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Research Article

Bäcklund Transformation and Quasi-Periodic Solutions for a Variable-Coefficient Integrable Equation

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Binary Bell polynomials are applied to construct bilinear formalism, bilinear Bäcklund transformation, Lax pair, and infinite conservation laws of the generalized variable-coefficient fifth-order Korteweg-de Vries equation. In the meantime, quasi-periodic wave solutions for the equation are obtained by using the Riemann theta function. The asymptotic properties of one-periodic wave solution and two-periodic wave solutions are also established, respectively.

1. Introduction

Nonlinear evolution equations (NLEEs) have attracted intensive attention in the past few decades, since they occur in a variety of physical applications. It is always important to search for explicit and exact solutions. Various kinds of exact solutions such as soliton, peakon, complexiton, rational, periodic, and quasi-periodic solutions have been presented for NLEEs. Successful methods include the inverse scattering method [1], the Darboux transformation [2–4] and the Bäcklund transformation [5, 6], the Hirota method [7, 8], and algebrogeometrical approach [9–11]. Among the abovementioned methods, the Hirota method is a powerful approach to construct exact solution of nonlinear equations. By applying the Hirota method, people obtained a series of multisoliton solutions and rational solutions of many nonlinear equations in a systematic way. Unfortunately, this method relies on particular skills, appropriate exchange formulas, and complex calculations. On the other hand, in recent years, Lambert, Gilson et al. proposed an alternative procedure based on the use of the Bell polynomials to obtain parameter families of bilinear Bäcklund transformation and Lax pairs for soliton equations in quick and short way [12–14]. Fan developed this method to find infinite conservation laws of soliton equations [15–17] and proposed the super Bell polynomials [18, 19]. Ma systematically analyzed the connection between Bell polynomials and new bilinear equations [20].

From bilinear forms, Nakamura proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equation in his two serial papers [21, 22], where the quasi-periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by using the Riemann theta function. Recently, Hon et al. have extended this method to investigate the discrete Toda lattice [23], (2 + 1)-dimensional Bogoyavlenskiis breaking soliton equation [24], and the asymmetrical Nizhnik-Novikov-Veselov equation [25]. Ma et al. constructed one-periodic and two-periodic wave solutions to a class of (2 + 1)-dimensional Hirota bilinear equations [26]. Zhang et al. applied this method to get periodic wave solutions of the variable-coefficient mKdV equation [27].

Due to the inhomogeneities of media and nonuniformities of boundaries in various real physical situations, the variable-coefficient NLEEs are considered to be more realistic than constant-coefficient equations in describing a large variety of real phenomena; for example, many physical and mechanical situations are governed by variable-coefficient KdV equation, for example, the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement, the nonlinear waves in types of rods [28–30]. Obviously, equations with variable-coefficient are much more complicated than constant-coefficient forms, and much attention has been paid to this subject [31–35]. In this paper, we will focus our study on the generalized variable-coefficient fifth-order Korteweg-de Vries equation such as the one given below:

$$u_t + a(t) u u_{xxx} + b(t) u_x u_{xx} + c(t) u^2 u_x + d(t) u u_x + e(t) u_{xxx} + l(t) u_{xxxxx} + m(t) u + n(t) u_x = 0, \quad (1)$$

where u is a function of x and t and $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $l(t)$, $m(t)$, and $n(t)$ are analytic functions of t. Since there are choices for the parameters, the variable-coefficient NLEEs can be considered as generalizations of the constant coefficient ones. Under certain constraint conditions, the variable-coefficient models may be proved to be integrable and given explicit analytic solutions [36]. The corresponding constraint conditions on (1) in this paper, which are obtained by the Painlev analysis [37] and conditions from the variable-coefficient models mapped to the completely integrable constant-coefficient counterparts [38], will be

$$a(t) = b(t) = \frac{15 l(t)}{\rho} e^{\int m(t) dt}, \qquad c(t) = \frac{45 l(t)}{\rho^2} e^{\int 2m(t) dt}, \qquad d(t) = e(t) = 0, \quad (2)$$

where $\rho \neq 0$ is an arbitrary constant. The main goal of this paper is twofold. First, we apply the binary Bell polynomials to construct bilinear formalism, bilinear Bäcklund transformation, Lax pairs, and infinite conservation laws of (1) under condition (2). Second, we obtain the periodic wave solutions of the equation by using the Riemann theta function and discussing their asymptotic properties.

The organization of this paper is as follows. In Section 2, we briefly present necessary notations on binary Bell polynomial that will be used in this paper. In Section 3, we get bilinear formalism, bilinear Bäcklund transformation, Lax pairs, and infinite conservation laws of the generalized variable-coefficient fifth-order Korteweg-de Vries equation by utilizing the binary Bell polynomials. In Section 4, we apply Hirota's bilinear method to construct one- and two-periodic wave solutions (1), respectively. Further we use a limiting procedure to analyze asymptotic behavior of the periodic wave solutions in detail. Finally, some conclusions are given in Section 5.

2. Binary Bell Polynomials

To begin with, we will give some basic concepts and notations about the Bell polynomials. For details, please refer to [11–13].

