi)$ and $S_2(\xi)$, the first when $|\eta| \le c|\xi|$ and the second when $|\eta| \ge c|\xi|$, for a constant $c > 0$ to be chosen.

First, we estimate w_N defined in formula (76). Proceeding in a similar way as before with the expression of e_N, if we take $|\xi| > mN$ and $|\beta| \le 2m^2N$ and estimate the binomials as in (85), we find a constant $A > 0$ such that

$$\begin{aligned} \left|D_x^\beta w_N\right| &\le \left|\sum_{j=0}^{mN}\binom{-N}{j}(-1)^j D_x^\beta\left(R^j\chi_{3m^2N}\right)\right| \\ &\le \sum_{j=0}^{mN}\left|\binom{-N}{j}\right|\sum_{j_1+\cdots+j_m=j}\frac{j!}{j_1!\cdots j_m!} \\ &\qquad\qquad \times\left|D_x^\beta\left(R_1^{j_1}\cdots R_m^{j_m}\chi_{3m^2N}\right)\right| \\ &\le C^{N+1}\left(3m^2N\right)^{|\beta|}\sum_{j=0}^{mN}\left|\binom{-N}{j}\right| \\ &\quad\times\sum_{j_1+\cdots+j_m=j}\frac{j!}{j_1!\cdots j_m!}|\xi|^{-jm}\left(3m^2N\right)^{jm} \\ &\le A^N(mN)^{|\beta|}. \end{aligned} \tag{90}$$

At this point, we have to separate Beurling and Roumieu cases.

Roumieu Case. From Definition 7(ii)(a), we have

$$\left|\widehat{f}_N(\xi)\right| \le \widetilde{C}^N\left(e^{(1/Nmk)\varphi^*(Nmk)}+|\xi|\right)^{Nm}(1+|\xi|)^M, \qquad N\in\mathbb{N},\quad \xi\in\mathbb{R}^n, \tag{91}$$

for some constants $\widetilde{C} > 0$, $M > 0$, and $k \in \mathbb{N}$. Now, as $\omega_N \in \mathscr{D}(U)$, by (90), we have, as in [22, Lemma 3.5],

$$\left|\widehat{w_N}(\eta)\right| \le C^{N+1}\frac{e^{(1/k)\varphi^*(Nmk)}}{\left(e^{(1/Nmk)\varphi^*(Nmk)}+|\eta|\right)^{Nm}}(1+|\eta|)^{-n-1-M}, \qquad \eta\in\mathbb{R}^n. \tag{92}$$

We proceed now as in the proof of (ii)(b) of Proposition 9 in order to estimate $H_2(\xi) = S_1(\xi) + S_2(\xi)$. In $S_2(\xi)$, we have $|\xi - \eta| \le (1 + c^{-1})|\eta|$ and, by (92), we deduce

$$\begin{aligned} \left|S_2(\xi)\right| &\le (2\pi)^{-n}\left|P_m(\xi)\right|^{-N} \\ &\quad\times\int_{|\eta|\ge c|\xi|}\left|\widehat{w_N}(\eta)\,\widehat{f_N}(\xi-\eta)\right|d\eta \\ &\le D^N|\xi|^{-Nm}\left(1+c^{-1}\right)^{Nm+M} \\ &\quad\times\int_{|\eta|\ge c|\xi|}\left|\widehat{w_N}(\eta)\right|\left(e^{(1/Nmk)\varphi^*(Nmk)}+|\eta|\right)^{Nm} \\ &\qquad\times(1+|\eta|)^M d\eta \\ &\le B^N e^{(1/k)\varphi^*(Nmk)}|\xi|^{-Nm}, \end{aligned} \tag{93}$$

for some constants $D, B > 0$.

For $S_1(\xi)$ we have

$$\left|S_1(\xi)\right| \le \left|P_m(\xi)\right|^{-N}\left\|\widehat{w_N}\right\|_{L_1}\cdot\sup_{|\eta|\le c|\xi|}\left|\widehat{f_N}(\xi-\eta)\right|. \tag{94}$$

As in the proof of Proposition 9, we can estimate $S_1(\xi)$, in the Roumieu case, with the use of (ii)(b) of Definition 7 in the following way: we select $c > 0$ for which there are $C > 0$ and $k \in \mathbb{N}$ such that for ξ in some neighborhood Γ' of ξ_0 (see the argument before inequality (58)),

$$\begin{aligned} &\sup_{|\eta|\le c|\xi|}\left|\widehat{f_N}(\xi-\eta)\right| \\ &\quad\le C^{N+1}e^{(1/k)\varphi^*(Nkm)}\sup_{|\eta|\le c|\xi|}(1+|\xi-\eta|)^M \\ &\quad\le C^{N+1}e^{(1/k)\varphi^*(Nkm)}(1+(1+c)|\xi|)^M. \end{aligned} \tag{95}$$

Consequently, since $\|\widehat{w_N}\|_{L_1} \le A^N$ for some constant $A > 0$,

$$\begin{aligned} \left|S_1(\xi)\right| &\le D^{N+1}e^{(1/k)\varphi^*(Nkm)}|\xi|^M\left|P_m(\xi)\right|^{-N} \\ &\le E^{N+1}e^{(1/k)\varphi^*(Nkm)}|\xi|^{M-Nm}. \end{aligned} \tag{96}$$

Therefore, if we combine (96) and (93), we obtain two constants $C > 0$ and $h \in \mathbb{N}$ such that for ξ in some conic neighborhood of ξ_0 and $|\xi| \ge e^{(1/2N(m-1)h)\varphi^*(2N(m-1)h)}$, by (89),

$$\begin{aligned} \left|H_2(\xi)\right| &\le C^{N+1}e^{(1/h)\varphi^*(Nmh)}|\xi|^{M-Nm} \\ &\le C^{N+1}e^{(1/2h)\varphi^*(2Nh)+(1/2h)\varphi^*(2N(m-1)h)}|\xi|^{M-Nm} \\ &\le C^{N+1}e^{(1/2h)\varphi^*(2hN)}|\xi|^{M-N}. \end{aligned} \tag{97}$$

As in (50), we have $N^N \le Ae^{\varphi^*(N)}$ for some constant $A > 0$ and every $N \in \mathbb{N}$. Then, from (88), we deduce a similar estimate to the one of $|H_2(\xi)|$ for $|H_1(\xi)|$. Now, from the bounds for $H_1(\xi)$ and $H_2(\xi)$, there are constants $C, h > 0$ such that, for ξ in some conic neighborhood of ξ_0 and $|\xi| \ge e^{(1/2N(m-1)h)\varphi^*(2N(m-1)h)}$,

$$\left|\widehat{u}_N(\xi)\right| \le C^N(1+|\xi|)^M e^{(1/h)\varphi^*(hN)}|\xi|^{-N}. \tag{98}$$

We have a similar estimate when $|\xi| \le e^{(1/2N(m-1)h)\varphi^*(2N(m-1)h)}$. In fact, since the sequence u_N is bounded in $\mathscr{E}'(\Omega)$, there are constants $D > 0$ and $M' > 0$ which satisfy

$$\left|\widehat{u}_N(\xi)\right| \le D(1+|\xi|)^{M'}, \quad \xi\in\mathbb{R}^n. \tag{99}$$

