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Research Article

Generalized *n* Dimensional Ostrowski Type and Grüss Type Inequalities on Time Scales

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Some new generalized n dimensional Ostrowski type and Grüss type integral inequalities on time scales are established in this paper. The present results unify continuous and discrete analysis, and extend some known results in the literature.

1. Introduction

In recent years, the research for the Ostrowski type and Grüss type inequalities has been an interesting topic in the literature. The Ostrowski type inequality can be used to estimate the absolute deviation of a function from its integral mean, and it was originally presented by Ostrowski in [1] as follows.

Theorem 1. *Let $f : I \to R$ be a differentiable mapping in the interior $\mathrm{Int}\, I$ of I, where $I \subset R$ is an interval, and let $a, b \in \mathrm{Int}\, I$. $a < b$. If $|f'(t)| \le M, \forall t \in [a, b]$, then one has*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{(x-((a+b)/2))^2}{(b-a)^2}\right](b-a)M, \quad \text{for } x \in [a,b]. \tag{1}$$

The Grüss inequality, which can be used to estimate the absolute deviation of the integral of the product of two functions from the product of their respective integral, was originally presented by Grüss in [2] as follows.

Theorem 2. *Let f, g be integrable functions on $[a, b]$ and satisfy the condition $m \le f(x) \le M$, $n \le g(x) \le N$. Then one has*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \times \left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{4}(M-m)(N-n). \tag{2}$$

In the last few decades, various generalizations of the Ostrowski inequality and the Grüss inequality including continuous and discrete versions have been established (e.g., see [3–14] and the references therein). On the other hand, Hilger [15] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type and Grüss type inequalities on time scales. For example, in [16], Bohner and Matthews established the following Ostrowski type inequality on time scales for the first time.

Theorem 3 (see [16, Theorem 3.5]). *Let $a, b, s, t \in \mathbb{T}, a < b$, and let $f : [a,b] \to R$ be differentiable, where $\mathbb{T}$ is an arbitrary time scale. Then*

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\,\Delta s\right| \leq \frac{M}{b-a}\left(h_2(t,a) + h_2(t,b)\right), \tag{3}$$

where $M = \sup_{a<t<b}|f^{\Delta}(t)|$. This inequality is sharp in the sense that the right-hand side of it cannot be replaced by a smaller one.

In [17], Özkan and Yildirim established the following Grüss type inequality for double integrals on time scales.

Theorem 4 (see [17, Theorem 2.2]). *Let $a, b \in \mathbb{T}_1$, $c, d \in \mathbb{T}_2$ and $f, \in CC^1_{rd}([a,b] \times [c,d], R)$, where $\mathbb{T}_1$ and $\mathbb{T}_2$are two arbitrary time scales. Then*

$$\begin{aligned}
&\left|\frac{1}{k}\int\int_R f(x,y)\,g(x,y)\,\Delta_2 y\Delta_1 x\right.\\
&\quad+\frac{1}{2}\left(\frac{1}{k}\int\int_R f(x,y)\,\Delta_2 y\Delta_1 x\right)\\
&\quad\times\left(\frac{1}{k}\int\int_R g(\sigma_1(x),\sigma_2(y))\,\Delta_2 y\Delta_1 x\right)\\
&\quad+\frac{1}{2}\left(\frac{1}{k}\int\int_R f(\sigma_1(x),\sigma_2(y))\,\Delta_2 y\Delta_1 x\right)\\
&\quad\times\left(\frac{1}{k}\int\int_R g(x,y)\,\Delta_2 y\Delta_1 x\right)\\
&\quad\left.-\frac{1}{2k^2}\int\int_R \left(g(x,y)\,F(x,y) + f(x,y)\,G(x,y)\right)\Delta_2 y\Delta_1 x\right|\\
&\leq \frac{1}{2k^2}\int\int_R M_2(x,y)\,H_1(x)\,H_2(y)\,\Delta_2 y\Delta_1 x,
\end{aligned} \tag{4}$$

where $k = (b-a)(d-c)$, $F(x,y) = (d-c)\int_a^b f(\sigma_1(t),y)\Delta_1 t + (b-a)\int_c^d f(x,\sigma_2(s))\Delta_2 s$, and $G(x,y) = (d-c)\int_a^b g(\sigma_1(t),y)\Delta_1 t + (b-a)\int_c^d g(x,\sigma_2(s))\Delta_2 s$.

In [18], Liu and Ngô established an Ostrowski-Grüss type inequality in one independent variable as follows.

Theorem 5 (see [18, Theorem 4]). *Suppose $a, b, s, t \in \mathbb{T}$ and $f : [a,b] \to R$ is differentiable, where $\mathbb{T}$ is an arbitrary time scale, f^{Δ} is rd-continuous, and $\gamma \leq f^{\Delta}(t) \leq \Gamma$, $\forall t \in [a,b]$. Then one has*

$$\begin{aligned}
&\left|f(t) - \frac{1}{b-a}\int_a^b f^{\sigma}(s)\,\Delta s\right.\\
&\quad\left.-\frac{f(b)-f(a)}{(b-a)^2}\left(h_2(t,a) - h_2(t,b)\right)\right|\\
&\leq \frac{1}{4}(b-a)(\Gamma-\gamma),
\end{aligned} \tag{5}$$

$\forall t \in [a,b]$.

Other results on the Ostrowski type and Grüss type inequalities on time scales can be found in [19–25]. These inequalities on time scales unify continuous and discrete analysis and can be used to provide explicit error bounds for some known and some new numerical quadrature formulae.