Let $f = f(x_1, x_2, \ldots, x_l)$ be a C^∞ function with multivariables; the following polynomials

$$Y_{n_1 x_1, \ldots, n_l x_l}(f) = Y_{n_1, \ldots, n_l}\left(f_{r_1 x_1, \ldots, r_l x_l}\right) = \exp(-f)\, \partial_{x_1}^{n_1} \cdots \partial_{x_l}^{n_l} \exp(f) \quad (3)$$

are called the multidimensional Bell polynomials, where

$$f_{r_1 x_1, \ldots, r_l x_l} = \partial_{x_1}^{r_1} \cdots \partial_{x_l}^{r_l} f, \quad r_1 = 0, \ldots, n_1, \quad r_l = 0, \ldots, n_l. \quad (4)$$

For convenience, we denote the multidimensional Bell polynomials by Y-polynomials.

For example, for the simplest case $f = f(x)$, the one-dimensional Bell polynomials are

$$Y_1 = f_x, \qquad Y_2 = f_{2x} + f_x^2, \qquad Y_3 = f_{3x} + 3 f_x f_{3x} + f_x^3, \ldots. \quad (5)$$

For $f = f(x, t)$, the two-dimensional Bell polynomials are

$$\begin{aligned} Y_x(f) &= f(x), \\ Y_{2x}(f) &= f_{2x} + f_x^2, \\ Y_{2x}(f) &= f_{3x} + 3 f_x f_{2x} + f_x^3, \\ Y_{x,t}(f) &= f_{x,t} + f_x f_t, \\ Y_{2x,t}(f) &= f_{2x,t} + f_{2x} f_t + 2 f_{x,t} f_x + f_x^2 f_t, \ldots. \end{aligned} \quad (6)$$

Based on the above Bell polynomials, the multidimensional binary Bell polynomials ($\mathscr{Y}$-polynomials) can be defined as follows:

$$\mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}(v, w) = Y_{n_1 x_1, \ldots, n_l x_l}(f) \big|_{f_{r_1 x_1, \ldots, r_l x_l}} = \begin{cases} v_{r_1 x_1, \ldots, r_l x_l}, & r_1 + \cdots + r_l \text{ is odd}, \\ w_{r_1 x_1, \ldots, r_l x_l}, & r_1 + \cdots + r_l \text{ is even}. \end{cases} \quad (7)$$

The $\mathscr{Y}$-polynomials inherit the easily recognizable partial structure of the Bell polynomials. The first few lowest order binary Bell polynomials are

$$\begin{aligned} y_x(v) &= v_x, \qquad y_{2x}(v, w) = w_{2x} + v_x^2, \\ y_{x,t}(v, w) &= w_{x,t} + v_x v_t, \\ y_{3x}(v, w) &= v_{3x} + 3 v_x w_{2x} + v_x^3, \\ y_{4x}(v, w) &= w_{4x} + 3 w_{2x}^2 + 4 v_x v_{3x} + 6 v_x^2 w_{2x} + v_x^4, \\ y_{5x}(v, w) &= v_{5x} + 5 v_x w_{4x} + 10 v_{3x} w_{2x} + 10 v_x^2 v_{3x} + 15 v_x w_{2x}^2 + 10 v_x^3 w_{2x} + v_x^5. \end{aligned} \quad (8)$$

Theorem 1 (see [11]). *The link between binary Bell polynomials $\mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}(v, w)$ and the standard Hirota bilinear equation $D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G$ can be given by an identity*

$$\mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}\left(v = \ln \frac{F}{G},\ w = \ln FG\right) = (FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G, \quad (9)$$

in which $n_1 + n_2 + \cdots + n_l \geq 1$ and operators $D_{x_1} \cdots D_{x_l}$ are classical Hirotas bilinear operators defined by

$$\begin{aligned} &D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G \\ &\quad = \left(\partial_{x_1} - \partial_{x_1'}\right)^{n_1} \cdots \left(\partial_{x_l} - \partial_{x_l'}\right)^{n_l} \\ &\qquad \times F\left(x_1, \ldots, x_l\right) \times G(x_1', \ldots, x_l')\big|_{x_1'=x_1,\ldots,x_l'=x_l}. \end{aligned} \tag{10}$$

In the particular case, when $F = G$, formula (9) becomes

$$\begin{aligned} G^{-2} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} G \cdot G &= \mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}\left(0, q = 2 \ln G\right) \\ &= \begin{cases} 0, & n_1 + \cdots + n_l \text{ is odd}, \\ P_{n_1 x_1, \ldots, n_l x_l}(q), & n_1 + \cdots + n_l \text{ is even}, \end{cases} \end{aligned} \tag{11}$$

in which the P-polynomials can be characterized by an equally recognizable even part partitional structure

$$\begin{aligned} &P_{2x}(q) = q_{2x}, \qquad P_{x,t}(q) = q_{xt}, \\ &P_{4x}(q) = q_{4x} + 3q_{2x}^2, \\ &P_{6x}(q) = q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3, \ldots. \end{aligned} \tag{12}$$

This formulae will be used to obtain the bilinear Bäcklund transformations of the NLEEs. It means that once an NLEE is written in a combination form of the $\mathscr{Y}$-polynomials, then it can be easily transformed into the corresponding bilinear Bäcklund transformation form.

