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Research Article

Integrated Fractional Resolvent Operator Function and Fractional Abstract Cauchy Problem

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We firstly prove that β-times integrated α-resolvent operator function ((α,β)-ROF) satisfies a functional equation which extends that of β-times integrated semigroup and α-resolvent operator function. Secondly, for the inhomogeneous α-Cauchy problem ${}^cD_t^\alpha u(t) = Au(t) + f(t)$, $t \in (0,T)$, $u(0) = x_0$, $u'(0) = x_1$, if A is the generator of an (α,β)-ROF, we give the relation between the function $v(t) = S_{\alpha,\beta}(t)x_0 + (g_1 * S_{\alpha,\beta})(t)x_1 + (g_{\alpha-1} * S_{\alpha,\beta} * f)(t)$ and mild solution and classical solution of it. Finally, for the problem ${}^cD_t^\alpha v(t) = Av(t) + g_{\beta+1}(t)x$, $t > 0$, $v^{(k)}(0) = 0$, $k = 0, 1, \ldots, N-1$, where A is a linear closed operator. We show that A generates an exponentially bounded (α,β)-ROF on a Banach space X if and only if the problem has a unique exponentially bounded classical solution v_x and $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+, X)$. Our results extend and generalize some related results in the literature.

1. Introduction

This paper is concerned with the properties of β-integrated α-resolvent operator function ((α,β)-ROF) and two inhomogeneous fractional Cauchy problems.

Throughout this paper, $\mathbb{R}^+ = [0,\infty)$, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X, Y be Banach spaces, $B(X,Y)$ denote the space of all bounded linear operators from X to Y, $B(X) = B(X,X)$. If A is a closed linear operator, $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. $L^1(\mathbb{R}^+, X)$ denotes the space of X-valued Bochner integrable functions: $u : \mathbb{R}^+ \to X$ with the norm $\|u\|_{L^1(\mathbb{R}^+,X)} = \int_0^\infty \|u(t)\| dt$, it is a Banach space. By $*$ we denote the convolution of functions

$$(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau, \quad t \ge 0. \tag{1}$$

g_α denotes the function

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{2}$$

and $g_0(t) = \delta_0(t)$, the Dirac delta function.

In 1997, Mijatović et al. [1] introduced the concept of β-times integrated semigroup ($\beta \in \mathbb{R}^+$) which extends k-times integrated semigroup ($k \in \mathbb{N}_0$) [2], they showed an $R(\lambda)$ to be the pseudoresolvent of a β-times ($\beta > 0$) integrated semigroup $\{S(t)\}$ if and only if $\{S(t)\}$ satisfies the following functional equation:

$$\begin{aligned} &\int_t^{t+s} (s+t-r)^{\beta-1} S(r)\, dr - \int_0^s (s+t-r)^{\beta-1} S(r)\, dr \\ &\quad = \Gamma(\beta)\, S(t)\, S(s), \quad t, s \ge 0. \end{aligned} \tag{3}$$

In the special case of $\beta = k \in \mathbb{N}$, the corresponding result is summarized in [2].

For the inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \quad t \in [0,T], \; u(0) = x, \tag{4}$$

where $T > 0$, $f \in L^1([0,T], X)$, $x \in X$, and A is the generator of a k-times integrated semigroup $\{S(t)\}$ on a Banach space X for some $k \in \mathbb{N}_0$. Let $v(t) = S(t)x + \int_0^t S(t-s) f(s) ds$, $t \in [0,T]$. Lemmas 3.2.9 and 3.2.10 of [2] show that if there is a mild(classical) solution u of (4), then $v \in C^k([0,T], X)$ $(C^{k+1}([0,T], X))$ and $u = v^{(k)}$. On the other hand, if $v \in C^k([0,T],X)$ $(C^{k+1}([0,T],X))$, then $v^{(k)}$ is also a mild (classical) solution of it.

Furthermore, if A generates an exponential bounded k-times integrated semigroup on a Banach space X, then, for any $x \in X$, $v(t) = \int_0^t S(s)x\,ds$ is the unique exponential bounded classical solution of the following problem:

$$u'(t) = Au(t) + g_{k+1}(t)x, \quad t \geq 0, \ u(0) = 0. \tag{5}$$

In recent years, a considerable interest has been paid to fractional evolution equation due to its applications in different areas such as stochastic, finance, and physics; see [3–8]. One of the most important tools in the theory of fractional evolution equation is the solution operator (fractional resolvent family) [9–15]. The notion of solution operator was developed to study some abstract Volterra integral equations [16] and was first used by Bajlekova [17] to study a class of fractional order abstract Cauchy problem. In [9], Chen and Li introduced α-resolvent operator functions (α-ROF for short) defined by purely algebraic equation. They showed that a family $\{S_\alpha(t)\}_{t\geq 0} \subset B(X)$ is an α-ROF if and only if $\{S_\alpha(t)\}_{t\geq 0}$ is a solution of abstract fractional Cauchy problem

$$\begin{gathered} {}^cD_t^\alpha v(t) = Av(t) + g_{\beta+1}(t)x, \quad t > 0, \\ v(0) = x, \qquad v^{(k)}(0) = 0, \quad k = 1, \ldots, N-1. \end{gathered} \tag{6}$$

When $0 < \alpha < 1$, Peng and Li [18] proved that the solution operator $\{S_\alpha(t)\}_{t\geq 0}$ for (6) satisfies the following equality:

$$\begin{aligned} &\int_t^{t+s} \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha}d\tau - \int_0^s \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha}d\tau \\ &\quad = \alpha \int_0^t \int_0^s \frac{S_\alpha(\tau_1)S_\alpha(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}}d\tau_1 d\tau_2, \quad t,s \geq 0. \end{aligned} \tag{7}$$

We refer to [5, 15, 16, 19] for further information concerning general resolvent operator functions. In addition, Chen and Li [9] also introduced the concept of integrated fractional resolvent operator function in an algebraic notion as follows.

Definition 1 (see [9, Definition 3.7]). Let $\alpha > 0$, $\beta \geq 0$. A function $S_{\alpha,\beta} : \mathbb{R}^+ \to B(X)$ is called a β-times integrated α-resolvent operator function or an (α,β)-resolvent operator function ((α,β)-ROF for short) if the following conditions hold:

(a) $S_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}^+$ and $S_{\alpha,\beta}(0) = g_{\beta+1}(0)I$;

(b) $S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)S_{\alpha,\beta}(s)$ for all $s,t \geq 0$;

(c) the functional equation

$$\begin{aligned} &S_{\alpha,\beta}(s)J_t^\alpha S_{\alpha,\beta}(t) - J_s^\alpha S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) \\ &\quad = g_{\beta+1}(s)J_t^\alpha S_{\alpha,\beta}(t) - g_{\beta+1}(t)J_s^\alpha S_{\alpha,\beta}(s) \end{aligned} \tag{8}$$

holds for $s,t \geq 0$, where J_t^α is the Riemann-Liouville fractional integral of order α.