Then, we have

$$\begin{aligned} &\left|\widehat{u}_N(\xi)\right| \\ &\quad\le D(1+|\xi|)^{M'} \le C\left(e^{(1/2N(m-1)h)\varphi^*(2N(m-1)h)}\right)^{M'+N}|\xi|^{-N} \\ &\quad\le C\left(e^{(1/(N+M')h')\varphi^*((N+M')h')}\right)^{M'+N}|\xi|^{-N} \\ &\quad\le D'e^{(1/h'')\varphi^*(Nh'')}|\xi|^{-N}. \end{aligned} \tag{100}$$

Beurling Case. In this setting we will proceed in a similar way. We can select $0 < c < 1$ and apply now (iii)(b) of Definition 7 to obtain, for every $k \in \mathbb{N}$, a constant $C_k > 0$ such that, for all ξ in some neighborhood of ξ_0,

$$\begin{aligned} |S_1(\xi)| &\le |P_m(\xi)|^{-N} \|\widehat{w_N}\|_{L_1} \cdot \sup_{|\eta| \le c|\xi|} |\widehat{f_N}(\xi - \eta)| \\ &\le C_k E^N e^{k\varphi^*(Nm/k)} |\xi|^{M-Nm}. \end{aligned} \tag{101}$$

In a similar way to (92), we can obtain here

$$\begin{aligned} |\widehat{w_N}(\eta)| &\le C_k C^{N+1} \frac{e^{k\varphi^*(Nm/k)}}{(e^{(k/Nm)\varphi^*(Nm/k)} + |\eta|)^{Nm}} \\ &\quad \times (1 + |\eta|)^{-n-1-M}, \quad \eta \in \mathbb{R}^n, \end{aligned} \tag{102}$$

where the constant $M > 0$ comes from Definition 7(iii)(a).

Now, as in (93), we have a constant $C > 0$ and for every $k \in \mathbb{N}$ a constant $C_k > 0$ such that

$$|S_2(\xi)| \le C_k C^N e^{k\varphi^*(Nm/k)} |\xi|^{-Nm}, \quad N \in \mathbb{N},\ |\xi| > N. \tag{103}$$

Therefore, from (101) and (103), we have $C > 0$ and for a fixed $k \in \mathbb{N}$ a constant $C_k > 0$ such that for ξ in some conic neighborhood of ξ_0 and $|\xi| \ge e^{(k/N(m-1))\varphi^*(N(m-1)/k)}$,

$$\begin{aligned} |H_2(\xi)| &\le C_k C^N e^{2k\varphi^*(Nm/2k)} |\xi|^{M-Nm} \\ &\le C_k C^N e^{k\varphi^*(N/k)} |\xi|^{M-N}. \end{aligned} \tag{104}$$

As in the Roumieu case, we deduce a similar estimate for $|H_1(\xi)|$. Then, the bounds for $H_1(\xi)$ and $H_2(\xi)$ give a constant $C > 0$ and, for every $k \in \mathbb{N}$, a constant $C_k > 0$ such that for ξ in some conic neighborhood of x_0 and $|\xi| \ge e^{(k/N(m-1))\varphi^*(N(m-1)/k)}$ ($> N$) (if N is large enough),

$$|\widehat{u}_N(\xi)| \le C_k C^N e^{k\varphi^*(N/k)} |\xi|^{M-N}. \tag{105}$$

Finally, we also have a similar estimate when $|\xi| \le e^{(k/N(m-1))\varphi^*(N(m-1)/k)}$, which concludes the proof of the theorem. ☐

Remark 14. If $P(D)$ is elliptic, then $\Sigma = \emptyset$ and Theorem 13 and Remark 12 imply that

$$\mathrm{WF}_*(u) = \mathrm{WF}_*^P(u). \tag{106}$$

Example 15. We show that the inclusions

$$\begin{aligned} \mathrm{WF}_*^P(u) &\subset \mathrm{WF}_*(u), \\ \mathrm{WF}_*^P(u) &\subset \mathrm{WF}_*(Pu) \end{aligned} \tag{107}$$

of Remark 12 are strict. As in [14] (see [26]), we consider a nonquasianalytic weight function ω satisfying the following condition: there exists a constant $H \ge 1$ such that for all $t \ge 0$,

$$2\omega(t) \le \omega(Ht) + H. \tag{108}$$

For example, if ω is a Gevrey weight, then it satisfies such a property. We consider now a polynomial P with constant complex coefficients such that it is hypoelliptic but not elliptic (for instance, the heat operator). Then by [14, Theorem 4.12], there is $u \in \mathcal{E}_{\{\omega\}}^P(\Omega) \setminus \mathcal{E}_{\{\omega\}}(\Omega)$ (for some open subset Ω of $\mathbb{R}^n$). Then, $\mathrm{WF}_{\{\omega\}}^P(u) = \emptyset$ but $\mathrm{WF}_{\{\omega\}}(u) \ne \emptyset$, which implies that the inclusion

$$\mathrm{WF}_{\{\omega\}}^P(u) \subsetneq \mathrm{WF}_{\{\omega\}}(u) \tag{109}$$

is strict.

On the other hand, if we consider now a $\{\omega\}$-hypoelliptic polynomial P which is not elliptic (e.g., the heat operator in $\mathbb{R}^n$ for $\omega(t) = t^{1/3}$), then as before there will be $u \in \mathcal{E}_{\{\omega\}}^P(\Omega) \setminus \mathcal{E}_{\{\omega\}}(\Omega)$. In particular, $\mathrm{WF}_{\{\omega\}}^P(u) = \emptyset$. Now, if $\mathrm{WF}_{\{\omega\}}(Pu) = \emptyset$, we will have $Pu \in \mathcal{E}_{\{\omega\}}(\Omega)$ and since P is $\{\omega\}$-hypoelliptic, $u \in \mathcal{E}_{\{\omega\}}(\Omega)$, which is a contradiction. Therefore, $\mathrm{WF}_{\{\omega\}}(Pu) \ne \emptyset$ and we conclude that the inclusion

$$\mathrm{WF}_{\{\omega\}}^P(u) \subsetneq \mathrm{WF}_{\{\omega\}}(Pu) \tag{110}$$

is strict.

Let us also remark that for the heat operator $Q(D) = \partial_t - \Delta_x$, we can explicitly write its characteristic set Σ, so that the previous considerations give, for $u \in \mathcal{E}_{\{\omega\}}^Q(\Omega) \setminus \mathcal{E}_{\{\omega\}}(\Omega)$, the following information on $\mathrm{WF}_{\{\omega\}}(u)$, because of Theorem 13:

$$\begin{aligned} \emptyset \ne \mathrm{WF}_{\{\omega\}}(u) &\subset \mathrm{WF}_{\{\omega\}}^Q(u) \cup \Sigma \\ &= \Sigma = \{(t, x, \tau, 0) \in \Omega \times \mathbb{R}^{n+1} : \tau \ne 0\}. \end{aligned} \tag{111}$$

In the Beurling setting, we can proceed in a similar way.

Let us finally notice that the inclusion

$$\mathrm{WF}_*(Pu) \subsetneq \mathrm{WF}_*(u) \tag{112}$$

of Remark 12 is strict in general.

4. Distributions with Prescribed Wave Front Set

The proof of the following lemma is straightforward.