Theorem 2 (see [11]). *The binary Bell polynomials $\mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}(v, w)$ can be separated into P-polynomials and Y-polynomials:*

$$\begin{aligned} &(FG)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} F \cdot G \\ &\quad = \mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}(v, w)\big|_{v=\ln F/G,\, w=\ln FG} \\ &\quad = \mathscr{Y}_{n_1 x_1, \ldots, n_l x_l}(v, v+q)\big|_{v=\ln F/G,\, q=2\ln G} \\ &\quad = \sum_{n_1+\cdots+n_l=\text{even}} \sum_{r_1=0}^{n_1} \cdots \sum_{r_l=0}^{n_l} \prod_{i=1}^{l} \binom{n_i}{r_i} P_{r_1 x_1, \ldots, r_l x_l}(q) \\ &\qquad \times Y_{(n_1-r_1)x_1, \ldots, (n_l-r_l)x_l}(v). \end{aligned} \tag{13}$$

Under the Hopf-Cole transformation $v = \ln \psi$, that is, $\psi = F/G$, multidimensional binary Bell polynomials $Y_{n_1 x_1, \ldots, n_l x_l}(v)$ can be linearized into the following form:

$$Y_{n_1 x_1, \ldots, n_l x_l}\big|_{v=\ln \psi} = \frac{\psi_{n_1 x_1, \ldots, n_l x_l}}{\psi}. \tag{14}$$

Formulae (13) *and* (14) *provide the shortest way to the associated Lax system of nonlinear equations.*

3. Bilinear Representation, Bäcklund Transformation, and Conservation Laws of (1)

In this section, we will systematically investigate bilinear representation, Bäcklund transformation, Lax pair, and infinite conservation laws of (1) based on the Bell polynomials.

3.1. Bilinear Representation. In order to detect the existence of the bilinear representation, we introduce a potential field q by setting

$$u = s q_{2x}, \tag{15}$$

with $s = s(t)$ being a free function with respect to t, which will be chosen appropriately so that (1) is related to the P-polynomials. Then substituting (15) into (1) and integrating with respect to x and noting condition (2) yield

$$\begin{aligned} E(q) = &\left(\frac{s'(t)}{s(t)} + m(t)\right) q_x + q_{xt} + n(t) q_{2x} \\ &+ a(t) s(t) q_{2x} q_{4x} + c(t) s^2(t) \frac{q_{2x}^3}{3} + l(t) q_{6x} = 0. \end{aligned} \tag{16}$$

Comparing the fourth and the sixth terms of the above equation with formula (12) implies that we should require $s(t) = \rho e^{-\int m(t)dt}$. The resulting equation is then cast into a combination form of the P-polynomials:

$$\begin{aligned} E(q) &= q_{xt} + n(t) q_{2x} + l(t)\left(q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3\right) \\ &= P_{xt}(q) + n(t) P_{2x}(q) + l(t) P_{6x}(q) = 0. \end{aligned} \tag{17}$$

Making a change of the dependent variable

$$q = 2 \ln F \Longleftrightarrow u = s q_{2x} = 2\rho e^{-\int m(t)dt} (\ln F)_{xx} \tag{18}$$

and noting property (11), we can obtain the bilinear representation of (1) as

$$\left(D_x D_t + l(t) D_x^6 + n(t) D_x^2\right) F \cdot F = 0. \tag{19}$$

Following the Hirota bilinear theory, one-soliton solution for (1) in explicit forms can be given as

$$u_1 = 2\rho e^{-\int m(t)dt} \frac{\partial^2}{\partial x^2}\left[\log\left(1 + e^{\xi_1}\right)\right], \tag{20}$$

and two-soliton solutions are denoted by

$$u_2 = 2\rho e^{-\int m(t)dt} \frac{\partial^2}{\partial x^2}\left[\log\left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}\right)\right], \tag{21}$$

with

$$\xi_j = k_j x - k_j^5 \int l(t)\, dt - k_j \int n(t)\, dt + \xi_j^0, \tag{22}$$

$$e^{A_{12}} = \frac{(k_1 - k_2)^2 \left(k_1^2 - k_1 k_2 + k_2^2\right)}{(k_1 + k_2)^2 \left(k_1^2 + k_1 k_2 + k_2^2\right)}, \tag{23}$$

where k_j and ξ_j^0, $j = 1, 2$, are arbitrary real constants.

3.2. Bäcklund Transformation and Lax Pair. In the following, we search for the bilinear Bäcklund transformation and the Lax pair of (1). Let $q' = 2\ln F$ and $q = 2\ln G$ be two different solutions of (17), respectively; we have the two-field condition

$$E(q') - E(q) = (q'-q)_{xt} + n(t)(q'-q)_{2x} + l(t) \times \left[(q'-q)_{6x} + 15(q'_{2x}q'_{4x} - q_{2x}q_{4x}) + 15(q'^3_{2x} - q^3_{2x})\right] = 0. \tag{24}$$

If set

$$v = \frac{(q'-q)}{2} = \ln\left(\frac{F}{G}\right), \quad w = \frac{(q'+q)}{2} = \ln(FG), \tag{25}$$

then (24) can be rewritten as

$$\begin{aligned}\frac{(E(q') - E(q))}{2} &= v_{xt} + n(t)v_{2x} + l(t)(v_{6x} + 15w_{4x}v_{2x} + 15w_{2x}v_{4x} + 45v_{2x}w^2_{2x} + 15v^3_{2x}) \\ &= \partial_x\left[\mathcal{Y}_t(v) + n(t)\mathcal{Y}_x(v) + l(t)\mathcal{Y}_{5x}(v,w)\right] + R(v,w) = 0,\end{aligned} \tag{26}$$

with

$$\begin{aligned}R(v,w) = l(t)\big(&10v_{2x}w_{4x} + 5v_{4x}w_{2x} + 30v_{2x}w^2_{2x} + 15v^3_{2x} - 5v_xw_{5x} - 10v_{3x}w_{3x} - 30v_xw_{2x}w_{3x} \\ &- 30v^2_xv_{xx}w_{2x} - 10v^3_xw_{3x} - 5v^4_xv_{2x} - 20v_xv_{2x}v_{3x} - 10v^2_xv_{4x}\big).\end{aligned} \tag{27}$$