The generator A of $S_{\alpha,\beta}(t)$ is defined by

$$\begin{gathered} D(A) := \left\{x \in X : \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_{\beta+1}(t)x}{g_{\alpha+\beta+1}(t)} \text{ exists}\right\}, \\ Ax := \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_{\beta+1}(t)x}{g_{\alpha+\beta+1}(t)}, \quad x \in D(A). \end{gathered} \tag{9}$$

Note that an $(\alpha,0)$-ROF is just an α-ROF.

In this paper, we firstly show that (α,β)-ROF satisfies an equality which extends (3) and (7) for β-integrated semigroup and α-ROF, respectively. Then, we consider the inhomogeneous fractional order abstract Cauchy problem

$$\begin{gathered} {}^cD_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,T), \\ u(0) = x_0, \qquad u'(0) = x_1, \end{gathered} \tag{10}$$

where $1 < \alpha < 2$, $T > 0$, $f \in L^1((0,T),X)$, and A is assumed to be the generator of an (α,β)-ROF $S_{\alpha,\beta}(t)$ on X. We give the relation between the function $v(t) = S_{\alpha,\beta}(t)x_0 + (g_1 * S_{\alpha,\beta})(t)x_1 + (g_{\alpha-1} * S_{\alpha,\beta} * f)(t)$ and solution of (10). We also study the problem

$$\begin{gathered} {}^cD_t^\alpha v(t) = Av(t) + g_{\beta+1}(t)x, \quad t > 0, \\ v^{(k)}(0) = 0, \quad k = 0, 1, \ldots, N-1, \end{gathered} \tag{11}$$

where $\alpha > 0$, $x \in X$, N is the smallest integer greater than or equal to α. We prove that if A generates an exponentially bounded (α,β)-ROF on X if and only if the problem (11) has a unique exponentially bounded classical solution v_x and $Av_x \in L^1_{\text{Loc}}(\mathbb{R}^+,X)$. If $\alpha \to 1^+$, $\beta = k \in \mathbb{N}$, our Theorem 13 reduces to Lemma 3.2.10 in [2]. When $\alpha = 1$, $\beta = k$, it is easy to see that our Theorem 15 extends and generalizes Theorem 3.2.13 in [2].

This paper is organized as follows. In Section 2, we provide some preliminaries of the fractional calculus and (α,β)-ROF. Section 3 is devoted to present an equality characteristic of the (α,β)-ROF. Finally, as an application of (α,β)-ROF, we discuss the solutions of fractional abstract Cauchy problem in Section 4.

2. Preliminary

Recall that the Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)\,ds, \tag{12}$$

and the Caputo fractional derivative of order $\alpha > 0$ of f can be written as

$${}^cD_t^\alpha f(t) = \frac{d^m}{dt^m}\left(g_{m-\alpha} * \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\,g_{k+1}(t)\right)\right), \tag{13}$$

where m is the smallest integer greater than or equal to α. For more details in fractional calculus, we refer to [5, 20, 21].

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\qquad E_\alpha(z)=E_{\alpha,1}(z),\qquad \operatorname{Re}\alpha>0,\ \beta,z\in\mathbb{C}. \tag{14}$$

And if $0<\alpha<2$, $\beta>0$, then

$$\begin{aligned}&E_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp\left(z^{1/\alpha}\right)+\varepsilon_{\alpha,\beta}(z),\\ &|\arg z|\le\frac{1}{2}\alpha\pi,\qquad E_{\alpha,\beta}(z)=\varepsilon_{\alpha,\beta}(z),\\ &|\arg(-z)|<\left(1-\frac{1}{2}\alpha\right)\pi,\end{aligned} \tag{15}$$

where

$$\varepsilon_{\alpha,\beta}(z)=-\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}+O\left(|z|^{-N}\right)\quad\text{as } z\longrightarrow\infty, \tag{16}$$

and the O-term is uniform in $\arg z$ if $|\arg(-z)|\le(1-(\alpha/2)-\epsilon)\pi$.

We now recall some properties of (α,β)-ROF.

Lemma 2 (see [9, Proposition 3.10]). *Let $S_{\alpha,\beta}:\mathbb{R}^+\to B(X)$ be an (α,β)-ROF generated by A. The following assertions hold:*

(a) *$S_{\alpha,\beta}(t)D(A)\subset D(A)$ and $AS_{\alpha,\beta}(t)x=S_{\alpha,\beta}(t)Ax$ for $x\in D(A)$ and $t\ge0$;*

(b) *for all $x\in X$, $J_t^\alpha S_{\alpha,\beta}(t)x\in D(A)$ and $S_{\alpha,\beta}(t)x=g_{\beta+1}(t)x+AJ_t^\alpha S_{\alpha,\beta}(t)x$, $t\ge0$;*

(c) *$x\in D(A)$ and $Ax=y$ if and only if $S_{\alpha,\beta}(t)x=g_{\beta+1}(t)x+J_t^\alpha S_{\alpha,\beta}(t)y$, $t\ge0$;*

(d) *A is closed.*

Lemma 3 (see [9, Proposition 3.5, Theorem 3.11]). *Let $\alpha>0$, $\beta\ge0$. A generates an (α,β)-ROF $S_{\alpha,\beta}$ satisfying $\|S_{\alpha,\beta}(t)\|\le Me^{\omega t}$, $t\ge0$, for some constants $M>0$ and $\omega\ge0$, if and only if $(\omega^\alpha,\infty)\subset\rho(A)$ and there exists a strongly continuous function $S:\mathbb{R}^+\to B(X)$ such that $\|S(t)\|\le Me^{\omega t}$ for all $t\ge0$ and $\int_0^\infty e^{-\lambda t}S(t)x\,dt=\lambda^{\alpha-\beta-1}R(\lambda^\alpha,A)x$, $\lambda>\omega$, for all $x\in X$. Furthermore, $S(t)$ is $S_{\alpha,\beta}(t)$.*

Lemma 4 (see [2, Proposition B.6]). *Let $U\subset\mathbb{C}$. If function $R:U\to B(X)$ satisfies $R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu)$, then there is an operator A on X such that $R(\lambda)=(\lambda I-A)^{-1}$ for all $\lambda\in U$ if and only if $\ker R(\lambda)=\{0\}$.*

3. An Novel Equality Characteristic for (α,β)-ROF

The following theorem shows that an (α,β)-ROF satisfies a functional equation and the treatment bases on the technique of Laplace transform. For convenience, we drop the subscript α,β from $\{S_{\alpha,\beta}\}_{t\ge0}$ in this theorem.