Lemma 16. *Let ω be a weight function. Then, for every $a > 0$ and $m \in \mathbb{N}$*

(i) $t^m e^{-a\omega(t)} \le e^{a\varphi^*(m/a)}$ $\forall t \ge 1$;

(ii) $\inf_{j \in \mathbb{N}_0} t^{-jm} e^{a\varphi^*(jm/a)} \le t^m e^{-a\omega(t)}$ $\forall t \ge 1$.

Now, we show that the product of a Gevrey function with a function in $\mathcal{E}_*^P(\Omega)$ belongs to the last space.

Proposition 17. *Let ω be a nonquasianalytic weight function such that $\omega(t^\gamma) = o(\sigma(t))$ as $t \to \infty$, where $\gamma > 0$ is the constant in* (28) *and $\sigma(t) = t^{1/s}$ is a Gevrey weight, with $s > 1$. If $\chi \in \mathcal{E}_{\{\sigma\}}(\Omega)$ and $u \in \mathcal{E}_*^P(\Omega)$, where $* = \{\omega\}$ or (ω), then the multiplication $\chi u \in \mathcal{E}_*^P(\Omega)$.*

Proof. We will analyse the L^2-norms of $P(D)^j(\chi u)$ on a compact set K in Ω. First, we observe that, by the generalized Leibniz rule over $P(D)$ applied j times,

$$P(D)^j(\chi u) = P(D)\left[P(D) \overset{(j-1)}{\cdots} P(D)(\chi u)\right] = \sum_{|\alpha_1|,\dots,|\alpha_j|\le m} \frac{1}{\alpha_1!\cdots\alpha_j!} D^{\alpha_1+\cdots+\alpha_j}\chi \cdot P^{(\alpha_1)}(D)\left(P^{(\alpha_2)}(D)\cdots\left(P^{(\alpha_j)}(D)u\right)\right). \tag{113}$$

We fix now a compact set K in Ω such that $\operatorname{dist}(K,\partial\Omega) \ge r > 0$. We apply L^2-norms in the compact set K

$$\left\|P(D)^j(\chi u)\right\|_{2,K} \le \sum_{|\alpha_1|\le m}\cdots\sum_{|\alpha_j|\le m}\frac{1}{\alpha_1!\cdots\alpha_j!} \times \left\|D^{\alpha_1}\cdots D^{\alpha_j}\chi \cdot P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\right\|_{2,K}. \tag{114}$$

Since $\chi \in \mathscr{E}_{\{\sigma\}}(\Omega)$, there is a constant $A > 0$ such that, for each $\alpha \in \mathbb{N}_0^n$ and $x \in K$ we have

$$\left|D^\alpha\chi(x)\right| \le A^{|\alpha|}|\alpha|^{s|\alpha|}. \tag{115}$$

Consequently,

$$\sup_{x\in K}\left|D^{\alpha_1}\cdots D^{\alpha_j}\chi(x)\right| \le A^{|\alpha_1+\cdots+\alpha_j|}\left|\alpha_1+\cdots+\alpha_j\right|^{s|\alpha_1+\cdots+\alpha_j|} \le A^{jm}(jm)^{s(|\alpha_1|+\cdots+|\alpha_j|)}. \tag{116}$$

Therefore,

$$\begin{aligned}\left\|P(D)^j(\chi u)\right\|_{2,K} &\le \sum_{|\alpha_1|\le m,\dots,|\alpha_j|\le m}\frac{1}{\alpha_1!\cdots\alpha_j!}\sup_K\left|D^{\alpha_1}\cdots D^{\alpha_j}\chi\right| \cdot \left\|P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\right\|_{2,K} \\ &\le \sum_{|\alpha_1|\le m,\dots,|\alpha_j|\le m}\frac{A^{jm}(jm)^{s(|\alpha_1|+\cdots+|\alpha_j|)}}{\alpha_1!\cdots\alpha_j!} \cdot \left\|P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\right\|_{2,K}.\end{aligned} \tag{117}$$

Now, we apply (28) j times to the factor $\|P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\|_{2,K}$. We will use the notation $K(\varepsilon) = K + B(0,\varepsilon)$, for $\varepsilon > 0$. In the first step,

$$\left\|P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\right\|_{2,K} \le C\left(\varepsilon_1^{|\alpha_1|}\left\|P\, P^{(\alpha_2)}\cdots P^{(\alpha_j)}u\right\|_{2,K(\varepsilon_1)} + \varepsilon_1^{|\alpha_1|-\gamma}\left\|P^{(\alpha_2)}\cdots P^{(\alpha_j)}u\right\|_{2,K(\varepsilon_1)}\right). \tag{118}$$

In the second step, $K(\varepsilon_1)$ is replaced by $K(\varepsilon_1+\varepsilon_2)$ and so on in the next steps. Therefore, to avoid that, after j steps, the set $K(\varepsilon_1+\cdots+\varepsilon_j)$ leaves Ω and to keep it bounded for all j, we may take ε_k depending on k for all $1 \le k \le j$. We take $\varepsilon_k = Bk^{-s}$ with $B > 0$ a constant such that

$$\varepsilon_1+\cdots+\varepsilon_j = B\left(1+\frac{1}{2^s}+\cdots+\frac{1}{j^s}\right) < \frac{r}{2} \tag{119}$$

for all j. It is obvious that $\varepsilon_k^{-\gamma} \le \varepsilon_{k+1}^{-\gamma}$ for all $1 \le k \le j-1$. Moreover, we can assume that $\varepsilon_k < 1$ for all $1 \le k \le j$.

After j steps we get

$$\begin{aligned}&\left\|P^{(\alpha_1)}\cdots P^{(\alpha_j)}u\right\|_{2,K} \\ &\le C^j2^j\varepsilon_1^{|\alpha_1|}\cdots\varepsilon_j^{|\alpha_j|}\left(\left\|P^ju\right\|_{2,K(\varepsilon_1+\cdots+\varepsilon_j)} + \varepsilon_j^{-\gamma}\left\|P^{j-1}u\right\|_{2,K(\varepsilon_1+\cdots+\varepsilon_j)} + \varepsilon_{j-1}^{-\gamma}\varepsilon_j^{-\gamma}\left\|P^{j-2}u\right\|_{2,K(\varepsilon_1+\cdots+\varepsilon_j)} + \cdots + \varepsilon_1^{-\gamma}\varepsilon_2^{-\gamma}\cdots\varepsilon_j^{-\gamma}\|u\|_{2,K(\varepsilon_1+\cdots+\varepsilon_j)}\right).\end{aligned} \tag{120}$$