Taking

$$\mathcal{Y}_{3x}(v,w) = \lambda, \tag{28}$$

where λ is an arbitrary parameter. Then from (28), $R(v,w)$ can be rewritten in the form

$$R(v,w) = -\frac{5}{2}l(t)\partial_x\left[\mathcal{Y}_{5x}(v,w) + 3\lambda\mathcal{Y}_{3x}(v,w)\right]. \tag{29}$$

Then from (26)–(29), we deduce a coupled system of $\mathcal{Y}$-polynomials:

$$\begin{aligned}&\mathcal{Y}_{3x}(v,w) = \lambda, \\ &\partial_x\mathcal{Y}_t(v) + \partial_x\left[n(t)\mathcal{Y}_x(v) - \frac{3}{2}l(t)\mathcal{Y}_{5x}(v,w) - \frac{15}{2}l(t)\lambda\mathcal{Y}_{2x}(v,w)\right] = 0.\end{aligned} \tag{30}$$

By application of the identity (13), the system (30) immediately leads to the following bilinear Bäcklund transformation:

$$\begin{aligned}&(D^3_x - \lambda)F\cdot G = 0, \\ &\left[D_t + n(t)D_x - \frac{3}{2}l(t)D^5_x - \frac{15}{2}\lambda l(t)D^2_x + \beta\right] \times F\cdot G = 0,\end{aligned} \tag{31}$$

where β is an arbitrary parameter.

By using the Hopf-Cole transformation $v = \ln\psi$, it follows from formulas (13) and (14) that

$$\begin{aligned}&\mathcal{Y}_x(v) = \frac{\psi_x}{\psi}, \quad \mathcal{Y}_{2x}(v,w) = q_{2x} + \frac{\psi_{2x}}{\psi}, \\ &\mathcal{Y}_t(v) = \frac{\psi_t}{\psi}, \quad \mathcal{Y}_{3x}(v,w) = \frac{3q_{2x}\psi_x}{\psi} + \frac{\psi_{3x}}{\psi}, \\ &\mathcal{Y}_{5x}(v,w) = \frac{5q_{4x}\psi_x}{\psi} + \frac{15q^2_{2x}\psi_x}{\psi} + \frac{10q_{2x}\psi_{3x}}{\psi} + \frac{\psi_{5x}}{\psi};\end{aligned} \tag{32}$$

therefore, system (30) is linearized into the corresponding Lax representation

$$\begin{aligned}&L_1\psi \equiv (3q_{2x}\partial_x + \partial^3_x)\psi = \lambda\psi, \\ &(\partial_t + L_2)\psi \equiv \Big[\partial_t + \Big(n(t) - \frac{15}{2}l(t)q_{4x}\Big)\partial_x - \frac{15}{2}\lambda l(t)\partial^2_x - 15l(t)q_{2x}\partial^3_x - \frac{3}{2}l(t)\partial^5_x + \Big(-\frac{15}{2}\lambda l(t)q_{2x} + \beta\Big)\Big]\psi.\end{aligned} \tag{33}$$

It is easy to check that the integrability condition

$$[L_1 - \lambda, \partial_t + L_2]\psi = 0 \tag{34}$$

is satisfied if $q_{2x} = (e^{\int m(t)dt}/\rho)u$ and u is a solution of the generalized variable-coefficient fifth-order Korteweg-de Vries (1).

3.3. Infinite Conservation Laws. Next, through the Bell-polynomial-type Bäcklund transformation, we will perform the procedure of deriving the infinite sequence of conservation laws of (1) in the following form:

$$I_{n,t} + F_{n,x} = 0, \quad n = 1, 2, \ldots. \tag{35}$$

Let

$$\eta = \frac{q'_x - q_x}{2}; \tag{36}$$

it follows from relation (25) that

$$v_x = \eta, \quad w_x = q_x + \eta. \tag{37}$$

Rewrite (30) in the conserved form

$$\mathscr{Y}_{3x}(v,w)=\lambda,$$
$$\partial_t\mathscr{Y}_x(v)+\partial_x\left[n(t)\mathscr{Y}_x(v)-\frac{3}{2}l(t)\mathscr{Y}_{5x}(v,w)-\frac{15}{2}\lambda l(t)\mathscr{Y}_{2x}(v,w)\right]=0. \tag{38}$$

Substituting (37) into (38), we can obtain

$$\eta_{2x}+3\eta\eta_x+3q_{2x}\eta+\eta^3=\lambda=\varepsilon^3, \tag{39}$$

$$\begin{aligned}\eta_t+\partial_x\Big[n(t)\eta-\frac{3}{2}l(t)&\big(15\lambda q_{2x}+15\lambda\eta_x+\eta_{4x}+5q_{4x}\eta\\&-15q_{2x}^2\eta+5\eta\eta_{3x}+5\lambda\eta^2-30q_{2x}^2\eta\eta_x\\&+10\eta^2\eta_{2x}-15\eta\eta_x^2+\eta^5\big)\Big]=0,\end{aligned} \tag{40}$$

where we have used (39) to get (40).