Motivated by the above works, in this paper, we establish some new generalized Ostrowski type and Grüss type inequalities on time scales involving functions of n independent variables, which extend some known results in the literature to n dimensional case.

We first give the following definition for further use.

Definition 6. $h_k : \mathbb{T}^2 \to \mathbb{R}, k = 0, 1, 2, \ldots$ are defined by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau, \quad \forall s,t \in \mathbb{T}, \tag{6}$$

where $\mathbb{T}$ is an arbitrary time scale and $h_0(t,s) = 1$.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$, while $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}_0$ denotes the set of nonnegative integers. For a function f and two integers m_0 and m_1, one has $\sum_{s=m_0}^{m_1} f = 0$, provided that $m_0 > m_1$. $\mathbb{T}_i$ denotes an arbitrary time scale, $i = 1, 2, \ldots, n$. For an interval $[a,b]$, $[a,b]_{\mathbb{T}_i} := [a,b] \bigcap \mathbb{T}_i$, $i = 1, 2, \ldots, n$.

2. Main Results

For the sake of convenience, we present the following notations:

$$\begin{aligned}
&L = \prod_{i=1}^{n}(b_i - a_i), \qquad L_k = \frac{L}{\prod_{i=1}^{k}(b_{mi} - a_{mi})},\\
&A_1(x_1, x_2, \ldots, x_n)\\
&= \sum_{k=1}^{n-1}(-1)^{k+1}L_k\\
&\quad\times\left\{\sum_{1\leq m1\leq m2\cdots\leq mk\leq n}\int_{a_{m1}}^{b_{m1}}\int_{a_{m2}}^{b_{m2}}\right.\\
&\quad\cdots\int_{a_{mk}}^{b_{mk}} f(x_1,\ldots,\\
&\quad\sigma_{m1}(s_{m1}),\ldots,\\
&\quad\sigma_{m2}(s_{m2}),\ldots,\\
&\quad\sigma_{mk}(s_{mk}),\ldots,x_n)\\
&\quad\left.\times\Delta_{mk}s_{mk},\ldots,\Delta_{m2}s_{m2}\Delta_{m1}s_{m1}\right\},
\end{aligned}$$

$$\begin{aligned}
&A_2(x_1, x_2, \ldots, x_n) \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} L_k \\
&\quad \times \Bigg\{ \sum_{1 \le m1 \le m2 \cdots \le mk \le n} \int_{a_{m1}}^{b_{m1}} \int_{a_{m2}}^{b_{m2}} \cdots \int_{a_{mk}}^{b_{mk}} g(x_1, \ldots, \\
&\qquad \sigma_{m1}(s_{m1}), \ldots, \\
&\qquad \sigma_{m2}(s_{m2}), \ldots, \\
&\qquad \sigma_{mk}(s_{mk}), \ldots, x_n) \\
&\qquad \times \Delta_{mk} s_{mk}, \ldots, \Delta_{m2} s_{m2} \Delta_{m1} s_{m1} \Bigg\}, \\
&A_3(x_1, x_2, \ldots, x_n) \\
&= (-1)^{n+1} \Bigg\{ f(x_1, x_2, \ldots, x_n) \\
&\quad \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} g(\sigma_1(s_1), \\
&\qquad \sigma_2(s_2), \ldots, \\
&\qquad \sigma_{n-1}(s_{n-1}), \\
&\qquad \sigma_n(s_n)) \\
&\quad \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \\
&\quad + g(x_1, x_2, \ldots, x_n) \\
&\quad \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(\sigma_1(s_1),\ \sigma_2(s_2), \ldots, \\
&\qquad \sigma_{n-1}(s_{n-1}), \sigma_n(s_n)) \\
&\quad \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \Bigg\}.
\end{aligned} \tag{7}$$

Lemma 7 (generalized n dimensional Montgomery identity). *Let*

$$p_i(x_i, s_i) = \begin{cases} s_i - a_i, & s_i \in [a_i, x_i], \\ s_i - b_i, & s_i \in (x_i, b_i], \end{cases} \qquad i = 1, 2, \ldots, n. \tag{8}$$

Then one has

$$\begin{aligned}
&f(x_1, x_2, \ldots, x_n) L - A_1(x_1, x_2, \ldots, x_n) \\
&= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} x_{n-1} \Delta_n x_n} \\
&\quad \times \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 + (-1)^{n+1} \\
&\quad \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(\sigma_1(s_1), \sigma_2(s_2), \ldots, \\
&\qquad \sigma_{n-1}(s_{n-1}), \sigma_n(s_n)) \\
&\quad \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1.
\end{aligned} \tag{9}$$

Proof. Suppose that $n = 2$. Then we have

$$\begin{aligned}
&\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^2 f(s_1, s_2)}{\Delta_1 s_1 \Delta_2 s_2} \prod_{i=1}^{2} p_i(x_i, s_i) \Delta_2 s_2 \Delta_1 s_1 \\
&= \int_{a_1}^{b_1} (b_2 - a_2) \frac{\partial f(s_1, x_2)}{\Delta_1 s_1} p_1(x_1, s_1) \Delta_1 s_1 \\
&\quad - \int_{a_1}^{b_1} \left[p_1(x_1, s_1) \int_{a_2}^{b_2} \frac{\partial f(s_1, \sigma_2(s_2))}{\Delta_1 s_1} \Delta_2 s_2 \right] \Delta_1 s_1 \\
&= \int_{a_1}^{b_1} (b_2 - a_2) \frac{\partial f(s_1, x_2)}{\Delta_1 s_1} p_1(x_1, s_1) \Delta_1 s_1 \\
&\quad - \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left[p_1(x_1, s_1) \frac{\partial f(s_1, \sigma_2(s_2))}{\Delta_1 s_1} \Delta_1 s_1 \right] \Delta_2 s_2 \\
&= (b_2 - a_2)(b_1 - a_1) f(x_1, x_2) - (b_2 - a_2) \\
&\quad \times \int_{a_1}^{b_1} f(\sigma_1(s_1), x_2) \Delta_1 s_1 \\
&\quad - (b_1 - a_1) \int_{a_2}^{b_2} f(x_1, \sigma_2(s_2)) \Delta_2 s_2 \\
&\quad + \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(\sigma_1(s_1), \sigma_2(s_2)) \Delta_2 s_2 \Delta_1 s_1.
\end{aligned} \tag{10}$$