With our selection of ε_k for $1 \le k \le j$, we have

$$\begin{aligned}\varepsilon_1^{|\alpha_1|}\cdots\varepsilon_j^{|\alpha_j|} &= \frac{B^{|\alpha_1|+\cdots+|\alpha_j|}}{2^{s|\alpha_2|}\cdots j^{s|\alpha_j|}}, \\ (\varepsilon_{k+1}\cdots\varepsilon_j)^{-\gamma} &= \frac{(k+1)^{s\gamma}\cdots j^{s\gamma}}{B^{(j-k)\gamma}},\end{aligned} \tag{121}$$

for all $k = 0,1,\dots,j-1$. Moreover, for all j, $K(\varepsilon_1+\cdots+\varepsilon_j) \subset K(r/2)$, which is compact and a subset of Ω. Consequently, since $j^j \le e^jj!$ for all $j = 1,2,\dots$, we have (we can assume that the constant $B < 1$ and then $B^{|\alpha_k|} < 1$ for all $1 \le k \le j$)

$$\varepsilon_1^{|\alpha_1|}\cdots\varepsilon_j^{|\alpha_j|}j^{s|\alpha_1+\cdots+\alpha_j|} \le j^{s|\alpha_1|}\frac{j^{s|\alpha_2|}}{2^{s|\alpha_2|}}\cdots\frac{j^{s|\alpha_j|}}{j^{s|\alpha_j|}} \le \frac{j^{smj}}{(j!)^{sm}} \le e^{smj}. \tag{122}$$

Summing up, we obtain

$$\begin{aligned}&\left\|P(D)^j(\chi u)\right\|_{2,K} \\ &\le \left(\left\|P^ju\right\|_{2,K(r/2)} + \frac{j^{s\gamma}}{B^\gamma}\left\|P^{j-1}u\right\|_{2,K(r/2)} + \frac{(j(j-1))^{s\gamma}}{B^{2\gamma}}\left\|P^{j-2}u\right\|_{2,K(r/2)} + \cdots + \frac{(j!)^{s\gamma}}{B^{j\gamma}}\|u\|_{2,K(r/2)}\right) \\ &\times \sum_{|\alpha_1|\le m,\dots,|\alpha_j|\le m}\frac{(2Ce^{sm}A^m)^jm^{s(|\alpha_1|+\cdots+|\alpha_j|)}}{\alpha_1!\cdots\alpha_j!}.\end{aligned} \tag{123}$$

If we use the multinomial theorem,

$$\sum_{|\alpha_k|\le m}\frac{m^{s|\alpha_k|}}{\alpha_k!}\le\sum_{|\alpha|=0}^{\infty}\frac{m^{s|\alpha|}}{\alpha!}\le e^{m^s n},\tag{124}$$

where n is the dimension of the multi-index $|\alpha_k|$ or $|\alpha|$. Then, it is clear that

$$\sum_{|\alpha_1|\le m,\dots,|\alpha_j|\le m}\frac{(2C/B^{\gamma}e^{sm}A^m)^j m^{s(|\alpha_1|+\cdots+|\alpha_j|)}}{\alpha_1!\cdots\alpha_j!}\le E^j\tag{125}$$

for some constant $E>0$ that depends on $P(D)$, χ, and the compact set $K(r/2)$.

Now, we control the sequence $(j(j-1)\cdots(j-k+1))^{s\gamma}$ for $k=1,\dots,j$, which is the factor of $\|P^{j-k}u\|_{2,K(r/2)}$ and less than or equal to

$$\binom{j}{k}^{s\gamma}k!^{s\gamma}\le 2^{js\gamma}k!^{s\gamma}.\tag{126}$$

For $*=\{\omega\}$, since $\omega(t^{\gamma})=o(t^{1/s})$ as $t\to+\infty$, there is a constant $F>0$ such that

$$(k!)^{s\gamma}\le Fe^{\varphi^*(k)},\qquad k\in\mathbb{N}.\tag{127}$$

Since $\varphi^*(x)/x\to\infty$ as $t\to\infty$, for any constant $h\in\mathbb{N}$,

$$(k!)^{s\gamma}\le Fe^{(1/h)\varphi^*(kh)}\le Fe^{(1/h)\varphi^*(kmh)}.\tag{128}$$

On the other hand, since $u\in\mathscr{E}^P_{\{\omega\}}(\Omega)$, there are constants $G>0$ and $h\in\mathbb{N}$ that depend on $K(r/2)$ such that

$$\left\|P^{j-k}u\right\|_{2,K(r/2)}\le Ge^{(1/h)\varphi^*((j-k)mh)},\quad k=0,1,\dots,j,\ j\in\mathbb{N}.\tag{129}$$

Then, from the convexity of φ^*,

$$\begin{aligned}&\left\|P(D)^j(\chi u)\right\|_{2,K}\\&\le E^j2^{js\gamma}\Big(\left\|P^ju\right\|_{2,K(r/2)}+Fe^{(1/h)\varphi^*(mh)}\left\|P^{j-1}u\right\|_{2,K(r/2)}\\&\quad+Fe^{(1/h)\varphi^*(2mh)}\left\|P^{j-2}u\right\|_{2,K(r/2)}\\&\quad+\cdots+Fe^{(1/h)\varphi^*(jmh)}\|u\|_{2,K(r/2)}\Big)\\&\le(j+1)\,2^{js\gamma}E^jFGe^{(1/h)\varphi^*(jmh)}.\end{aligned}\tag{130}$$

If $*=(\omega)$, since $\omega(t^{\gamma})=o(t^{1/s})$ as $t\to+\infty$ for every $\ell\in\mathbb{N}$, there is $D_\ell>0$ such that

$$(k!)^{s\gamma}\le D_\ell e^{\ell\varphi^*(k/\ell)},\quad k\in\mathbb{N}.\tag{131}$$

Moreover, if $u\in\mathscr{E}^P_{(\omega)}(\Omega)$ for each $\ell\in\mathbb{N}$, there is $C_\ell>0$ such that

$$\left\|P^{j-k}u\right\|_{2,K(r/2)}\le C_\ell e^{\ell\varphi^*((j-k)/\ell)},\quad k=0,1,\dots,j,\ j\in\mathbb{N}.\tag{132}$$

Now, we can proceed as in the Roumieu case to obtain

$$\left\|P(D)^j(\chi u)\right\|_{2,K(r/2)}\le(j+1)\,2^{js\gamma}E^jC_\ell D_\ell e^{\ell\varphi^*(j/\ell)},\quad j\in\mathbb{N},\tag{133}$$

which concludes the proof. □

Let us recall that, by Proposition 9 and Theorem 13 if ω is a nonquasianalytic weight and $P(D)$ is elliptic, then

$$\mathrm{WF}^P_*u=\mathrm{WF}_*u\quad\forall u\in\mathscr{D}',\tag{134}$$

for $*$ being equal to $\{\omega\}$ or (ω). Let us then assume $P(D)$ is not elliptic and prove the following result, which generalizes Theorems 8.1.4 and 8.4.14 of [25].

Theorem 18. *Let ω be a nonquasianalytic weight function such that $\omega(t^b)=o(\overline{\sigma}(t))$ as t tends to infinity, where $\overline{\sigma}(t)=t^{1/s}$ is a Gevrey weight function, with $s>1$ and $b=\max(\gamma,3/2)$, with γ the constant in (28). Let $P(D)$ be a linear partial differential operator with constant coefficients which is hypoelliptic but not elliptic. Given an open subset Ω of $\mathbb{R}^n$ and a closed conic subset S of $\Omega\times(\mathbb{R}^n\setminus\{0\})$, then there is a distribution $u\in\mathscr{D}'(\Omega)$ with $\emptyset\ne WF^P_*u\subset S$. In particular, if $S=\{(x_0,t\xi_0),t>0\}$ for some $x_0\in\Omega$ and $\xi_0\in\mathbb{R}^n$ with $|\xi_0|=1$, we have $WF^P_*u=S$.*

Proof. Let us first remark that it is sufficient to prove the statement when $\Omega=\mathbb{R}^n$.