To proceed, inserting the expansion

$$\eta=\varepsilon+\sum_{n=1}^{\infty}I_n(q,q_x,\ldots)\varepsilon^{-n} \tag{41}$$

into (39) and equating the coefficients for power of ε, we then obtain the recursion relations for the conserved densities I_n:

$$\begin{aligned}I_1&=-q_{2x}=-\frac{e^{\int m(t)dt}}{\rho}u,\\I_2&=q_{3x}=\frac{e^{\int m(t)dt}}{\rho}u_x,\\I_3&=-\frac{2}{3}q_{4x}=-\frac{2}{3}\frac{e^{\int m(t)dt}}{\rho}u_{2x},\\I_4&=\frac{1}{3}q_{5x}=\frac{1}{3}\frac{e^{\int m(t)dt}}{\rho}u_{3x},\end{aligned} \tag{42}$$

and the recursion relation is given as

$$\begin{aligned}I_n=&-\frac{1}{3}I_{n-2,xx}-I_{n-1,x}-uI_{n-2}-\sum_{i=1}^{n-3}\left(I_iI_{n-2-i,x}\right)\\&-\sum_{i=1}^{n-2}\left(I_iI_{n-1-i}\right)-\frac{1}{3}\sum_{i+j+k=n-2}\left(I_iI_jI_k\right),\\&\qquad n=5,6,7,\ldots.\end{aligned} \tag{43}$$

In addition, substituting (41) into (40) yields

$$\begin{aligned}F_1=n(t)I_1-\frac{3}{2}l(t)&\times\Big[15I_{4,x}+I_{1,4x}+5\frac{e^{\int m(t)dt}}{\rho}u_{2x}I_1\\&-15\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_1+5I_{2,3x}\\&+5\left(2I_1I_3+I_2^2+2I_5\right)\\&-30\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_{2,x}+10\left(2I_1I_{1,2x}+I_{3,2x}\right)\\&-15I_{1,x}^2+10I_1^3+15I_2^2+5I_5\Big],\end{aligned}$$

$$\begin{aligned}F_2=n(t)I_2-\frac{3}{2}l(t)&\times\Big[15I_{5,x}+I_{2,4x}+5\frac{e^{\int m(t)dt}}{\rho}u_{2x}I_2\\&-15\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_2+5\left(I_1I_{1,3x}+I_{3,3x}\right)\\&+10\left(I_1I_4+I_2I_3+I_6\right)-30\frac{e^{2\int m(t)dt}}{\rho^2}\\&\times u^2\left(I_{3,x}+I_1I_{1,x}\right)+10\left(2I_1I_{2,2x}+I_{4,2x}\right)\\&-30I_{1,x}I_{2,x}+30I_1^2I_2+20I_2I_3\\&+20I_1I_4+5I_6\Big],\end{aligned}$$

$$\begin{aligned}F_3=n(t)I_3-\frac{3}{2}l(t)&\times\Big[15I_{6,x}+I_{3,4x}+5\frac{e^{\int m(t)dt}}{\rho}u_{2x}I_3\\&-15\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_3\\&+5\left(I_1I_{2,3x}+I_2I_{1,3x}+I_{4,3x}\right)\\&+5\left(2I_1I_5+2I_2I_4+I_3^2+2I_7\right)\\&-30\frac{e^{2\int m(t)dt}}{\rho^2}u^2\left(I_{4,x}+I_1I_{2,x}+I_2I_{1,x}\right)\\&+10\left(2I_1I_{3,2x}+2I_3I_{1,2x}+I_1^2I_{1,2x}+I_{5,2x}\right)\\&-15\left(I_{2,x}^2+I_1I_{2,x}^2\right)+5I_1^4+30I_1^2I_3\end{aligned}$$

$$+30I_1I_2^2+10I_3^2+20I_2I_4+20I_1I_5+5I_7\Bigg],$$

$$\begin{aligned}F_4 &= n(t)I_4-\frac{3}{2}l(t)\\ &\times\Bigg[15I_{7,x}+I_{4,4x}+5\frac{e^{\int m(t)dt}}{\rho}u_{2x}I_4\\ &-15\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_4\\ &+5\left(I_1I_{3,3x}+I_2I_{2,3x}+I_3I_{1,3x}+I_{5,3x}\right)\\ &+10\left(2I_1I_6+I_2I_5+I_3I_4+I_8\right)\\ &-30\frac{e^{2\int m(t)dt}}{\rho^2}u^2\\ &\times\left(I_{5,x}+I_1I_{3,x}+I_3I_{1,x}+I_2I_{2,x}\right)\\ &+10\left(2I_1I_{4,2x}+2I_{1,2x}\left(I_1I_2+I_4\right)\right.\\ &\quad\left.+I_{2,2x}\left(I_1^2+2I_3\right)+I_{6,2x}\right)\\ &-15\left(2I_{2,x}I_{3,x}+2I_{1,x}I_{4,x}^2+2I_1I_{1,x}I_{2,x}\right.\\ &\quad\left.+I_2I_{1,x}^2\right)+20I_1^3I_2+60I_1I_2I_3\\ &+10I_2^3+30I_1^2I_4+20I_2I_5+20I_1I_6\\ &+20I_3I_4+5I_8\Bigg],\end{aligned}$$