Theorem 5. *Let $\alpha\in\mathbb{R}^+\setminus\mathbb{N}_0$, $\beta\in\mathbb{R}^+$ satisfy $\beta-\alpha>-1$. If $\{S(t)\}_{t\ge0}$ is an (α,β)-ROF, then it satisfies the following equality:*

$$\begin{aligned}&\int_t^{t+s}(s+t-r)^{\beta-\alpha}S(r)\,dr-\int_0^s(s+t-r)^{\beta-\alpha}S(r)\,dr\\ &\quad=\frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}\int_0^s\int_0^t\frac{S(r_1)S(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2.\end{aligned} \tag{17}$$

Proof. Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of equality (17), respectively, and denote by $f_a(t)$ the truncation of $f(t)$ at a, that is, $f_a(t)=f(t)$ for $0\le t\le a$ and $f_a(t)=0$ otherwise.

We will show that the Laplace transform of $L_a(t,s)$ and $R_a(t,s)$ with respect to t and s is equivalent, and by the uniqueness of Laplace transform, we can get that $L_a(t,s)=R_a(t,s)$.

Taking Laplace transform of $L_a(t,s)$ with respect to s as follows

$$\begin{aligned}\widehat{L}_a(t,\lambda)&=\int_0^\infty e^{-\lambda s}\left[\int_t^{t+s}(s+t-r)^{\beta-\alpha}S_a(r)\,dr\right.\\ &\qquad\left.-\int_0^s(s+t-r)^{\beta-\alpha}S_a(r)\,dr\right]ds\\ &=\int_t^\infty S_a(r)\int_{r-t}^\infty e^{-\lambda s}(s+t-r)^{\beta-\alpha}ds\,dr\\ &\quad-\int_0^\infty S_a(r)\int_r^\infty e^{-\lambda s}(s+t-r)^{\beta-\alpha}ds\,dr\\ &=\int_t^\infty S_a(r)e^{-\lambda(r-t)}\int_0^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau\,dr\\ &\quad-\int_0^\infty S_a(r)e^{-\lambda(r-t)}\int_t^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau\,dr\\ &=\frac{\Gamma(\beta-\alpha+1)}{\lambda^{(\beta-\alpha+1)}}\int_t^\infty S_a(r)e^{-\lambda(r-t)}dr\\ &\quad-e^{\lambda t}\widehat{S}_a(\lambda)\int_t^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau,\end{aligned} \tag{18}$$

then taking Laplace transform with respect to t, we have

$$\begin{aligned}\widehat{L}_a(\mu,\lambda)&=\int_0^\infty e^{-\mu t}\left[\frac{\Gamma(\beta-\alpha+1)}{\lambda^{(\beta-\alpha+1)}}\int_t^\infty S_a(r)e^{-\lambda(r-t)}dr\right.\\ &\qquad\left.-e^{\lambda t}\widehat{S}_a(\lambda)\int_t^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau\right]dt\\ &=\frac{\Gamma(\beta-\alpha+1)}{\lambda^{(\beta-\alpha+1)}}\int_0^\infty e^{-\mu t}\int_t^\infty S_a(r)e^{-\lambda(r-t)}dr\,dt\\ &\quad-\widehat{S}_a(\lambda)\int_0^\infty e^{(\lambda-\mu)t}\int_t^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau\,dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\beta-\alpha+1)}{\lambda^{(\beta-\alpha+1)}}\int_0^\infty e^{-\lambda r} S_a(r)\int_0^r e^{(\lambda-\mu)t}\,dt\,dr \\
&\quad - \int_0^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}\int_0^\tau e^{(\lambda-\mu)t}\,dt\,d\tau\,\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)\lambda^{(\beta-\alpha+1)}} \\
&\quad \times\left(\int_0^\infty e^{-\mu r} S_a(r)\,dr - \int_0^\infty e^{-\lambda r} S_a(r)\,dr\right) \\
&\quad - \frac{1}{\lambda-\mu} \\
&\quad \times\left(\int_0^\infty e^{-\mu\tau}\tau^{\beta-\alpha}d\tau - \int_0^\infty e^{-\lambda\tau}\tau^{\beta-\alpha}d\tau\right)\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)\lambda^{(\beta-\alpha+1)}}\left(\widehat{S}_a(\mu)-\widehat{S}_a(\lambda)\right) \\
&\quad - \frac{1}{\lambda-\mu}\left(\frac{\Gamma(\beta-\alpha+1)}{\mu^{(\beta-\alpha+1)}} - \frac{\Gamma(\beta-\alpha+1)}{\lambda^{(\beta-\alpha+1)}}\right)\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{\lambda-\mu}\left(\lambda^{(\alpha-\beta-1)}\widehat{S}_a(\mu) - \mu^{(\alpha-\beta-1)}\widehat{S}_a(\lambda)\right) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)(\lambda\mu)^{\beta-\alpha+1}} \\
&\quad \times\left(\mu^{(\beta-\alpha+1)}\widehat{S}_a(\mu) - \lambda^{(\beta-\alpha+1)}\widehat{S}_a(\lambda)\right) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)(\lambda\mu)^{\beta-\alpha+1}}\left(R(\mu^\alpha, A) - R(\lambda^\alpha, A)\right),
\end{aligned}
\tag{19}
$$

where the last equality follows from Lemma 3.

On the other hand, observing that

$$
R_a(t,s) = \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}\int_0^t \frac{S_a(r)}{(t+s-r)^{1+\alpha}}dr * S_a(s),
\tag{20}
$$

Then taking Laplace transform with respect to t and s, respectively, we deduce

$$
\begin{aligned}
\widehat{R}_a(t,\lambda) &= \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} \\
&\quad \times\int_0^\infty e^{-\lambda s}\int_0^t \frac{S_a(r)}{(t+s-r)^{1+\alpha}}dr\,ds\,\widehat{S}_a(\lambda) \\
&= \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} \\
&\quad \times\int_0^\infty e^{-\lambda s}(t+s)^{-\alpha-1} * S_a(t)\,ds\,\widehat{S}_a(\lambda),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{R}_a(\mu,\lambda) &= \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} \\
&\quad \times\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}(t+s)^{-\alpha-1} * S_a(t)\,ds\,dt\,\widehat{S}_a(\lambda) \\
&= \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} \\
&\quad \times\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}(t+s)^{-\alpha-1} * S_a(t)\,ds\,dt\,\widehat{S}_a(\lambda) \\
&= \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)} \\
&\quad \times\int_0^\infty e^{-\lambda s}\int_0^\infty e^{-\mu t}(t+s)^{-\alpha-1}\,dt\,ds\,\widehat{S}_a(\mu)\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)}{\lambda-\mu}(\lambda^\alpha-\mu^\alpha)\widehat{S}_a(\mu)\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{\lambda-\mu}(\lambda^\alpha-\mu^\alpha)\widehat{S}_a(\mu)\widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)}(\lambda^\alpha-\mu^\alpha) \\
&\quad \times\frac{1}{(\lambda\mu)^{\beta-\alpha+1}}R(\mu^\alpha,A)R(\lambda^\alpha,A) \\
&= \frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)(\lambda\mu)^{\beta-\alpha+1}}\left(R(\mu^\alpha,A)-R(\lambda^\alpha,A)\right),
\end{aligned}
\tag{21}
$$

where the last equality follows from the resolvent identity. In view of (19), (21), and the uniqueness of Laplace transform, we obtain $L_a(t,s) = R_a(t,s)$, $t,s \geq 0$. The arbitrariness of a implies $L(t,s) = R(t,s)$ for $t,s \geq 0$. □

Remark 6. (a) If $\beta = 0$, then $(\alpha,0)$-ROF $S_{\alpha,0}(t)$ is an α-ROF and the equality (17) degenerates to be equality (7).