Moreover, since P is hypoelliptic but not elliptic, we can find $\delta>0$ and $0<d<m$ such that

$$|P(\xi)|\ge\delta|\xi|^d,\tag{135}$$

for ξ big enough. Choose a sequence $(x_k,\theta_k)\in S$ with $|\theta_k|=1$ so that every $(x,\theta)\in S$ with $|\theta|=1$ is the limit of a subsequence.

Let us now set $\sigma(t):=\omega(t^{3/2})$ and separate Beurling and Roumieu cases.

Roumieu Case. Take $\phi\in\mathscr{D}_{\{\sigma\}}(\mathbb{R}^n)$ with $\widehat{\phi}(0)=1$.

Then, there exist $c>0$ and $h\in\mathbb{N}$ such that

$$\left|\widehat{\phi}(\xi)\right|\le ce^{-(1/h)\sigma(\xi)}\qquad\forall\xi\in\mathbb{R}^n.\tag{136}$$

Since $\log t=o(\sigma(t))$ as $t\to+\infty$, by definition of weight function, by Lemma 1.7 of [15], there exists a weight function α such that $\log t=o(\alpha(t))$ and $\alpha(t)=o(\sigma(t))$ for $t\to+\infty$.

Note that for every $\ell\in\mathbb{N}$, there is $k_\ell\in\mathbb{N}$ such that

$$\exp\left\{-\frac{\sigma\left(k^{d/m}\right)}{\alpha\left(k^{d/m}\right)}\log k\right\}<k^{-\ell}\quad\forall k\ge k_\ell\tag{137}$$

and define then

$$u(x)=\sum_{k=1}^{+\infty}e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k}\phi\left(k\left(x-x_k\right)\right)e^{ik^3\langle x,\theta_k\rangle}.\tag{138}$$

This is a continuous function in $\mathbb{R}^n$ and we will prove that $\emptyset\ne\mathrm{WF}^P_{\{\omega\}}u\subset S$.

To prove first that $\mathrm{WF}^P_{\{\omega\}}u \subset S$, we take $(x_0, \xi_0) \notin S$ and prove that $(x_0, \xi_0) \notin \mathrm{WF}^P_{\{\omega\}}u$. To this aim, we choose an open neighborhood U of x_0 and an open conic neighborhood Γ of ξ_0 such that

$$(U \times \Gamma) \cap S = \emptyset. \tag{139}$$

Write $u = u_1 + u_2$, where u_1 is the sum of terms in (138) with $x_k \notin U$ and u_2 is the sum of terms with $x_k \in U$.

Therefore, there is a neighborhood U_1 of x_0 with $\overline{U}_1 \subset U$ such that u_1 is in $\mathscr{E}_{\{\sigma\}}(U_1)$ since all but a finite number of terms vanish in U_1. Moreover, every weight function ω is increasing by definition, so that $\omega \le \sigma$, $\mathscr{E}_{\{\sigma\}} \subset \mathscr{E}_{\{\omega\}}$ and hence $u_1 \in \mathscr{E}_{\{\omega\}}(U_1)$.

Consider then

$$\begin{aligned} f_N &= P(D)^N u_2(x) \\ &= \sum_{x_k \in U} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} P(D)^N \\ &\quad \times \left[\phi(k(x - x_k)) e^{ik^3\langle x, \theta_k\rangle}\right]. \end{aligned} \tag{140}$$

Note that it is a totally convergent series since

$$\sup_{x \in \mathbb{R}^n} \left| P(D)^N \left[\phi(k(x - x_k)) e^{ik^3\langle x, \theta_k\rangle}\right]\right| \le C_N k^{3mN} \tag{141}$$

for some $C_N > 0$ and because of (137) with $\ell \ge 3mN + 2$.

Let us then compute the Fourier transform

$$\begin{aligned} \widehat{f}_N(\xi) &= \sum_{x_k \in U} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} P(\xi)^N \\ &\quad \times \mathscr{F}\left(\phi(k(x - x_k)) e^{ik^3\langle x, \theta_k\rangle}\right) \\ &= \sum_{x_k \in U} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} k^{-n} P(\xi)^N \\ &\quad \times \widehat{\phi}\left(\frac{\xi - k^3\theta_k}{k}\right) e^{i\langle x_k, k^3\theta_k - \xi\rangle} \end{aligned} \tag{142}$$

with $\theta_k \notin \Gamma$ because of (139).

If Γ_1 is a conic neighborhood of ξ_0 with $\overline{\Gamma}_1 \subset \Gamma \cup \{0\}$, then $|\xi - \eta| \ge c_0(|\xi| + |\eta|)$ when $\xi \in \Gamma_1$ and $\eta \notin \Gamma$, for some $c_0 > 0$, since this is true when $|\xi| + |\eta| = 1$. Thus,

$$\begin{aligned} \left|\xi - k^3\theta_k\right| &\ge c_0\left(|\xi| + k^3\right) \\ &\ge c_0\frac{1}{3}\left(|\xi| + |\xi| + k^3\right) \\ &\ge c_0\sqrt[3]{|\xi| \cdot |\xi| \cdot k^3} \\ &= c_0|\xi|^{2/3}k, \quad \xi \in \Gamma_1. \end{aligned} \tag{143}$$

It follows from (136) that

$$\begin{aligned} \left|\widehat{\phi}\left(\frac{\xi - k^3\theta_k}{k}\right)\right| &\le c\exp\left\{-\frac{1}{h}\sigma\left(\frac{\xi - k^3\theta_k}{k}\right)\right\} \\ &\le ce^{-(1/h)\sigma(c_0\xi^{2/3})} \\ &\le c'e^{-(1/h)\omega(\xi)}, \quad \xi \in \Gamma_1, \end{aligned} \tag{144}$$

for some $c' > 0$, since $\omega(2t) \le L(\omega(t) + 1)$ for some $L > 0$ by definition of weight function. Therefore, by (142) and Lemma 16(i), if we fix $\ell \in \mathbb{N}$, for $\xi \in \Gamma_1$, $|\xi| \ge 1$,

$$\begin{aligned} (1 + |\xi|)^\ell \left|\widehat{f}_N(\xi)\right| &\le (1 + |\xi|)^\ell \sum_{x_k \in U} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} k^{-n} \\ &\quad \times |P(\xi)|^N c' e^{-(1/h)\omega(\xi)} \\ &\le c''_\ell |\xi|^{mN+\ell} e^{-(1/h)\omega(\xi)} \\ &\le c''_\ell e^{(1/h)\varphi^*(mNh + \ell h)}, \end{aligned} \tag{145}$$

for some $c''_\ell > 0$. Now, from the convexity of φ^*, it follows easily that condition (ii)(b) of Definition 7 is satisfied. But also condition (ii)(a) of Definition 7 is satisfied

$$\begin{aligned} &\left|\widehat{f}_N(\xi)\right| \\ &\quad \le \sum_{x_k \in U} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} \\ &\qquad \times k^{-n}|P(\xi)|^N c e^{-(1/h)\sigma((\xi - k^3\theta_k)/k)} \\ &\quad \le c'|\xi|^{mN}, \quad \xi \in \mathbb{R}^n, \end{aligned} \tag{146}$$

for some $c' > 0$. This, together with $u_1 \in \mathscr{E}_{\{\omega\}}(U_1)$, proves that $(x_0, \xi_0) \notin \mathrm{WF}^P_{\{\omega\}}u$.