$$\begin{aligned}F_n &= n(t)I_n-\frac{3}{2}l(t)\\ &\times\Bigg[15I_{n+3,x}+I_{n,4x}+5\frac{e^{\int m(t)dt}}{\rho}u_{2x}I_n\\ &-15\frac{e^{2\int m(t)dt}}{\rho^2}u^2I_n+5\left(\sum_{k=1}^{n-1}I_kI_{n-k,3x}+I_{n+1,3x}\right)\\ &+5\left(\sum_{k=1}^{n+2}I_kI_{n+3-k}+2I_{n+4}\right)-30\frac{e^{2\int m(t)dt}}{\rho^2}u^2\\ &\times\left(\sum_{k=1}^{n-1}I_kI_{n-k,x}+I_{n+1,x}\right)\\ &+10\times\left(\sum_{i+j+k=n}I_iI_jI_{k,x}+2\sum_{k=1}^{n}I_kI_{n+1-k,x}+I_{n+2,x}\right)\\ &-15\left(\sum_{i+j+k=n}I_iI_{j,x}I_{k,x}+\sum_{k=1}^{n}I_{k,x}I_{n+1-k,x}\right)\\ &+\sum_{i+j+k+l+m=n}I_iI_jI_kI_lI_m+\sum_{i+j+k+l=n+1}I_iI_jI_kI_l\\ &+\sum_{i+j+k=n+2}I_iI_jI_k+\sum_{i+j=n+3}I_iI_j+I_{n+4}\Bigg],\\ &\qquad n=4,5,6,\ldots.\end{aligned}\tag{44}$$

With the recursion formulae of I_n and F_n presented previously, the infinite conservation laws for (1) can be constructed.

4. Quasi-Periodic Wave Solutions and Asymptotic Properties

The quasi-periodic wave solutions of (1) are based on the following multidimensional Riemann theta function of genus N:

$$\vartheta(\xi)=\vartheta(\xi,\tau)=\sum_{n\in Z^N}e^{-\pi\langle\tau n,n\rangle+2\pi i\langle\xi,n\rangle}.\tag{45}$$

Here the integer value vector $n=(n_1,\ldots,n_N)^T\in Z^N$, and complex phase variables $\xi=(\xi_1,\ldots,\xi_N)^T\in C^N$. Moreover, for two vectors $f=(f_1,\ldots,f_N)^T$ and $g=(g_1,\ldots,g_N)^T$, their inner product is defined by

$$\langle f,g\rangle=f_1g_1+f_2g_2+\cdots+f_Ng_N.\tag{46}$$

The $\tau=(\tau_{ij})$ is a positive definite and real-valued symmetric $N\times N$ matrix, which we call the period matrix of the theta function. The entries τ_{ij} of the period matrix τ can be considered as free parameters of the theta function (45).

Now, we consider the solution for (1) in the following bilinear form:

$$G\left(D_x,D_t\right)=\left(D_xD_t+l(t)D_x^6+n(t)D_x^2+c\right)f\cdot f=0,\tag{47}$$

where c is the constant of integration.

4.1. Construction of One-Periodic Waves. In this section, we consider the one-periodic wave solutions for (1). When $N=1$, the theta function reduces the following Fourier series in n:

$$\vartheta(\xi,\tau)=\sum_{n=-\infty}^{\infty}e^{2\pi in\xi-\pi n^2\tau},\tag{48}$$

where the phase variable $\xi=kx+\int\omega dt+\xi^{(0)}$ and the parameter $\tau>0$.

Substituting (48) into (47), we obtain

$$\begin{aligned}
&G\left(D_x, D_t\right)\vartheta(\xi,\tau)\cdot\vartheta(\xi,\tau)\\
&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G\left(D_x, D_t\right)e^{2\pi in\xi-\pi n^2\tau}e^{2\pi im\xi-\pi m^2\tau}\\
&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G\left[2\pi i(n-m)k, 2\pi i(n-m)\omega\right]\\
&\qquad\times e^{2\pi(n+m)\xi-\pi(n^2+m^2)\tau}\\
&\overset{m=m'-n}{=}\sum_{m'=-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}G\left[2\pi i\left(2n-m'\right)k,\right.\right.\\
&\qquad\left.2\pi i\left(2n-m'\right)\omega\right]\\
&\qquad\left.\times e^{-\pi[n^2+(n-m')^2]\tau}\right\}e^{2\pi im'\xi}\\
&=\sum_{m'=-\infty}^{\infty}\overline{G}\left(m'\right)e^{2\pi im'\xi},
\end{aligned}\tag{49}$$

where

$$\overline{G}\left(m'\right)=\sum_{n=-\infty}^{\infty}G\left[2\pi i\left(2n-m'\right)k, 2\pi i\left(2n-m'\right)\omega\right]\times e^{-\pi[n^2+(n-m')^2]\tau}.\tag{50}$$

By shifting sum index as $n=n'+1$, we conclude that

$$\begin{aligned}
\overline{G}\left(m'\right)&=\left(\sum_{n'=-\infty}^{\infty}G\left\{2\pi i\left[2n'-\left(m'-2\right)\right]k,\right.\right.\\
&\qquad\left.2\pi i\left[2n'-\left(m'-2\right)\right]\omega\right\}\\
&\qquad\left.\times e^{-\pi\{n'^2+[n'-(m'-2)]^2\}\tau}\right)e^{[-2\pi(m'-1)\tau]}\\
&=\overline{G}\left(m'-2\right)e^{-2\pi(m'-1)\tau}=\cdots\\
&=\begin{cases}\overline{G}(0)\,e^{-\pi m'^2(\tau/2)}, & m' \text{ is even},\\ \overline{G}(1)\,e^{-\pi(m'^2-1)(\tau/2)}, & m' \text{ is odd},\end{cases}
\end{aligned}\tag{51}$$