(b) If we assume that, for each $x \in X$, the map $t \to S_{\alpha,\beta}(t)x$ is continuously differentiable on $[0, \infty)$ and the limit of (α,β)-ROF $S_{\alpha,\beta}(t)$ exists as $\alpha \to 1^-$, then multiplying both sides of (17) with $1-\alpha$ and integrating by parts to the right side of (17) and letting $\alpha \to 1^-$, we can get that (3) is just the limit state of (17).

By Lemma 3, (α,β)-ROF generated by operator A is exactly operator valued functions whose Laplace transforms are $\lambda^{\alpha-\beta-1}R(\lambda, A)$. In the following theorem, we show that this property corresponds to the functional equation (17) for $S_{\alpha,\beta}(t)$. The proof of this theorem is proved by Ardent [2, proposition 3.2.4] for $\alpha \to 1^-$, $\beta = k \in \mathbb{N}$. Our proof is different since we could not use the binomial formula as in [2].

Theorem 7. *Let $\widetilde{S}:\mathbb{R}^+\to B(X)$ be a strongly continuous function satisfying $\|\widetilde{S}(t)\|\le Me^{\omega t}$ $(t\ge 0)$ for some $M,\ \omega\ge 0$. Let $\alpha\in\mathbb{R}^+\setminus\mathbb{N}_0,\ \beta\in\mathbb{R}^+$ satisfy that $\beta-\alpha>-1$, set*

$$R(\lambda^\alpha):=\lambda^{-\alpha+\beta+1}\int_0^\infty e^{-\lambda t}\widetilde{S}(t)\,dt. \tag{22}$$

Then the following assertions are equivalent.

(i) *There exists an operator A such that $(\omega^\alpha,\infty)\subset\rho(A)$ and $R(\lambda^\alpha)=(\lambda^\alpha I-A)^{-1}$ for $\lambda>\omega$.*

(ii) *For $s,t\ge 0$, the equality*

$$\int_t^{t+s}(s+t-r)^{\beta-\alpha}\widetilde{S}(r)\,dr-\int_0^s(s+t-r)^{\beta-\alpha}\widetilde{S}(r)\,dr = \frac{\alpha\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)}\int_0^s\int_0^t\frac{\widetilde{S}(r_1)\widetilde{S}(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2 \tag{23}$$

holds and $\widetilde{S}(t)x=0$ for all $t\ge 0$ implies that $x=0$.

Proof. Assume that (i) holds; then $(\omega^\alpha,\infty)\subset\rho(A)$, $(\lambda^\alpha I-A)^{-1}=\lambda^{-\alpha+\beta+1}\int_0^\infty e^{-\lambda t}\widetilde{S}(t)\,dt$ for $\lambda>\omega$; from Lemma 3, we know that $\widetilde{S}(t)$ is the (α,β)-ROF generated by A; then Theorem 5 shows that equality (23) holds. It follows from $(\omega^\alpha,\infty)\subset\rho(A)$ and $R(\lambda^\alpha)=(\lambda^\alpha I-A)^{-1}$ for $\lambda>\omega$ that $R(\lambda^\alpha)$ is injective. If $\widetilde{S}(t)x=0$ for all $t\ge 0$, from $R(\lambda^\alpha):=\lambda^{-\alpha+\beta+1}\int_0^\infty e^{-\lambda t}\widetilde{S}(t)\,dt$, we have $R(\lambda^\alpha)x=0$; thus $x=0$.

If (ii) is satisfied, similar as the calculations of (19) and (21), we can get that the Laplace transform of the left side and the right side of (17) are

$$\begin{gathered}\frac{\Gamma(\beta-\alpha+1)}{(\lambda-\mu)(\lambda\mu)^{\beta-\alpha+1}}\left(R(\mu^\alpha)-R(\lambda^\alpha)\right),\\ \frac{\Gamma(\beta-\alpha+1)}{\lambda-\mu}\frac{\lambda^\alpha-\mu^\alpha}{(\lambda\mu)^{\beta-\alpha+1}}R(\mu^\alpha)R(\lambda^\alpha),\end{gathered} \tag{24}$$

respectively. So,

$$R(\mu^\alpha)-R(\lambda^\alpha)=(\lambda^\alpha-\mu^\alpha)R(\mu^\alpha)R(\lambda^\alpha). \tag{25}$$

On the other hand, if $R(\lambda^\alpha)x=0$, by $R(\lambda^\alpha)=\lambda^{-\alpha+\beta+1}\int_0^\infty e^{-\lambda t}\widetilde{S}(t)\,dt$ and uniqueness of Laplace transform, we have $\widetilde{S}(t)x=0$ for all $t\ge 0$, then from (ii) we know $x=0$, so, $\operatorname{Ker}R(\lambda^\alpha)=0$, by (25) and Lemma 4, we get the conclusion. □

4. Fractional Abstract Cauchy Problems

In this section, we study the following inhomogeneous fractional abstract Cauchy problem:

$$\begin{gathered}{}^cD_t^\alpha u(t)=Au(t)+f(t),\quad t\in(0,T),\\ u(0)=x_0,\qquad u'(0)=x_1,\end{gathered} \tag{26}$$

where $1<\alpha<2,\ T>0,\ f\in L^1((0,T),X),\ x_0,x_1\in X$, A is a linear closed operator.

First, we give the definitions of solutions to (26).

Definition 8. A function $u\in C([0,T);X)$ is called a mild solution of (26), if $(g_\alpha*u)(t)\in D(A)$ and $u(t)=x_0+tx_1+A(g_\alpha*u)(t)+(g_\alpha*f)(t),\ t\in[0,T)$.

Definition 9. A function $u\in C([0,T);X)$ is called a classical solution of (26) if u satisfies the following.

(a) $u\in C([0,T);D(A))\cap C^1([0,T);X)$.