Let us now prove that $\mathrm{WF}^P_{\{\omega\}}u \ne \emptyset$.

Choose $\chi \in \mathscr{D}_{\{\overline{\sigma}\}}(\mathbb{R}^n)$ equal to 1 near $x_0 \in \Omega$, where $\overline{\sigma}$ is the Gevrey weight of the hypotheses. To prove that $\mathrm{WF}^P_{\{\omega\}}u \ne \emptyset$, we proceed by contradiction and assume that the wave front set is empty. Then, $u \in \mathscr{E}^P_{\{\omega\}}(\Omega)$.

Set

$$\phi_k(k(x - x_k)) := \chi(x)\phi(k(x - x_k)). \tag{147}$$

By hypothesis $\sigma = o(\overline{\sigma})$ which implies in particular that $\mathscr{D}_{\{\overline{\sigma}\}}(\mathbb{R}^n) \subset \mathscr{D}_{\{\sigma\}}(\mathbb{R}^n)$. Then, the sequence $\phi_k(y) = \chi(y/k + x_k)\phi(y)$ is a bounded set in $\mathscr{D}_{\{\sigma\}}(\mathbb{R}^n)$ and, in fact, the supports $\operatorname{supp}\phi_k \subset \operatorname{supp}\phi$ for all k. We can use [15, Proposition 3.4] to obtain constants $c, h > 0$ such that

$$\left|\widehat{\phi}_j(\xi)\right| \le ce^{-(1/h)\sigma(\xi)} \tag{148}$$

for all $j \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$.

The Fourier transform of $P(D)^N(\chi u)$ is a sum of the form (142) with ϕ replaced by ϕ_k. We observe that

$$\left|k^3\theta_k - j^3\theta_j\right| \ge \left|k^3 - j^3\right| \ge k^2 + kj + j^2 \ge kj, \quad \text{if } k \ne j. \tag{149}$$

Moreover, for x_k close to x_0 and k large enough, the equality $\phi_k = \phi$ is satisfied. Consequently, from (135), we have, for some $c' > 0$,

$$\begin{aligned}
&\left|\mathcal{F}\left[P(D)^N(\chi u)\right]\left(k^3\theta_k\right)\right| \\
&= \Bigg| e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} k^{-n} P(k^3\theta_k)^N \\
&\quad + \sum_{j\neq k} e^{-(\sigma(j^{d/m})/\alpha(j^{d/m}))\log j} j^{-n} P\left(k^3\theta_k\right)^N \\
&\quad \times \widehat{\phi}_j\left(\frac{k^3\theta_k - j^3\theta_j}{j}\right) e^{i\langle x_j, j^3\theta_j - k^3\theta_k\rangle}\Bigg| \\
&\geq \left|P\left(k^3\theta_k\right)\right|^N \Bigg(e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} k^{-n} \\
&\quad - \sum_{j\neq k} e^{-(\sigma(j^{d/m})/\alpha(j^{d/m}))\log j} j^{-n} \\
&\quad \times c e^{-(1/h)\sigma((k^3\theta_k - j^3\theta_j)/j)}\Bigg) \\
&\geq \delta^N k^{3Nd}\left(e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} k^{-n} - c' e^{-(1/h)\sigma(k)}\right) \\
&\geq \delta^N k^{3Nd}\frac{1}{2} k^{-n} e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k}.
\end{aligned} \tag{150}$$

In fact, for k large enough

$$\frac{1}{h}\sigma(k) \geq -\log\left(\frac{1}{2c'}\right) + \frac{\sigma\left(k^{d/m}\right)}{\alpha\left(k^{d/m}\right)}\log k + n\log k, \tag{151}$$

since, for $k \to +\infty$, $\sigma(k) \to +\infty$, $\sigma(k^{d/m})/\sigma(k)$ is bounded ($d < m$ in (135)), $\log k = o(\alpha(k))$, and $\log k = o(\sigma(k))$.

On the other hand, by Proposition 17, the product $\chi u \in \mathcal{E}^P_{\{\omega\}}(\Omega)$. We obtain $C > 0$ and $h' \in \mathbb{N}$ such that, for all $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\left|\mathcal{F}\left(P(D)^N(\chi u)\right)(\xi)\right| &= \left|\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} P(D)^N(\chi u)(x)\,dx\right| \\
&\leq D\left\|P(D)^N(\chi u)\right\|_{2,\operatorname{supp}\chi} \\
&\leq CDe^{(1/h')\varphi^*(Nmh')},
\end{aligned} \tag{152}$$

where $D > 0$ is a constant that depends on the Lebesgue measure of supp χ. Consequently, from (150), we have

$$\begin{aligned}
&\frac{\delta^N}{2}k^{3Nd-n}e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} \\
&\quad \leq \left|\mathcal{F}\left(P(D)^N(\chi u)\right)\left(k^3\theta_k\right)\right| \\
&\quad \leq Ce^{(1/h')\varphi^*(Nmh')},
\end{aligned} \tag{153}$$

for every $N \in \mathbb{N}$ and k.

Now, (153) implies, by Lemma 16(ii),

$$\begin{aligned}
&e^{-(\omega(k^{3d/2m})/\alpha(k^{d/m}))\log k} = e^{-(\sigma(k^{d/m})/\alpha(k^{d/m}))\log k} \\
&\quad \leq 2Ck^n \inf_{N\in\mathbb{N}}\left\{\left(\delta^{1/m}k^{3d/m}\right)^{-Nm} e^{(1/h')\varphi^*(Nmh')}\right\} \\
&\quad \leq 2C\delta k^{n+3d} e^{-(1/h')\omega(\delta^{1/m}k^{3d/m})}.
\end{aligned} \tag{154}$$

But for every fixed h', there is k large enough so that

$$\begin{aligned}
&\frac{\omega\left(k^{3d/2m}\right)}{\alpha\left(k^{d/m}\right)}\log k \\
&\quad < \frac{1}{h'}\omega\left(\delta^{1/m}k^{3d/m}\right) - (n+3d)\log k - \log(2C\delta),
\end{aligned} \tag{155}$$

since we can argue as in (151), which is a contradiction. Therefore, $\mathrm{WF}^P_{\{\omega\}}u \neq \emptyset$.

Beurling Case. Take $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^n)$ with $\widehat{\phi}(0) = 1$.

For every $h \in \mathbb{N}$, there exists then a constant $c_h > 0$ such that

$$\left|\widehat{\phi}(\xi)\right| \leq c_h e^{-h\sigma(\xi)} \quad \forall\xi \in \mathbb{R}^n. \tag{156}$$

Note that for every fixed $\ell \in \mathbb{N}$,

$$\exp\left\{-\sigma\left(k^{d/m}\right)\right\} = \exp\left\{-\frac{\sigma\left(k^{d/m}\right)}{\log\left(k^{d/m}\right)}\cdot\frac{m}{\ell d}\cdot\log k^{\ell}\right\} < k^{-\ell}, \tag{157}$$

for k large enough since $\log k = o(\sigma(k))$ as $k \to \infty$. Define then

$$u(x) = \sum_{k=1}^{+\infty} e^{-\sigma(k^{d/m})}\phi\left(k\left(x - x_k\right)\right)e^{ik^3\langle x,\theta_k\rangle}. \tag{158}$$

This is a continuous function in $\mathbb{R}^n$ and we will prove that $\emptyset \neq \mathrm{WF}^P_{(\omega)}u \subset S$.