which imply that if $\overline{G}(0)=\overline{G}(1)=0$, then it follows that

$$\overline{G}\left(m'\right)=0,\quad m'\in Z,\tag{52}$$

and thus the theta function (48) is the exact solution of (47). In this way, we may let

$$\begin{aligned}
\overline{G}(0)&=\sum_{n=-\infty}^{\infty}\left[-16\pi^2n^2k\omega-4096\pi^6n^6k^6l(t)\right.\\
&\qquad\left.-16\pi^2n^2k^2n(t)+c\right]e^{-2\pi n^2\tau}=0,\\
\overline{G}(1)&=\sum_{n=-\infty}^{\infty}\left[-4\pi^2(2n-1)^2k\omega-64\pi^6(2n-1)^6k^6l(t)\right.\\
&\qquad\left.-4\pi^2(2n-1)^2k^2n(t)+c\right]e^{-\pi(2n^2-2n+1)\tau}=0.
\end{aligned}\tag{53}$$

Denote

$$\begin{aligned}
&\lambda=e^{-\pi\tau},\qquad a_{11}=-\sum_{n=-\infty}^{\infty}16\pi^2n^2k\lambda^{2n^2},\\
&a_{12}=\sum_{n=-\infty}^{\infty}\lambda^{2n^2},\qquad a_{22}=\sum_{n=-\infty}^{\infty}\lambda^{2n^2-2n+1},\\
&a_{21}=-\sum_{n=-\infty}^{\infty}4\pi^2(2n-1)^2k\lambda^{2n^2-2n+1},\\
&b_1=\sum_{n=-\infty}^{\infty}\left(4096\pi^6n^6k^6l(t)+16\pi^2n^2k^2n(t)\right)\lambda^{2n^2},\\
&b_2=\sum_{n=-\infty}^{\infty}\left(64\pi^6(2n-1)^6k^6l(t)+4\pi^2(2n-1)^2\right.\\
&\qquad\left.\times k^2n(t)\right)\lambda^{2n^2-2n+1}.
\end{aligned}\tag{54}$$

Then (53) can be written as

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\omega\\ c\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}.\tag{55}$$

Notice that there are a lot of choices for the angular wave number k. The determinant of the coefficient matrix $A(k)=(a_{ij}(k))_{2\times 2}$ is a polynomial in k; if $\det(A(k))\not\equiv 0$, then

$$A_0:=\{k\in R\mid\det(A(k))=0\}\tag{56}$$

is either an empty set or a finite set, and so, there are real solutions (ω,c) to the system (55) for $k\notin A_0$. Solving this system, we have

$$\omega=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}.\tag{57}$$

Therefore we get a one-periodic wave solution of (1):

$$u=2\rho e^{-\int m(t)dt}\left(\log\vartheta(\xi,\tau)\right)_{xx},\tag{58}$$

where the parameter ω is given by (57).

4.2. Asymptotic Property of One-Periodic Waves. In the following, we further consider asymptotic properties of the one-periodic wave solution. It is shown that the soliton solution of (1) can be obtained as a limit of the one-periodic wave solution. The relation between these two solutions can be established as Theorem 3.

Theorem 3. *Suppose that the vector* $(\omega, c)^T$ *is a solution of the system* (55)*, and for the quasi-periodic wave solution* (58)*, we let*

$$\xi = \frac{\xi'}{2\pi i} + \frac{\tau}{2i}, \qquad \xi' = k'x + \int \omega' dt + \xi^{(0)'}, \tag{59}$$

where $k' = 2\pi i k, \omega' = 2\pi i \omega, \xi^{(0)'}$ *and* $= 2\pi i \xi^{(0)} - \pi\tau$*. Then the one-periodic solution* (58) *tends to the one-soliton solution* (20) *under a small amplitude limit; that is,*

$$u \longrightarrow u_1, \qquad \text{as } \lambda \longrightarrow 0. \tag{60}$$

Proof. By using (54), we write functions a_{ij}, b_j, $i, j = 1, 2$, as the series about λ:

$$\begin{aligned}
a_{11} &= -32\pi^2 k\left(\lambda^2 + 4\lambda^8 + 9\lambda^{18} + \cdots\right),\\
a_{12} &= 1 + 2\lambda^2 + 2\lambda^8 + 2\lambda^{18} + 2\lambda^{32} + \cdots,\\
a_{21} &= -8\pi^2 k\left(\lambda + 9\lambda^5 + 25\lambda^{13} + \cdots\right),\\
a_{22} &= 2\lambda + 2\lambda^5 + 2\lambda^{13} + 2\lambda^{25} + \cdots,\\
b_1 &= \left(8192\pi^6 k^6 l(t) + 32\pi^2 k^2 n(t)\right)\lambda^2 + \cdots,\\
b_2 &= \left(128\pi^6 k^6 l(t) + 8\pi^2 k^2 n(t)\right)\lambda + \cdots.
\end{aligned} \tag{61}$$

Suppose that the solution of system (55) has the following form:

$$\omega = \omega_0 + \omega_1\lambda^1 + \omega_2\lambda^2 + \cdots = \omega_0 + o(\lambda), \tag{62}$$

substituting expansions (61) and (62) into system (55), and let $\lambda \to 0$; we can obtain the following relation immediately:

$$\omega_0 = -16\pi^4 k^5 l(t) - kn(t); \tag{63}$$

combining (59) and (63), we then obtain

$$\begin{aligned}
\omega' &= 2\pi i\omega \longrightarrow -32\pi^5 l(t)\, ik^5 - 2\pi n(t)\, ik,\\
&= -l(t)\, k'^5 - n(t)\, k'.
\end{aligned} \tag{64}$$