That is, (9) holds for $n = 2$. □

Suppose that (9) holds for $n - 1$. Then for n we have

$$\begin{aligned}
&\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1} \Delta_n s_n} \\
&\quad \times \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1
\end{aligned}$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}} (b_n - a_n)\frac{\partial^{n-1} f(s_1, s_2, \ldots, s_{n-1}, x_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1}}$$

$$\times \prod_{i=1}^{n-1} p_i(x_i, s_i)\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$

$$- \int_{a_1}^{b_1}\int_{a_2}^{b_2}$$

$$\cdots\int_{a_{n-1}}^{b_{n-1}}\left[\prod_{i=1}^{n-1} p_i(x_i, s_i)\right.$$

$$\times \int_{a_n}^{b_n}\frac{\partial^{n-1} f(s_1, s_2, \ldots, s_{n-1}, \sigma_n(s_n))}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1}}$$

$$\left.\times\,\Delta_n s_n\right]$$

$$\times\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$

$$= (b_n - a_n)$$

$$\times \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}} \frac{\partial^{n-1} f(s_1, s_2, \ldots, s_{n-1}, x_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1}}$$

$$\times \prod_{i=1}^{n-1} p_i(x_i, s_i)\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$

$$- \int_{a_n}^{b_n}\left[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}} \frac{\partial^{n-1} f(s_1, s_2, \ldots, s_{n-1}, \sigma_n(s_n))}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1}}\right.$$

$$\left.\times \prod_{i=1}^{n-1} p_i(x_i, s_i)\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1\right]\Delta_n s_n$$

$$= (b_n - a_n)\left\{f(x_1, x_2, \ldots, x_{n-1}, x_n)\right.$$

$$\times \prod_{i=1}^{n-1}(b_i - a_i) + \sum_{k=1}^{n-2}(-1)^k \frac{L_k}{b_n - a_n}$$

$$\times \left[\sum_{1\le m1\le m2\cdots\le mk\le n-1}\int_{a_{m1}}^{b_{m1}}\int_{a_{m2}}^{b_{m2}}\right.$$

$$\cdots\int_{a_{mk}}^{b_{mk}} f(x_1, \ldots,$$

$$\sigma_{m1}(s_{m1}),$$

$$\ldots, \sigma_{m2}(s_{m2}),$$

$$\ldots, \sigma_{mk}(s_{mk}),$$

$$\ldots, x_{n-1}, x_n)$$

$$\left.\times\,\Delta_{mk}s_{mk}, \ldots, \Delta_{m2}s_{m2}\Delta_{m1}s_{m1}\right] + (-1)^{n-1}$$

$$\times \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}} f(\sigma_1(s_1), \sigma_2(s_2), \ldots,$$

$$\sigma_{n-1}(s_{n-1}), x_n)$$

$$\left.\times\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1\right\}$$

$$- \int_{a_n}^{b_n}\left\{f(x_1, x_2, \ldots, x_{n-1}, \sigma_n(s_n))\right.$$

$$\times \prod_{i=1}^{n-1}(b_i - a_i) + \sum_{k=1}^{n-2}(-1)^k \frac{L_k}{b_n - a_n}$$

$$\times \left[\sum_{1\le m1\le m2\cdots\le mk\le n-1}\int_{a_{m1}}^{b_{m1}}\int_{a_{m2}}^{b_{m2}}\right.$$

$$\cdots\int_{a_{mk}}^{b_{mk}} f(x_1, \ldots,$$

$$\sigma_{m1}(s_{m1}),$$

$$\ldots, \sigma_{m2}(s_{m2}),$$

$$\ldots, \sigma_{mk}(s_{mk}),$$

$$\ldots, x_{n-1}, \sigma_n(s_n))$$

$$\left.\times\,\Delta_{mk}s_{mk}, \ldots, \Delta_{m2}s_{m2}\Delta_{m1}s_{m1}\right]$$

$$+ (-1)^{n-1}\int_{a_1}^{b_1}\int_{a_2}^{b_2}$$

$$\cdots\int_{a_{n-1}}^{b_{n-1}} f(\sigma_1(s_1),$$

$$\sigma_2(s_2), \ldots,$$

$$\sigma_{n-1}(s_{n-1}), \sigma_n(s_n))$$

$$\left.\times\,\Delta_{n-1}s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1\right\}\Delta_n s_n$$

$$= f(x_1, x_2, \ldots, x_n)\prod_{i=1}^{n}(b_i - a_i)$$

$$+ \sum_{k=1}^{n-1}(-1)^k L_k$$

$$\times \left\{\sum_{1\le m1\le m2\cdots\le mk\le n}\int_{a_{m1}}^{b_{m1}}\int_{a_{m2}}^{b_{m2}}\right.$$

$$\cdots\int_{a_{mk}}^{b_{mk}} f(x_1, \ldots, \sigma_{m1}(s_{m1}),$$

$$\ldots, \sigma_{m2}(s_{m2}),$$

$$\begin{aligned}&\ldots,\sigma_{mk}(s_{mk}),\ldots,x_n)\\&\times\Delta_{mk}s_{mk},\ldots,\Delta_{m2}s_{m2}\Delta_{m1}s_{m1}\Big\}+(-1)^n\\&\times\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}f(\sigma_1(s_1),\sigma_2(s_2),\ldots,\\&\qquad\sigma_{n-1}(s_{n-1}),\sigma_n(s_n))\\&\times\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1,\end{aligned}\tag{11}$$

which is the desired result.