The proof of the inclusion $\mathrm{WF}^P_{(\omega)}u \subset S$ is similar to that in the Roumieu case. We take $(x_0, \xi_0) \notin S$, choose an open neighborhood U of x_0 and an open conic neighborhood Γ of ξ_0 such that $(U \times \Gamma) \cap S \neq \emptyset$, and write $u = u_1 + u_2$, where u_1 is the sum of terms in (158) with $x_k \notin U$ and u_2 is the sum of terms with $x_k \in U$.

We choose a neighborhood U_1 of x_0 with $\overline{U}_1 \subset U$ such that u_1 is in $\mathcal{E}_{(\sigma)}(U_1) \subset \mathcal{E}_{(\omega)}(U_1)$ since all but a finite number of terms vanish in U_1.

Then, we consider the totally convergent series (because of (157) with ℓ large enough)

$$\begin{aligned}
f_N &= P(D)^N u_2(x) \\
&= \sum_{x_k\in U} e^{-\sigma(k^{d/m})} P(D)^N\left[\phi\left(k\left(x - x_k\right)\right)e^{ik^3\langle x,\theta_k\rangle}\right]
\end{aligned} \tag{159}$$

and compute its Fourier transform

$$\widehat{f}_N(\xi) = \sum_{x_k \in U} e^{-\sigma(k^{d/m})} k^{-n} P(\xi)^N \widehat{\phi}\left(\frac{\xi - k^3\theta_k}{k}\right) e^{i\langle x_k, k^3\theta_k - \xi\rangle}, \tag{160}$$

with $\theta_k \notin \Gamma$.

For a conic neighborhood Γ_1 of ξ_0 with $\overline{\Gamma}_1 \subset \Gamma \cup \{0\}$, we have that (143) is satisfied and hence, from (156),

$$\begin{aligned} \left|\widehat{\phi}\left(\frac{\xi - k^3\theta_k}{k}\right)\right| &\leq c_h \exp\left\{-h\sigma\left(\frac{\xi - k^3\theta_k}{k}\right)\right\} \\ &\leq c_h e^{-h\sigma(c_0 \xi^{2/3})} \leq c_h' e^{-h\omega(\xi)}, \quad \xi \in \Gamma_1, \end{aligned} \tag{161}$$

for some $c_h' > 0$, since $\omega(2t) \leq L(\omega(t) + 1)$ for some $L > 0$. Now, we fix $\ell \in \mathbb{N}$. By Lemma 16(i),

$$\begin{aligned} &(1 + |\xi|)^\ell \left|\widehat{f}_N(\xi)\right| \\ &\quad \leq (1 + |\xi|)^\ell \sum_{x_k \in U} e^{-\sigma(k^{d/m})} k^{-n} |P(\xi)|^N c_h' e^{-h\omega(\xi)} \\ &\quad \leq c_{h,\ell}'' |\xi|^{mN+\ell} e^{-h\omega(\xi)} \\ &\quad \leq c_{h,\ell}'' e^{h\varphi^*((mN+\ell)/h)}, \quad \xi \in \Gamma_1, \end{aligned} \tag{162}$$

for some $c_{h,\ell}'' > 0$. From the convexity of φ^*, we conclude that condition (iii)(b) of Definition 7 is satisfied. But also condition (iii)(a) of Definition 7 is satisfied

$$\begin{aligned} \left|\widehat{f}_N(\xi)\right| &\leq \sum_{x_k \in U} e^{-\sigma(k^{d/m})} k^{-n} |P(\xi)|^N c_h \\ &\quad \times e^{-h\sigma((\xi - k^3\theta_k)/k)} \leq c_h' |\xi|^{mN}, \\ &\quad \xi \in \mathbb{R}^n, \end{aligned} \tag{163}$$

for some $c_h' > 0$. This, together with $u_1 \in \mathscr{E}_{(\omega)}(U_1)$, proves that $(x_0, \xi_0) \notin \mathrm{WF}^P_{(\omega)} u$ and hence $\mathrm{WF}^P_{(\omega)} u \subset S$.

Let us prove now that $\mathrm{WF}^P_{(\omega)} u \neq \emptyset$.

Choose $\chi \in \mathscr{D}_{\{\overline{\sigma}\}}(\mathbb{R}^n)$ equal to 1 near x_0. We proceed by contradiction and assume that $\mathrm{WF}^P_{(\omega)} u = \emptyset$. Then, $u \in \mathscr{E}^P_{(\omega)}(\Omega)$.

Set $\phi_k(k(x - x_k)) := \chi(x)\phi(k(x - x_k))$ as in the Roumieu case. Since $\sigma = o(\overline{\sigma})$, $\mathscr{D}_{\{\overline{\sigma}\}}(\mathbb{R}^n) \subset \mathscr{D}_{(\sigma)}(\mathbb{R}^n)$ ([15, Proposition 4.7]). Then the sequence $\{\phi_k\}$ is a bounded set in $\mathscr{D}_{(\sigma)}(\mathbb{R}^n)$ and $\operatorname{supp}\phi_k \subset \operatorname{supp}\phi$ for all k, as in the Roumieu case. By [15, Proposition 3.4], for each $h \in \mathbb{N}$, there is $c_h > 0$ such that for all $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$,

$$\left|\widehat{\phi}_j(\xi)\right| \leq c_h e^{-h\sigma(\xi)}. \tag{164}$$

If x_k is close to x_0 and k is large enough, then $\phi_k = \phi$ and by (149), we have

$$\begin{aligned} &\left|\mathscr{F}\left(P(D)^N(\chi u)\right)\left(k^3\theta_k\right)\right| \\ &\quad = \left| e^{-\sigma(k^{d/m})} k^{-n} P\left(k^3\theta_k\right)^N + \sum_{j \neq k} e^{-\sigma(j^{d/m})} j^{-n} P\left(k^3\theta_k\right)^N \right. \\ &\qquad \left. \times \widehat{\phi}_j\left(\frac{k^3\theta_k - j^3\theta_j}{j}\right) e^{i\langle x_j, j^3\theta_j - k^3\theta_k\rangle} \right| \\ &\quad \geq \left|P\left(k^3\theta_k\right)\right|^N \left(e^{-\sigma(k^{d/m})} k^{-n} - \sum_{j \neq k} e^{-\sigma(j^{d/m})} j^{-n} \right. \\ &\qquad \left. \times c_h e^{-h\sigma((k^3\theta_k - j^3\theta_j)/j)} \right) \\ &\quad \geq \delta^N k^{3Nd} \left(e^{-\sigma(k^{d/m})} k^{-n} - c_h' e^{-h\sigma(k)} \right) \\ &\quad \geq \delta^N k^{3Nd} \frac{1}{2} k^{-n} e^{-\sigma(k^{d/m})}. \end{aligned} \tag{165}$$

On the other hand, by Proposition 17, $\chi u \in \mathscr{E}^P_{(\omega)}(\Omega)$ and proceeding as in the Roumieu case, we obtain that for every $h \in \mathbb{N}$, there would exist $C_h > 0$ such that

$$\left|\mathscr{F}\left(P(D)^N(\chi u)\right)\left(k^3\theta_k\right)\right| \leq C_h e^{h\varphi^*(Nm/h)} \quad \forall k. \tag{166}$$

But (166) and (165) give a contradiction since they imply, by Lemma 16(ii), that

$$\begin{aligned} e^{-\omega(k^{3d/2m})} &= e^{-\sigma(k^{d/m})} \\ &\leq 2C_h k^n \inf_{N \in \mathbb{N}} \left\{ \left(\delta^{1/m} k^{3d/m}\right)^{-Nm} e^{h\varphi^*(Nm/h)} \right\} \\ &\leq 2C_h \delta k^{n+3d} e^{-h\omega(\delta^{1/m} k^{3d/m})} \end{aligned} \tag{167}$$

must hold for every $h > 0$ and k large enough.