(b) $g_{2-\alpha}*(u-x_0-tx_1)\in C^2([0,T);X)$.

(c) u satisfies (26).

From the above definitions, it is clear that a classical solution of (26) is a mild solution of it. The following assertion shows that a mild solution of the problem (26) with suitable regularity is also a classical solution.

Theorem 10. *Let u be a mild solution of (26) and $f\in C([0,T);X)$, if $g_{2-\alpha}*(u-x_0-tx_1)\in C^2([0,T);X)$, and for any $t\in(0,\ T)$, $g_\alpha*u\in L^1((0,t),\ D(A))$; then u is also a classical solution of (26).*

Proof. Since u is a mild solution of (26), we have

$$\begin{gathered}(g_\alpha*u)(t)\in D(A),\\ u(t)=x_0+tx_1+A(g_\alpha*u)(t)+(g_\alpha*f)(t),\\ t\in[0,T).\end{gathered} \tag{27}$$

If we denote $w(t):=u(t)-x_0-tx_1$, then it follows from (27) that

$$\begin{aligned}(g_{2-\alpha}*w)(t)&=g_{2-\alpha}*\left(A(g_\alpha*u)(t)+(g_\alpha*f)(t)\right)\\ &=A(g_2*u)(t)+(g_2*f)(t).\end{aligned} \tag{28}$$

Since $g_{2-\alpha}*w\in C^2([0,T);X)$, then ${}^cD_t^\alpha u(t)=(d^2/dt^2)(g_{2-\alpha}*w)(t)$ is well defined, and by (28), we have

$$\begin{aligned}{}^cD_t^\alpha u(t)&=\frac{d^2}{dt^2}(g_{2-\alpha}*w)(t)\\ &=\lim_{h\to 0}\frac{1}{h^2}\left[(g_{2-\alpha}*w)(t)-2(g_{2-\alpha}*w)(t-h)\right.\\ &\qquad\left.+(g_{2-\alpha}*w)(t-2h)\right]\\ &=\lim_{h\to 0}\frac{1}{h^2}\left[A(g_2*u)(t)-2A(g_2*u)(t-h)\right.\\ &\qquad\left.+A(g_2*u)(t-2h)\right]\\ &\quad+\lim_{h\to 0}\frac{1}{h^2}\left[(g_2*f)(t)-2(g_2*f)(t-h)\right.\\ &\qquad\left.+(g_2*f)(t-2h)\right]\\ &=\lim_{h\to 0}\frac{1}{h^2}\left[A(g_2*u)(t)-2A(g_2*u)(t-h)\right.\\ &\qquad\left.+A(g_2*u)(t-2h)\right]+f(t).\end{aligned} \tag{29}$$

Thus,

$$\lim_{h\to 0}\frac{1}{h^2}\left[A\left(g_2 * u\right)(t) - 2A\left(g_2 * u\right)(t-h) + A\left(g_2 * u\right)(t-2h)\right] = {}^{c}D_t^{\alpha}u(t) - f(t). \tag{30}$$

On the other hand, from the closeness of A and $g_\alpha * u \in L^1((0,t), D(A))$ for $t \in [0,T)$, by Proposition 1.1.7 in [2], we have

$$\left(g_2 * u\right)(t) = \left(g_{2-\alpha} * \left(g_\alpha * u\right)\right)(t) \in D(A), \tag{31}$$

Then from (30) and the closeness of A, we obtain

$$\begin{aligned} u(t) &= \lim_{h\to 0}\frac{1}{h^2}\left[\left(g_2 * u\right)(t) - 2\left(g_2 * u\right)(t-h) + \left(g_2 * u\right)(t-2h)\right] \in D(A), \\ {}^{c}D_t^{\alpha}u(t) &= Au(t) + f(t), \quad t \in [0,T). \end{aligned} \tag{32}$$

It is clear that $u(0) = x_0$, $u'(0) = x_1$. Thus, u is a classical solution of (26). □

Lemma 11. *Let $1 < \alpha < 2$, $f \in L^1((0,T), X)$. Suppose A is the generator of an (α,β)-ROF $S_{\alpha,\beta}(t)$ on X for some $\beta \in \mathbb{R}^+$. Then, for every $t \in [0,T)$, $(g_{\alpha-1} * S_{\alpha,\beta} * f)(t)$ exists, and $(g_{\alpha-1} * S_{\alpha,\beta} * f) \in C([0,T), X)$.*

Proof. For every $t \in [0,T)$, since $g_{\alpha-1} \in L^1((0,t),\mathbb{R}^+)$, $f \in L^1((0,t), X)$, we get $g_{\alpha-1} * f \in L^1((0,t), X)$, hence, from

$$\begin{aligned} \left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t) &= \left(S_{\alpha,\beta} * g_{\alpha-1} * f\right)(t) \\ &= \int_0^t S_{\alpha,\beta}(t-s)\left(g_{\alpha-1} * f\right)(s)\,ds, \end{aligned} \tag{33}$$

we obtain that $(g_{\alpha-1} * S_{\alpha,\beta} * f)(t)$ exists.

For $h \in \mathbb{R}$, $|h| \ll 1$ and $t+h \in [0,T)$, we have

$$\begin{aligned} &\left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t+h) - \left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t) \\ &\quad = \int_0^{t+h} S_{\alpha,\beta}(t+h-s)\left(g_{\alpha-1} * f\right)(s)\,ds \\ &\qquad - \int_0^{t} S_{\alpha,\beta}(t-s)\left(g_{\alpha-1} * f\right)(s)\,ds \\ &\quad = \int_0^{t+h} \left(S_{\alpha,\beta}(t+h-s) - S_{\alpha,\beta}(t-s)\right)\left(g_{\alpha-1} * f\right)(s)\,ds \\ &\qquad - \int_t^{t+h} S_{\alpha,\beta}(t-s)\left(g_{\alpha-1} * f\right)(s)\,ds. \end{aligned} \tag{34}$$

From the dominated convergence theorem and absolute continuity of integral, we deduce

$$\lim_{h\to 0}\left(\left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t+h) - \left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t)\right) = 0. \tag{35}$$

So, $(g_{\alpha-1} * S_{\alpha,\beta} * f) \in C([0,T), X)$. □

Let

$$v(t) = S_{\alpha,\beta}(t)x_0 + \left(g_1 * S_{\alpha,\beta}\right)(t)x_1 + \left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t). \tag{36}$$

From Lemma 11, we know that v is well defined, and $v \in C([0,T), X)$.

The following theorem is proved by Arendt [2, Lemma 3.2.9] for $\alpha = 1$, $\beta = l \in \mathbb{N}$. Our proof is different because we could not use the formula of integration by parts as [2, Lemma 3.2.9].