Remark 8. Lemma 7 is the n dimensional extension of [17, Lemma 2.3].

Based on Lemma 7, we present two generalized n dimensional Ostrowski type inequalities on time scales.

Theorem 9. *Let $a_i, b_i \in \mathbb{T}_i$, $x_i \in [a_i,b_i]_{\mathbb{T}_i}$, $i = 1,2,\ldots,n$, and $f \in C_{rd}([a_1,b_1]_{\mathbb{T}_1}\times[a_2,b_2]_{\mathbb{T}_2}\times\cdots\times[a_n,b_n]_{\mathbb{T}_n},\mathbb{R})$ such that the partial delta derivative of order n exists and there exists a constant K such that $\sup_{a_i<x_i<b_i,\ i=1,2,\ldots,n}|\partial^n f(x_1,x_2,\ldots,x_n)/(\Delta_1x_1\Delta_2x_2,\ldots,\Delta_nx_n)| = K$. Then one has*

$$\begin{aligned}&\Bigg|f(x_1,x_2,\ldots,x_n)L\\&-\sum_{k=1}^{n-1}(-1)^{k+1}L_k\\&\times\Bigg\{\sum_{1\le m1\le m2\cdots\le mk\le n}\int_{a_{m1}}^{b_{m1}}\int_{a_{m2}}^{b_{m2}}\\&\qquad\cdots\int_{a_{mk}}^{b_{mk}}f(x_1,\ldots,\sigma_{m1}(s_{m1}),\\&\qquad\ldots,\sigma_{m2}(s_{m2}),\ldots,\\&\qquad\sigma_{mk}(s_{mk}),\ldots,x_n)\\&\qquad\times\Delta_{mk}s_{mk},\ldots,\Delta_{m2}s_{m2}\Delta_{m1}s_{m1}\Bigg\}+(-1)^n\\&\times\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}f(\sigma_1(s_1),\sigma_2(s_2),\\&\qquad\ldots,\sigma_{n-1}(s_{n-1}),\\&\qquad\sigma_n(s_n))\\&\times\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1\Bigg|\\&\le K\prod_{i=1}^{n}[h_2(x_i,a_i)+h_2(x_i,b_i)].\end{aligned}\tag{12}$$

The inequality (12) *is sharp in the sense that the right-hand side of it cannot be replaced by a smaller one.*

Proof. From the definition of $p_i(x_i,s_i)$ we can obtain

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\left|\prod_{i=1}^{n}p_i(x_i,s_i)\right|\\&\quad\times\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1\\&=\prod_{i=1}^{n}[h_2(x_i,a_i)+h_2(x_i,b_i)].\end{aligned}\tag{13}$$

Then by Lemma 7 we have

$$\begin{aligned}&\Bigg|f(x_1,x_2,\ldots,x_n)L-A_1(x_1,x_2,\ldots,x_n)+(-1)^n\\&\times\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}f(\sigma_1(s_1),\sigma_2(s_2),\\&\qquad\ldots,\sigma_{n-1}(s_{n-1}),\\&\qquad\sigma_n(s_n))\\&\times\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1\Bigg|\\&=\Bigg|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\frac{\partial^n f(s_1,s_2,\ldots,s_{n-1},s_n)}{\Delta_1 s_1\Delta_2 s_2,\ldots,\Delta_{n-1}x_{n-1}\Delta_n x_n}\\&\quad\times\prod_{i=1}^{n}p_i(x_i,s_i)\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1\Bigg|\\&\le\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\left|\frac{\partial^n f(s_1,s_2,\ldots,s_{n-1},s_n)}{\Delta_1 s_1\Delta_2 s_2,\ldots,\Delta_{n-1}x_{n-1}\Delta_n x_n}\right|\\&\quad\times\left|\prod_{i=1}^{n}p_i(x_i,s_i)\right|\Delta_n s_n\Delta_{n-1}s_{n-1},\ldots,\Delta_2 s_2\Delta_1 s_1\\&\le K\prod_{i=1}^{n}[h_2(x_i,a_i)+h_2(x_i,b_i)],\end{aligned}\tag{14}$$

which is the desired inequality. □

The proof of the sharpness of (12) can be referred to [16, Theorem 3.5], which is equivalent to the case in Theorem 9 with $n = 1$.