However, since $\omega(2t) \leq L(\omega(t) + 1)$ for some $L > 0$, there exists a constant $c_1 > 0$ such that

$$\omega\left(k^{3d/2m}\right) \leq c_1\left(\omega\left(\delta^{1/m} k^{3d/2m}\right) + 1\right), \tag{168}$$

contradicting (167) for k large enough. Then $\mathrm{WF}^P_{(\omega)} u \neq \emptyset$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research of the first and the second authors was partially supported by Grants PRIN2008 (MIUR) and FAR2009 (University of Ferrara). The research of the second and third authors was partially supported by MEC and FEDER, Project MTM2010-15200. The research of the second author was partially supported by Programa de Apoyo a la Investigación y Desarrollo de la UPV PAID-06-12. The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Instituto Nazionale di Alta Matematica (INdAM).

References

[1] H. Komatsu, "A characterization of real analytic functions," *Proceedings of the Japan Academy*, vol. 36, pp. 90–93, 1960.

[2] T. Kotake and M. S. Narasimhan, "Regularity theorems for fractional powers of a linear elliptic operator," *Bulletin de la Société Mathématique de France*, vol. 90, pp. 449–471, 1962.

[3] E. Newberger and Z. Zielezny, "The growth of hypoelliptic polynomials and Gevrey classes," *Proceedings of the American Mathematical Society*, vol. 39, no. 3, pp. 547–552, 1973.

[4] G. Métivier, "Propriété des itérés et ellipticité," *Communications in Partial Differential Equations*, vol. 3, no. 9, pp. 827–876, 1978.

[5] L. Zanghirati, "Iterates of a class of hypoelliptic operators and generalized Gevrey classes," *Bollettino della Unione Matematica Italiana*, supplement 1, pp. 177–195, 1980.

[6] M. Langenbruch, "P-Funktionale und Randwerte zu Hypoelliptischen Differentialoperatoren," *Mathematische Annalen*, vol. 239, no. 1, pp. 55–74, 1979.

[7] M. Langenbruch, "Fortsetzung von Randwerten zu hypoelliptischen Differentialoperatoren und partielle Differentialgleichungen," *Journal für die Reine und Angewandte Mathematik*, vol. 311/312, pp. 57–79, 1979.

[8] M. Langenbruch, "On the functional dimension of solution spaces of hypoelliptic partial differential operators," *Mathematische Annalen*, vol. 272, no. 2, pp. 217–229, 1985.

[9] M. Langenbruch, "Bases in solution sheaves of systems of partial differential equations," *Journal für die Reine und Angewandte Mathematik*, vol. 373, pp. 1–36, 1987.

[10] C. Bouzar and R. Chaili, "Une généralisation du problème des itérés," *Archiv der Mathematik*, vol. 76, no. 1, pp. 57–66, 2001.

[11] C. Bouzar and R. Chaili, "Gevrey vectors of multi-quasi-elliptic systems," *Proceedings of the American Mathematical Society*, vol. 131, no. 5, pp. 1565–1572, 2003.

[12] D. Calvo and G. H. Hakobyan, "Multianisotropic Gevrey regularity and iterates of operators with constant coefficients," *Bulletin of the Belgian Mathematical Society*, vol. 12, no. 3, pp. 461–474, 2005.

[13] D. Calvo and L. Rodino, "Iterates of operators and Gelfand-Shilov classes," *Integral Transforms and Special Functions*, vol. 22, no. 4-5, pp. 269–276, 2011.

[14] J. Juan-Huguet, "Iterates and hypoellipticity of partial differential operators on non-quasianalytic classes," *Integral Equations and Operator Theory*, vol. 68, no. 2, pp. 263–286, 2010.

[15] R. W. Braun, R. Meise, and B. A. Taylor, "Ultradifferentiable functions and Fourier analysis," *Results in Mathematics*, vol. 17, pp. 206–237, 1990.

[16] J. Juan-Huguet, "A Paley-Wiener type theorem for generalized non-quasianalytic classes," *Studia Mathematica*, vol. 208, no. 1, pp. 31–46, 2012.

[17] P. Bolley, J. Camus, and C. Mattera, "Analyticité microlocale et itérés d'opérateurs hypoelliptiques," in *Séminaire Goulaouic-Schwartz*, Exp N. 13, École Polytech, Palaiseau, France, 1978-79.

[18] L. Hörmander, "Uniqueness theorems and wave front sets for solutions of linear partial differential equations with analytic coefficients," *Communications on Pure and Applied Mathematics*, vol. 24, pp. 671–704, 1971.

[19] P. Bolley and J. Camus, "Regularité Gevrey et itérés pour une classe d'opérateurs hypoelliptiques," *Communications in Partial Differential Equations*, vol. 10, no. 6, pp. 1057–1110, 1981.

[20] L. Zanghirati, "Iterati di operatori e regolarità Gevrey microlocale anisotropa," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 67, pp. 85–104, 1982.

[21] C. Bouzar and R. Chaili, "A Gevrey microlocal analysis of multi-anisotropic differential operators," *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, vol. 64, no. 3, pp. 305–317, 2006.

[22] A. A. Albanese, D. Jornet, and A. Oliaro, "Quasianalytic wave front sets for solutions of linear partial differential operators," *Integral Equations and Operator Theory*, vol. 66, no. 2, pp. 153–181, 2010.

[23] A. A. Albanese, D. Jornet, and A. Oliaro, "Wave front sets for ultradistribution solutions of linear partial differential operators with coefficients in non-quasianalytic classes," *Mathematische Nachrichten*, vol. 285, no. 4, pp. 411–425, 2012.

[24] C. Fernández, A. Galbis, and D. Jornet, "Pseudodifferential operators of Beurling type and the wave front set," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1153–1170, 2008.

[25] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin, Germany, 1990.

[26] J. Bonet, R. Meise, and S. N. Melikhov, "A comparison of two different ways to define classes of ultradifferentiable functions," *Bulletin of the Belgian Mathematical Society*, vol. 14, no. 3, pp. 425–444, 2007.

[27] L. Hörmander, "On interior regularity of the solutions of partial differential equations," *Communications on Pure and Applied Mathematics*, vol. 11, pp. 197–218, 1958.