Theorem 12. *Suppose that A is the generator of an (α,β)-ROF $S_{\alpha,\beta}(t)$ on X for some $\beta \in \mathbb{R}^+$. Let v be defined by (36). Then one has the following results.*

(a) *If (26) has a mild solution u, then $g_{m-\beta} * (v - \sum_{k=0}^{m-1} v^{(k)}(0)g_{k+1}(t)) \in C^m([0,T); X)$ and $u(t) = {}^{c}D_t^{\beta}v(t)$.*

(b) *If there is a classical solution u of (26), then $g_{2-\alpha} * ({}^{c}D_t^{\beta}v(t) - x_0 - tx_1) \in C^2([0,T); X)$ and $u(t) = {}^{c}D_t^{\beta}v(t)$.*

Proof. If u is a mild solution of (26), then $(g_\alpha * u)(t) \in D(A)$ and

$$u(t) = x_0 + tx_1 + A\left(g_\alpha * u\right)(t) + \left(g_\alpha * f\right)(t), \quad t \in [0,T). \tag{37}$$

Using Lemma 2(b) and the closeness of A, we have

$$\begin{aligned} \left(g_{\beta+1} * u\right)(t) &= \left(S_{\alpha,\beta} - A\left(g_\alpha * S_{\alpha,\beta}\right)\right) * u(t) \\ &= \left(S_{\alpha,\beta} * u\right)(t) - \left(A\left(g_\alpha * S_{\alpha,\beta}\right) * u\right)(t) \\ &= \left(S_{\alpha,\beta} * u\right)(t) - S_{\alpha,\beta} * A\left(g_\alpha * u\right)(t) \\ &= \left(S_{\alpha,\beta} * u\right)(t) \\ &\quad - S_{\alpha,\beta} * \left(u - x_0 - tx_1 - \left(g_\alpha * f\right)(t)\right) \\ &= S_{\alpha,\beta} * x_0 + S_{\alpha,\beta} * tx_1 + \left(S_{\alpha,\beta} * g_\alpha * f\right)(t); \end{aligned} \tag{38}$$

that is, $(g_{\beta+1} * u)(t) = (1 * S_{\alpha,\beta})(t)x_0 + (g_2 * S_{\alpha,\beta})(t)x_1 + (g_\alpha * S * f)(t)$. So

$$\begin{aligned} J_t^{\beta}u(t) &= \left(g_\beta * u\right)(t) = \frac{d}{dt}\left(g_{\beta+1} * u\right)(t) \\ &= S_{\alpha,\beta}(t)x_0 + \left(g_1 * S_{\alpha,\beta}\right)(t)x_1 \\ &\quad + \left(g_{\alpha-1} * S_{\alpha,\beta} * f\right)(t) = v(t). \end{aligned} \tag{39}$$

Thus, it follows from $u \in C([0,T), X)$ that $g_{m-\beta} * (v - \sum_{k=0}^{m-1} v^{(k)}(0)g_{k+1}(t)) \in C^m([0,T); X)$ and $u(t) = {}^{c}D_t^{\beta}v(t)$. Hence (a) holds. If u is a classical solution of (26), then u is a mild solution of (26). So, assertion (b) follows immediately from (a). □

Theorem 13. *Let v be defined by (36). Assume that $v \in C^{m-1}([0,T];X)$, $v^{(k)}(0) = 0$ for $k = 0,1,\ldots,m-1$, and $g_{m-\beta} * v \in C^m([0,T];X)$; then ${}^cD_t^\beta v(t)$ is a mild solution of the problem (26). Moreover, if $g_{2-\alpha} * ({}^cD_t^\beta v(t) - x_0 - tx_1) \in C^2([0,T];X)$, and for any $t \in (0,T)$, $g_\alpha * {}^cD_t^\beta v(t) \in L^1((0,t),D(A))$, then ${}^cD_t^\beta v(t)$ is also a classical solution of (26).*

Proof. Consider the following steps.
Step 1. We first claim that $J_t^\alpha v(t) \in D(A)$ and

$$ {}^cD_t^\beta AJ_t^\alpha v(t) = {}^cD_t^\beta v(t) - x_0 - tx_1 - g_\alpha * f. \tag{40}$$

In view of definition of $v(t)$, we have

$$\begin{aligned} J_t^\alpha v(t) &= (g_\alpha * S_{\alpha,\beta})(t)x_0 + (g_\alpha * g_1 * S_{\alpha,\beta})(t)x_1 \\ &\quad + (g_\alpha * g_{\alpha-1} * S_{\alpha,\beta} * f)(t) \\ &= (g_\alpha * S_{\alpha,\beta})(t)x_0 + \int_0^t (g_\alpha * S_{\alpha,\beta})(\tau)x_1 d\tau \\ &\quad + \int_0^t (g_\alpha * S_{\alpha,\beta})(t-\tau)(g_{\alpha-1} * f)(\tau)d\tau, \end{aligned} \tag{41}$$

for $t \in [0,T)$.

From Lemma 2(b), for $0 < \tau < t$, we have

$$\begin{aligned} &(g_\alpha * S_{\alpha,\beta})(t)x_0 \in D(A), \qquad (g_\alpha * S_{\alpha,\beta})(\tau)x_1 \in D(A), \\ &\quad (g_\alpha * S_{\alpha,\beta})(t-\tau)(g_{\alpha-1} * f)(\tau) \in D(A), \\ &A(g_\alpha * S_{\alpha,\beta})(\tau)x_1 = S_{\alpha,\beta}(\tau)x_1 - g_{\beta+1}(\tau)x_1 \in L^1(0,t), \\ &A(g_\alpha * S_{\alpha,\beta})(t-\tau)(g_{\alpha-1} * f)(\tau) \\ &\quad = S_{\alpha,\beta}(t-\tau)(g_{\alpha-1} * f)(\tau) \\ &\qquad - g_{\beta+1}(t-\tau)(g_{\alpha-1} * f)(\tau) \in L^1(0,t), \end{aligned} \tag{42}$$

combining with the closeness of A, one has

$$\begin{aligned} &\int_0^t (g_\alpha * S_{\alpha,\beta})(\tau)x_1 d\tau \in D(A), \\ &\int_0^t (g_\alpha * S_{\alpha,\beta})(t-\tau)(g_{\alpha-1} * f)(\tau)d\tau \in D(A). \end{aligned} \tag{43}$$