Theorem 10. *Under the conditions of Theorem 9, if there exist constants K_1 and K_2 such that $K_1 \le \partial^n f(x_1,x_2,\ldots,x_n)/(\Delta_1x_1\Delta_2x_2,\ldots,\Delta_nx_n) \le K_2$, then one has*

$$\begin{aligned}&\Bigg|f(x_1,x_2,\ldots,x_n)L\\&-\sum_{k=1}^{n-1}(-1)^{k+1}L_k\end{aligned}$$

$$\times \left\{ \sum_{1 \le m1 \le m2 \cdots \le mk \le n} \int_{a_{m1}}^{b_{m1}} \int_{a_{m2}}^{b_{m2}} \cdots \int_{a_{mk}}^{b_{mk}} f\left(x_1, \ldots, \sigma_{m1}\left(s_{m1}\right), \ldots, \sigma_{m2}\left(s_{m2}\right), \ldots, \sigma_{mk}\left(s_{mk}\right), \ldots, x_n\right) \times \Delta_{mk} s_{mk}, \ldots, \Delta_{m2} s_{m2} \Delta_{m1} s_{m1} \right\} + (-1)^n$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f\left(\sigma_1\left(s_1\right), \sigma_2\left(s_2\right), \ldots, \sigma_{n-1}\left(s_{n-1}\right), \sigma_n\left(s_n\right)\right) \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 - \frac{K_1 + K_2}{2} \prod_{i=1}^{n} \left[h_2\left(x_i, a_i\right) - h_2\left(x_i, b_i\right)\right] \Bigg|$$
$$\le \frac{K_2 - K_1}{2} \prod_{i=1}^{n} \left[h_2\left(x_i, a_i\right) + h_2\left(x_i, b_i\right)\right]. \tag{15}$$

Proof. We notice that $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 = \prod_{i=1}^{n} [h_2(x_i, a_i) - h_2(x_i, b_i)]$.

We also have $|\partial^n f(x_1, x_2, \ldots, x_n)/(\Delta_1 x_1 \Delta_2 x_2, \ldots, \Delta_n x_n) - (K_1 + K_2)/2| \le (K_2 - K_1)/2$ and

$$\left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \left(\frac{\partial^n f\left(s_1, s_2, \ldots, s_{n-1}, s_n\right)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1} \Delta_n s_n} - \frac{K_1 + K_2}{2} \right) \prod_{i=1}^{n} p_i\left(x_i, s_i\right) \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \right|$$
$$\le \frac{K_2 - K_1}{2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \left| \prod_{i=1}^{n} p_i\left(x_i, s_i\right) \right| \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$= \frac{K_2 - K_1}{2} \prod_{i=1}^{n} \left[h_2\left(x_i, a_i\right) + h_2\left(x_i, b_i\right)\right]. \tag{16}$$

Collecting the information above and Lemma 7 we can get the desired result. □

Remark 11. Theorem 9 is the n dimensional extension of [16, Theorem 3.5].

In Theorem 9, if we take $\mathbb{T}_i$, $i = 1, 2, \ldots, n$, for some special time scales, then we immediately obtain the following three corollaries.

Corollary 12 (continuous case). *Let $\mathbb{T}_i = \mathbb{R}, i = 1, 2, \ldots, n$, in Theorem 9. Then $h_2(t, s) = (t - s)^2/2$, and one obtains*

$$\Bigg| f\left(x_1, x_2, \ldots, x_n\right) L - \sum_{k=1}^{n-1} (-1)^{k+1} L_k \times \left\{ \sum_{1 \le m1 \le m2 \cdots \le mk \le n} \int_{a_{m1}}^{b_{m1}} \int_{a_{m2}}^{b_{m2}} \cdots \int_{a_{mk}}^{b_{mk}} f\left(x_1, \ldots, s_{m1}, \ldots, s_{m2}, \ldots, s_{mk}, \ldots, x_n\right) \times ds_{mk}, \ldots, ds_{m2} ds_{m1} \right\} + (-1)^n$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f\left(s_1, s_2, \ldots, s_{n-1}, s_n\right) ds_n ds_{n-1}, \ldots, ds_2 ds_1 \Bigg|$$
$$\le K \prod_{i=1}^{n} \left[\frac{\left(x_i - a_i\right)^2}{2} + \frac{\left(x_i - b_i\right)^2}{2} \right], \tag{17}$$

where $K = \sup_{a_i < x_i < b_i,\ i = 1,2,\ldots,n} |\partial^n f(x_1, x_2, \ldots, x_n)/(\partial x_1 \partial x_2, \ldots, \partial x_n)|$.

Corollary 13 (discrete case). *Let $\mathbb{T}_i = \mathbb{Z}, i = 1, 2, \ldots, n$, in Theorem 9. Then $h_2(t, s) = (t - s)(t - s - 1)/2$ for $\forall t, s \in \mathbb{Z}$, and one has*

$$\Bigg| f\left(x_1, x_2, \ldots, x_n\right) L - \sum_{k=1}^{n-1} (-1)^{k+1} L_k \times \left\{ \sum_{1 \le m1 \le m2 \cdots \le mk \le n} \sum_{s_{m1} = a_{m1}+1}^{b_{m1}} \sum_{s_{m2} = a_{m2}+1}^{b_{m2}} \cdots \sum_{s_{mk} = a_{mk}+1}^{b_{mk}} f\left(x_1, \ldots, s_{m1}, \ldots, s_{m2},\right.\right.$$

$$\left.\ldots, s_{mk}, \ldots, x_n)\right\}$$

$$+(-1)^n \sum_{s_1=a_1+1}^{b_1} \sum_{s_2=a_2+1}^{b_2} \cdots \sum_{s_{n-1}=a_{n-1}+1}^{b_{n-1}} \sum_{s_n=a_n+1}^{b_n} \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} f(s_1, s_2, \ldots, s_{n-1}, s_n)\Bigg|$$

$$\leq K \prod_{i=1}^{n} \frac{(x_i-a_i)(x_i-a_i-1)+(x_i-b_i)(x_i-b_i-1)}{2}, \tag{18}$$

where K denotes the maximum value of the absolute value of the difference $\Delta_1\Delta_2,\ldots,\Delta_n f$ over $[a_1, b_1-1]_{\mathbb{Z}} \times [a_2, b_2-1]_{\mathbb{Z}} \cdots \times [a_n, b_n-1]_{\mathbb{Z}}$.

Corollary 14 (quantum calculus case). *Let $\mathbb{T}_i = q_i^{\mathbb{N}_0}$, $i = 1, 2, \ldots, n$, in Theorem 9, where $q_i > 1$, $i = 1, 2, \ldots, n$. Then one has*

$$\Bigg| f(x_1, x_2, \ldots, x_n) L - \sum_{k=1}^{n-1} (-1)^{k+1} L_k \times \Bigg\{ \sum_{1\leq m1 \leq m2 \cdots \leq mk \leq n} \left[\prod_{i=1}^{k} a_{mi}(q_{mi}-1) \right] \times \sum_{s_{m1}=0}^{\log_{q_{m1}}^{[b_{m1}/(q_{m1}a_{m1})]}} \sum_{s_{m2}=0}^{\log_{q_{m2}}^{[b_{m2}/(q_{m2}a_{m2})]}} \cdots \sum_{s_{mk}=0}^{\log_{q_{mk}}^{[b_{mk}/(q_{mk}a_{mk})]}} \left(\prod_{i=1}^{k} q_{mi}^{s_{mi}} \right)$$

$$\times f\left(x_1, \ldots, a_{m1} q_{m1}^{s_{m1}+1}, \ldots, a_{m2} q_{m2}^{s_{m2}+1}, \ldots, a_{mk} q_{mk}^{s_{mk}+1}, \ldots, x_n\right) \Bigg\}$$

$$+(-1)^n \left[\prod_{i=1}^{n} a_i(q_i-1)\right] \times \sum_{s_1=0}^{\log_{q_1}^{[b_1/(q_1a_1)]}} \sum_{s_2=0}^{\log_{q_2}^{[b_2/(q_2a_2)]}} \cdots \sum_{s_{n-1}=0}^{\log_{q_{n-1}}^{[(b_{n-1})/(q_{n-1}a_{n-1})]}} \sum_{s_{n-2}=0}^{\log_{q_n}^{[b_n/(q_na_n)]}} \left(\prod_{i=1}^{n} q_i^{s_i} \right)$$

$$\times f\left(q_1^{s_1+1}a_1, q_2^{s_2+1}a_2, \ldots, q_{n-1}^{s_{n-1}+1}a_{n-1}, q_n^{s_n+1}a_n\right)\Bigg|$$

$$\leq K \prod_{i=1}^{n} \left[\frac{(x_i-a_i)(x_i-q_ia_i)}{1+q_i} + \frac{(x_i-b_i)(x_i-q_ib_i)}{1+q_i} \right], \tag{19}$$

where K denotes the maximum value of the absolute value of the $q_1q_2,\ldots,q_n$-difference $D_{q_1q_2,\ldots,q_n} f(t_1, t_2, \ldots, t_n)$ over $[a_1, b_1/q_1]_{\mathbb{T}_1} \times [a_2, b_2/q_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n/q_n]_{\mathbb{T}_n}$.

Proof. Since $h_k(t,s) = \prod_{n=0}^{k-1}(t-q_i^n s)/\sum_{\mu=0}^{n} q_i^{\mu}$ for $\forall s, t \in q_i^{\mathbb{N}_0}$, $i = 1, 2, \ldots, n$, then we have

$$h_2(t,s) = \frac{(t-s)(t-q_is)}{1+q_i}, \quad i = 1, 2 \cdots, n. \tag{20}$$

Substituting (20) into (12) we get the desired result. □

Next we propose the n dimensional Grüss inequalities in the following two theorems.

Theorem 15. *Let $a_i, b_i \in \mathbb{T}_i$, $x_i \in [a_i, b_i]_{\mathbb{T}_i}$, $i = 1, 2, \ldots, n$, and $f, g \in C_{rd}([a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n}, \mathbb{R})$ such that the partial delta derivative of order n exists and there exist constants K_1 and K_2 such that $\sup_{a_i<x_i<b_i,\, i=1,2,\ldots,n} |\partial^n f(x_1, x_2, \ldots, x_n)/(\Delta_1 x_1 \Delta_2 x_2, \ldots, \Delta_n x_n)| = K_1$, $\sup_{a_i<x_i<b_i,\, i=1,2,\ldots,n} |\partial^n g(x_1, x_2, \ldots, x_n)/(\Delta_1 x_1 \Delta_2 x_2, \ldots, \Delta_n x_n)| = K_2$. Then one has*

$$\Bigg| \frac{1}{L} \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n) \times g(x_1, x_2, \ldots, x_n) \times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 - \frac{(-1)^{n+1}}{2L^2}$$

$$\times \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n) \times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1$$

$$\times \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} g(\sigma_1(s_1), \sigma_2(s_2), \ldots, \sigma_{n-1}(s_{n-1}), \sigma_n(s_n))$$

$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 - \frac{(-1)^{n+1}}{2L^2} \times \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} g(x_1, x_2, \ldots, x_n)$$

$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 x_2 \Delta_1 x_1$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(\sigma_1(s_1), \sigma_2(s_2), \dots, \sigma_{n-1}(s_{n-1}), \sigma_n(s_n))$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1 - \frac{1}{2L^2}$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} [g(x_1, x_2, \dots, x_n) \times A_1(x_1, x_2, \dots, x_n) + f(x_1, x_2, \dots, x_n) \times A_2(x_1, x_2, \dots, x_n)]$$
$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 x_2 \Delta_1 x_1 \Big|$$
$$\leq \frac{K_1}{2L^2} \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \left[|g(x_1, x_2, \dots, x_n)| \times \prod_{i=1}^{n} h_2(x_i, a_i) + h_2(x_i, b_i) \right]$$
$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1 + \frac{K_2}{2L^2}$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \left[|f(x_1, x_2, \dots, x_n)| \times \prod_{i=1}^{n} h_2(x_i, a_i) + h_2(x_i, b_i) \right]$$
$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1. \tag{21}$$