Thus $J_t^\alpha v(t) \in D(A)$, and

$$\begin{aligned} AJ_t^\alpha v(t) &= A(g_\alpha * S_{\alpha,\beta})(t)x_0 + g_1 * A(g_\alpha * S_{\alpha,\beta})(t)x_1 \\ &\quad + g_{\alpha-1} * A(g_\alpha * S_{\alpha,\beta} * f)(t) \\ &= S_{\alpha,\beta}(t)x_0 - g_{\beta+1}(t)x_0 + (g_1 * S_{\alpha,\beta})(t)x_1 \\ &\quad - (g_1 * g_{\beta+1})(t)x_1 \\ &\quad + g_{\alpha-1} * (S_{\alpha,\beta} * f - g_{\beta+1} * f)(t) \\ &= S_{\alpha,\beta}(t)x_0 + (g_1 * S_{\alpha,\beta})(t)x_1 \\ &\quad + (g_{\alpha-1} * S_{\alpha,\beta} * f)(t) - g_{\beta+1}(t)x_0 \\ &\quad - (g_1 * g_{\beta+1})(t)x_1 - (g_{\alpha+\beta} * f)(t) \\ &= v(t) - g_{\beta+1}(t)x_0 - (g_1 * g_{\beta+1})(t)x_1 \\ &\quad - (g_{\alpha+\beta} * f)(t). \end{aligned} \tag{44}$$

So

$$\begin{aligned} AJ_t^\alpha v(t) &= v(t) - g_{\beta+1}(t)x_0 - (g_1 * g_{\beta+1})(t)x_1 \\ &\quad - (g_{\alpha+\beta} * f)(t), \\ {}^cD_t^\beta AJ_t^\alpha v(t) &= {}^cD_t^\beta v(t) - x_0 - tx_1 - g_\alpha * f. \end{aligned} \tag{45}$$

Step 2. We prove ${}^cD_t^\beta J_t^\alpha v(t) \in D(A)$ and $A\,{}^cD_t^\beta J_t^\alpha v(t) = {}^cD_t^\beta AJ_t^\alpha v(t)$.

Since $v \in C^k([0,T];X)$, $v^{(k)}(0) = 0$ for $k = 0,1,\ldots,m-1$, we have

$$\frac{d^k}{dt^k}(g_\alpha * v)(t)|_{t=0} = (g_\alpha * v^{(k)})(t)|_{t=0} = 0. \tag{46}$$

So

$$\begin{aligned} {}^cD_t^\beta J_t^\alpha v(t) &= \frac{d^m}{dt^m}(g_{m-\beta} * g_\alpha * v)(t) \\ &= \lim_{h\to 0}\frac{1}{h^m}\sum_{r=0}^m C_m^r (g_{m-\beta} * g_\alpha * v)(t-rh), \end{aligned} \tag{47}$$

where $C_m^r = (m(m-1)\cdots(m-r+1))/r!$. From (45), we know that $AJ_t^\alpha v \in L^1((0,t),X)$, and the closeness of A implies that $(g_{m-\beta} * g_\alpha * v)(t) \in D(A)$, and $A(g_{m-\beta} * g_\alpha * v)(t) = g_{m-\beta} * A(g_\alpha * v)(t)$, by Step 1, ${}^cD_t^\beta AJ_t^\alpha v(t)$ exists, then ${}^cD_t^\beta J_t^\alpha v(t) \in D(A)$, and

$$A\,{}^cD_t^\beta J_t^\alpha v(t) = {}^cD_t^\beta AJ_t^\alpha v(t). \tag{48}$$

Step 3. We show that $v^{(k)}(0) = 0$ for $k = 0,1,\ldots,m-1$ implies

$${}^cD_t^\beta J_t^\alpha v(t) = J_t^\alpha\,{}^cD_t^\beta v(t). \tag{49}$$

In fact, if $\alpha \geq \beta$, we have ${}^cD_t^\beta J_t^\alpha v(t) = J_t^{\alpha-\beta}v(t)$, and

$$\begin{aligned} J_t^\alpha\,{}^cD_t^\beta v(t) &= J_t^{\alpha-\beta}J_t^\beta\,{}^cD_t^\beta v(t) \\ &= J_t^{\alpha-\beta}\left(v(t) - \sum_{k=0}^{m-1} v^{(k)}(0)g_{k+1}(t)\right). \end{aligned} \tag{50}$$

If $\alpha < \beta$, we have ${}^cD_t^{\beta} J_t^{\alpha} v(t) = {}^cD_t^{\beta-\alpha} v(t)$, and

$$J_t^{\alpha}\, {}^cD_t^{\beta} v(t) = {}^cD_t^{\beta-\alpha} J_t^{\beta}\, {}^cD_t^{\beta} v(t) = {}^cD_t^{\beta-\alpha} \left(v(t) - \sum_{k=0}^{m-1} v^{(k)}(0)\, g_{k+1}(t) \right). \tag{51}$$

From the above discussion and $v^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m-1$, we conclude that (49) holds.

Finally, in view of (40), (48), and (49), we have

$$AJ_t^{\alpha}\, {}^cD_t^{\beta} v(t) = A\, {}^cD_t^{\beta} J_t^{\alpha} v(t) = {}^cD_t^{\beta} A J_t^{\alpha} v(t) = {}^cD_t^{\beta} v(t) - x_0 - t x_1 - (g_{\alpha} * f)(t). \tag{52}$$

Therefore, ${}^cD_t^{\beta} v(t)$ is a mild solution of (26).

Moreover, if $g_{2-\alpha} * ({}^cD_t^{\beta} v(t) - x_0 - t x_1) \in C^2([0,T]; X)$, and for any $t \in (0,T)$, $g_{\alpha} * {}^cD_t^{\beta} v(t) \in L^1((0,t), D(A))$, applying Theorem 10, we have that ${}^cD_t^{\beta} v(t)$ is a classical solution of (26). □

Remark 14. If $\alpha \to 1^+$, $\beta = k$, then (26) becomes (4). Theorem 13 degenerated to Lemma 3.2.10 in [2]. Note that the condition $v^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k-1$ is not necessary in Lemma 3.2.10 of [2], since from its proof, it is easy to see that $v(0) = 0$ implies that $v^{(j)}(0) = 0$ for $j = 1, \ldots, k-1$.

Now, we turn our attention to the problem

$$\begin{aligned} {}^cD_t^{\alpha} v(t) &= A v(t) + g_{\beta+1} t(x), \quad t > 0, \\ v^{(k)}(0) &= 0, \quad k = 0, 1, \ldots, N-1, \end{aligned} \tag{53}$$

where $\alpha > 0$, $x \in X$, A is a linear closed operator on X and N is the smallest integer greater than or equal to α.

Theorem 15. *Let A be a closed operator on X and $\beta > 0$; the following two assertions are equivalent.*

(i) *A generates an exponentially bounded (α, β)-ROF $S_{\alpha,\beta}$ on X.*

(ii) *For every $x \in X$, there exists a unique classical solution v_x of (53) which is exponentially bounded and $Av_x \in L^1_{\mathrm{loc}}(\mathbb{R}^+; X)$.*

Proof. If (i) is satisfied, for every $x \in X$, define $v_x : \mathbb{R}^+ \to X$ by $v_x(t) = (g_{\alpha} * S_{\alpha,\beta})(t)x$, then $v_x^{(k)}(0) = 0$ for $k = 0, 1, \ldots, N-1$. By Lemma 2(b), we have $v_x(t) = (g_{\alpha} * S_{\alpha,\beta})(t)x \in D(A)$, and

$$\begin{aligned} {}^cD_t^{\alpha} v_x(t) &= S_{\alpha,\beta}(t)x = A\left(g_{\alpha} * S_{\alpha,\beta}\right)(t)x + g_{\beta+1}(t)x \\ &= A v_x(t) + g_{\beta+1}(t)x, \quad t > 0. \end{aligned} \tag{54}$$

Thus, v_x is a classical solution of (53); it is unique by Theorem 12. Since $S_{\alpha,\beta}$ is exponentially bounded, we have that v_x is exponentially bounded. From

$$A v_x(t) = A\left(g_{\alpha} * S_{\alpha,\beta}\right)(t)x = S_{\alpha,\beta}(t)x - g_{\beta+1}(t)x, \tag{55}$$

we know that $Av_x(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+; X)$. So (ii) is true.

Assume that (ii) holds. From linearity of (53) and the uniqueness of its solution, we get that v_x is linear in x. So, for each $t \geq 0$, there exists a linear mapping $V(t) : X \to D(A)$ such that $V(t)x = v_x(t)$ for any $x \in X$.

Next, we show that, for each $t \geq 0$, $V(t) \in B(X, D(A))$.

We consider the mapping $\Phi : X \to C(\mathbb{R}^+, D(A))$ by $\Phi(x) = v_x(\cdot) = V(\cdot)x$. Then, Φ is a linear operator defined on X. Now we show that Φ is closed, if $x_n \to x$ in X and $\Phi(x_n) \to u$ in $C(\mathbb{R}^+, D(A))$. For $t > 0$, by the dominated convergence theorem, we have that $J_t^{\alpha} v_{x_n}(t)$ converges to $J_t^{\alpha} u(t)$, since $v_{x_n}(\cdot) = g_{\alpha+\beta+1}(t)x_n + J_t^{\alpha} A v_{x_n}(t)$, from the closeness of A, it follows that as $n \to \infty$, $u(t) = g_{\alpha+\beta+1}(t)x + J_t^{\alpha} A u(t)$, which implies that $u = \Phi(x)$ and Φ is closed. Therefore, by the closed graph theorem, Φ is bounded. So, for each $t \geq 0$, $V(t) \in B(X, D(A))$. Then, the exponentially boundedness of $V(t)x$ and Lemma 3.2.14 in [2], imply that $\|V(t)\| \leq M e^{\omega t} (t \geq 0)$ for some constants $M,\ \omega \geq 0$. So $Q(\lambda)x = \lambda^{\beta+1} \int_0^{\infty} e^{-\lambda t} V(t)x\, dt$ is well defined for $\lambda > \omega$, $(\omega^{\alpha}, \infty) \subset \rho(A)$.

Since $AV(t)x \in L^1(\mathbb{R}^+, X)$, then the Laplace transform of $AV(t)x$ is well defined, and from the closeness of A, for $\lambda > \omega$, we have

$$\begin{aligned} (\lambda^{\alpha} - A) Q(\lambda)x &= \lambda^{\alpha+\beta+1} \int_0^{\infty} e^{-\lambda t} V(t)x\, dt - \lambda^{\alpha+\beta+1} \\ &\quad \times \int_0^{\infty} e^{-\lambda t} A V(t)x\, dt \\ &= \lambda^{\alpha+\beta+1} \int_0^{\infty} e^{-\lambda t} V(t)x\, dt - \lambda^{\beta+1} \\ &\quad \times \int_0^{\infty} e^{-\lambda t}\, {}^cD_t^{\alpha} V(t)x\, dt + \lambda^{\beta+1} \\ &\quad \times \int_0^{\infty} e^{-\lambda t} g_{\beta+1}(t)x\, dt \\ &= \lambda^{\alpha+\beta+1} \widehat{V}(\lambda)x - \lambda^{\beta+1} \lambda^{\alpha} \widehat{V}(\lambda)x \\ &\quad + \lambda^{\beta+1} \lambda^{-(\beta+1)} x = x. \end{aligned} \tag{56}$$

Now, we show that $(\lambda^{\alpha} - A)$ is injective for $\lambda > \omega$. Assume that $(\lambda^{\alpha} - A)x = 0$ for some $x \in D(A)$ and $\lambda > \omega$. Then, by the method of Laplace transform, we have that the solution of (53) is $t^{\alpha+\beta} E_{\alpha,\alpha+\beta+1}(\lambda^{\alpha} t^{\alpha})$. Since $\|v_x(t)\| \leq M e^{\omega t}$, for all $t \geq 0$, combine with (15), it follows that $x = 0$. Hence $(\lambda^{\alpha} - A)^{-1} = Q(\lambda)$ for $\lambda > \omega$ and $V(t)$ is an $(\alpha, \alpha+\beta)$-ROF. Let

$$S_{\alpha,\beta}(t)x := {}^cD_t^{\alpha} V(t)x = AV(t)x + g_{\beta+1}(t)x; \tag{57}$$

then $S_{\alpha,\beta}(t)x$ exists and $V(t)x = J_t^{\alpha} S_{\alpha,\beta}(t)x$ for all $t \geq 0$ and all $x \in X$. So

$$S_{\alpha,\beta}(t)x = A J_t^{\alpha} S_{\alpha,\beta}(t)x + g_{\beta+1}(t)x, \tag{58}$$

and taking the Laplace transform, we have

$$\widehat{S}_{\alpha,\beta}(\lambda)x = A\lambda^{-\alpha} \widehat{S}_{\alpha,\beta}(\lambda)x + \lambda^{-\beta-1}x, \quad \lambda > \omega; \tag{59}$$

that is,

$$\widehat{S}_{\alpha,\beta}(\lambda)x = \lambda^{\alpha-\beta-1}(\lambda^{\alpha} - A)^{-1}x, \quad \lambda > \omega. \tag{60}$$

From Lemma 3, we know that $S_{\alpha,\beta}$ is the (α,β)-ROF generated by A. □

Remark 16. Theorem 15 extends and generalizes Theorem 3.2.13 in [2]. In fact, when $\alpha = 1$ and $\beta = k$, (53) becomes (5), $S_{\alpha,\beta}(t)$ is a k-times integrated semigroup. For problem (5), the condition $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+; X)$ in (ii) is not necessary. Since from the proof of Theorem 3.2.13 in [2], it is easy to see that the assumption that exponentially boundedness of the unique classical solution to the problem (5) imply that $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+; X)$.

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