Proof. For a function $g(x_1, x_2, \dots, x_n)$, from Lemma 7, we have

$$g(x_1, x_2, \dots, x_n) L - A_2(x_1, x_2, \dots, x_n)$$
$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n g(s_1, s_2, \dots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \dots, \Delta_{n-1} x_{n-1} \Delta_n x_n}$$
$$\times \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1$$
$$+ (-1)^{n+1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} g(\sigma_1(s_1), \sigma_2(s_2), \dots, \sigma_{n-1}(s_{n-1}), \sigma_n(s_n))$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1. \tag{22}$$

Then multiplying (9) by $g(x_1, x_2, \dots, x_n)$ and (22) by $f(x_1, x_2, \dots, x_n)$, and adding the resulting identities, we obtain

$$2f(x_1, x_2, \dots, x_n) g(x_1, x_2, \dots, x_n) L$$
$$= g(x_1, x_2, \dots, x_n)$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \dots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \dots, \Delta_{n-1} x_{n-1} \Delta_n x_n}$$
$$\times \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1$$
$$+ f(x_1, x_2, \dots, x_n)$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n g(s_1, s_2, \dots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \dots, \Delta_{n-1} x_{n-1} \Delta_n x_n}$$
$$\times \prod_{i=1}^{n} p_i(x_i, s_i) \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1$$
$$+ g(x_1, x_2, \dots, x_n) A_1(x_1, x_2, \dots, x_n)$$
$$+ f(x_1, x_2, \dots, x_n) A_2(x_1, x_2, \dots, x_n)$$
$$+ A_3(x_1, x_2, \dots, x_n). \tag{23}$$

An integration for (23) on $[a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n}$ yields that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n) g(x_1, x_2, \dots, x_n)$$
$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 x_2 \Delta_1 x_1$$
$$= \frac{1}{2L} \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \left[g(x_1, x_2, \dots, x_n) \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \dots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \dots, \Delta_{n-1} x_{n-1} \Delta_n x_n} \times \prod_{i=1}^{n} p_i(x_i, s_i) \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \dots, \Delta_2 s_2 \Delta_1 s_1 \right]$$
$$\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \dots, \Delta_2 x_2 \Delta_1 x_1 + \frac{1}{2L}$$

$$
\begin{aligned}
&\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \Bigg[f(x_1, x_2, \ldots, x_n) \\
&\qquad \times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\qquad \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n g(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} x_{n-1} \Delta_n x_n} \\
&\qquad \times \prod_{i=1}^{n} p_i(x_i, s_i) \\
&\qquad \times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \Bigg] \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 + \frac{1}{2L} \\
&\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} [g(x_1, x_2, \ldots, x_n) \\
&\qquad \times A_1(x_1, x_2, \ldots, x_n) \\
&\qquad + f(x_1, x_2, \ldots, x_n) \\
&\qquad \times A_2(x_1, x_2, \ldots, x_n)] \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 \\
&+ \frac{1}{2L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} A_3(x_1, x_2, \ldots, x_n) \\
&\qquad \times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1.
\end{aligned} \tag{24}
$$

So we have

$$
\begin{aligned}
&\Bigg| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n) g(x_1, x_2, \ldots, x_n) \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 \\
&- \frac{1}{2L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} [g(x_1, x_2, \ldots, x_n) \\
&\qquad \times A_1(x_1, x_2, \ldots, x_n) \\
&\qquad + f(x_1, x_2, \ldots, x_n) \\
&\qquad \times A_2(x_1, x_2, \ldots, x_n)] \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 \\
&- \frac{1}{2L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \\
&\cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} A_3(x_1, x_2, \ldots, x_n) \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 \Bigg| \\
&\leq \frac{K_1}{2L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \Bigg[|g(x_1, x_2, \ldots, x_n)| \\
&\qquad \times \prod_{i=1}^{n} (h_2(x_i, a_i) \\
&\qquad + h_2(x_i, b_i)) \Bigg] \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1 \\
&+ \frac{K_2}{2L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \Bigg[|f(x_1, x_2, \ldots, x_n)| \\
&\qquad \times \prod_{i=1}^{n} (h_2(x_i, a_i) \\
&\qquad + h_2(x_i, b_i)) \Bigg] \\
&\times \Delta_n x_n \Delta_{n-1} x_{n-1}, \ldots, \Delta_2 x_2 \Delta_1 x_1.
\end{aligned} \tag{25}
$$

After simple computation we can get the desired result. □

Remark 16. Theorem 15 is the n dimensional extension of [17, Theorem 2.2].

Theorem 17. *Let f, $g \in C_{rd}([a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n}, \mathbb{R})$ such that $\phi \leq f(x_1, x_{2,n}) \leq \Phi$ and $\gamma \leq g(x_1, x_2, \ldots, x_n) \leq \Gamma$ for all $x_i \in [a_i, b_i]_{\mathbb{T}_i}$, $i = 1, 2, \ldots, n$, where ϕ, Φ, γ, and Γ are constants. Then one has*

$$
\begin{aligned}
&\Bigg| \frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n) g(x_1, x_2, \ldots, x_n) \\
&\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \\
&- \frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$\times \frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} g(x_1, x_2, \ldots, x_n)$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1 \Bigg|$$
$$\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma). \tag{26}$$

The proof for Theorem 17 is similar to [26, pp: 295–296], which is omitted here.

Finally, we present one Ostrowski-Grüss type inequality as follows.

Theorem 18. *Suppose that $f \in C_{rd}([a_1, b_1]_{\mathbb{T}_1} \times [a_2, b_2]_{\mathbb{T}_2} \times \cdots \times [a_n, b_n]_{\mathbb{T}_n}, \mathbb{R})$ such that the partial delta derivative of order n exists and there exist constants K_1 and K_2 such that $K_1 \leq \partial^n f(x_1, x_2, \ldots, x_n)/(\Delta_1 x_1 \Delta_2 x_2, \ldots, \Delta_n x_n) \leq K_2$. Then one has*

$$\Bigg| f(x_1, x_2, \ldots, x_n) - \frac{1}{L} \sum_{k=1}^{n-1} (-1)^{k+1} L_k$$
$$\times \Bigg\{ \sum_{1 \leq m1 \leq m2 \cdots \leq mk \leq n} \int_{a_{m1}}^{b_{m1}} \int_{a_{m2}}^{b_{m2}} \cdots \int_{a_{mk}}^{b_{mk}} f(x_1, \ldots, \sigma_{m1}(s_{m1}), \ldots, \sigma_{m2}(s_{m2}), \ldots, \sigma_{mk}(s_{mk}), \ldots, x_n)$$
$$\times \Delta_{mk} s_{mk}, \ldots, \Delta_{m2} s_{m2} \Delta_{m1} s_{m1} \Bigg\} + \frac{1}{L} (-1)^n$$
$$\times \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} f(\sigma_1(s_1), \sigma_2(s_2), \ldots, \sigma_{n-1}(s_{n-1}), \sigma_n(s_n))$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$- \Bigg[\frac{1}{L} \sum_{j^{(1)}=1}^{2} \sum_{j^{(2)}=1}^{2} \cdots \sum_{j^{(n)}=1}^{2} (-1)^{k_1 + k_2 + \cdots + k_n} \times f\left(\lambda_{1, j^{(1)}}, \lambda_{2, j^{(2)}}, \ldots, \lambda_{n, j^{(n)}}\right) \Bigg]$$
$$\times \frac{1}{L} \prod_{i=1}^{n} [h_2(x_i, a_i) - h_2(x_i, b_i)] \Bigg|$$
$$\leq \frac{L(K_2 - K_1)}{4}. \tag{27}$$

Proof. From the definition of $p_i(x_i, s_i)$ in (8) we obtain that $\max(\prod_{i=1}^n p_i(x_i, s_i)) - \min(\prod_{i=1}^n p_i(x_i, s_i)) \leq \prod_{i=1}^n (b_i - a_i) = L$. On the other hand, we have

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1} \Delta_n s_n}$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$= \sum_{j^{(1)}=1}^{2} \sum_{j^{(2)}=1}^{2} \cdots \sum_{j^{(n)}=1}^{2} (-1)^{k_1 + k_2 + \cdots + k_n} \times f\left(\lambda_{1, j^{(1)}}, \lambda_{2, j^{(2)}}, \ldots, \lambda_{n, j^{(n)}}\right), \tag{28}$$
$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \prod_{i=1}^{n} p_i(x_i, s_i)$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$= \prod_{i=1}^{n} [h_2(x_i, a_i) - h_2(x_i, b_i)],$$

where $\lambda_{i,1} = a_i$, $\lambda_{i,2} = b_i$, $k_i = \begin{cases} 1, & \lambda_{i, j^{(i)}} = a_i \\ 0, & \lambda_{i, j^{(i)}} = b_i, \end{cases}$ $i = 1, 2, \ldots, n$. So by Theorem 17 we deduce that

$$\frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1} \Delta_n s_n}$$
$$\times \prod_{i=1}^{n} p_i(x_i, s_i)$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$- \frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \frac{\partial^n f(s_1, s_2, \ldots, s_{n-1}, s_n)}{\Delta_1 s_1 \Delta_2 s_2, \ldots, \Delta_{n-1} s_{n-1} \Delta_n s_n}$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$\times \frac{1}{L} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \prod_{i=1}^{n} p_i(x_i, s_i)$$
$$\times \Delta_n s_n \Delta_{n-1} s_{n-1}, \ldots, \Delta_2 s_2 \Delta_1 s_1$$
$$\leq \frac{[\max(\prod_{i=1}^n p_i(x_i, s_i)) - \min(\prod_{i=1}^n p_i(x_i, s_i))]}{4} (K_2 - K_1)$$
$$\leq \frac{L}{4} (K_2 - K_1). \tag{29}$$

Then combining (9), (28), and (29) we get the desired result. □

Remark 19. Theorem 18 is the n dimensional extension of [18, Theorem 4] and is the n dimensional extension on time scales of [12, Theorem 2.1].

Remark 20. For Theorems 10, 15, 17, and 18, we can also obtain similar results as shown in Corollaries 12–14, which are omitted here.

3. Conclusions

In this paper, we have presented some new generalized n dimensional Ostrowski type and Grüss type integral inequalities on time scales. These inequalities are of new forms compared with the existing results in the literature and are n dimensional extension on time scales of some known continuous and discrete inequalities. Based on these inequalities, some new bounds as well as some sharp bounds for unknown functions are derived. These inequalities can also be used in the estimate of explicit error bounds for some numerical quadrature formulae.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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