□

It remains to show that the one-periodic wave (58) degenerates to the one-soliton solution (20) under the limit $\lambda \to 0$. We first expand the periodic function $\vartheta(\xi)$ in the form

$$\vartheta(\xi, \tau) = 1 + \lambda\left(e^{2\pi i\xi} + e^{-2\pi i\xi}\right) + \lambda^4\left(e^{4\pi i\xi} + e^{-4\pi i\xi}\right) + \cdots. \tag{65}$$

By using transformation (59), it follows that

$$\begin{aligned}
\vartheta(\xi, \tau) &= 1 + e^{\xi'} + \lambda^2\left(e^{-\xi'} + e^{2\xi'}\right) + \lambda^6\left(e^{-2\xi'} + e^{3\xi'}\right) + \cdots,\\
&\longrightarrow 1 + e^{\xi'}, \quad \text{as } \lambda \longrightarrow 0.
\end{aligned} \tag{66}$$

Thus, we conclude that the one-periodic solution (58) may go to one-soliton solution (20) as the amplitude $\lambda \to 0$.

4.3. Construction of Two-Periodic Waves. In the case when $N = 2$, the Riemann theta function (45) takes the form

$$\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \tau) = \sum_{n \in Z^2} e^{-\pi\langle\tau n, n\rangle + 2\pi i\langle\xi, n\rangle}, \tag{67}$$

where $n = (n_1, n_2)^T \in Z^2$, $\xi = (\xi_1, \xi_2)^T \in C^2$, and $\xi_j = k_j x + \int \omega_j dt + \xi_j^{(0)}$, $j = 1, 2$. τ is a positive definite and real-valued symmetric 2×2 matrix which can take the form

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}, \qquad \tau_{11} > 0, \tau_{22} > 0, \tau_{11}\tau_{22} - \tau_{12}^2 > 0. \tag{68}$$

Substituting (67) into (47), we have

$$\begin{aligned}
&G(D_x, D_t)\,\vartheta(\xi_1, \xi_2, \tau) \cdot \vartheta(\xi_1, \xi_2, \tau)\\
&\quad = \sum_{m,n \in Z^2} G(2\pi i\langle n-m, k\rangle, 2\pi i\langle n-m, \omega\rangle)\\
&\qquad \times e^{2\pi i\langle\xi, n+m\rangle - \pi(\langle\tau m, m\rangle + \langle\tau n, n\rangle)}\\
&\quad \overset{m=m'-n}{=} \sum_{m' \in Z^2}\sum_{n \in Z^2} G\left(2\pi i\left\langle 2n-m', k\right\rangle, 2\pi i\left\langle 2n-m', \omega\right\rangle\right)\\
&\qquad \times e^{-\pi[\langle\tau(n-m'), n-m'\rangle] + \langle\tau n, n\rangle]} \cdot e^{2\pi i\langle\xi, m'\rangle}\\
&\quad = \sum_{m' \in Z^2} \overline{G}\left(m_1', m_2'\right) e^{2\pi i\langle m', \xi\rangle},
\end{aligned} \tag{69}$$

where $\overline{G}(m'_1, m'_2)$ is the coefficient of $e^{2\pi i\langle m',\xi\rangle}$. For each fixed $l = 1, 2$, by shifting jth summation index as $n_j = n'_j + \delta_{j,l}$ with $\delta_{j,l}$ representing Kronecker's delta we obtain that

$$\begin{aligned}
&\overline{G}\left(m'_1, m'_2\right) \\
&= \sum_{n\in Z^2} G\left(2\pi i\left\langle 2n-m', k\right\rangle, 2\pi i\left\langle 2n-m', \omega\right\rangle\right) \\
&\qquad \times e^{-\pi[\langle \tau(n-m', n-m')\rangle+\langle \tau n, n\rangle]} \\
&= \sum_{n'\in Z^2} G\left(2\pi i\sum_{j=1}^{2}\left[2n'_j-\left(m'_j-2\delta_{jl}\right)\right]k_j,\right. \\
&\qquad\qquad \left. 2\pi i\sum_{j=1}^{2}\left[2n'_j-\left(m'_j-2\delta_{ij}\right)\right]\omega_j\right) \\
&\quad \times \exp\left\{-\pi\sum_{j,h=1}^{2}\left(n'_j+\delta_{jl}\right)\tau_{jh}\left(n'_h+\delta_{hl}\right)\right. \\
&\qquad -\pi\sum_{j,h=1}^{2}\left[\left(m'_j+2\delta_{jl}-n'_j\right)+\delta_{jl}\right] \\
&\qquad\qquad \left.\times\tau_{jh}\left[\left(m'_h-2\delta_{hl}-n'_h\right)+\delta_{hl}\right]\right\} \\
&= \begin{cases} \overline{G}\left(m'_1-2, m'_2\right)e^{-2\pi(\tau_{11}m'_1+\tau_{12}m'_2)+2\pi\tau_{11}}, & l=1, \\ \overline{G}\left(m'_1, m'_2-2\right)e^{-2\pi(\tau_{12}m'_1+\tau_{22}m'_2)+2\pi\tau_{22}}, & l=2, \end{cases}
\end{aligned} \tag{70}$$

which implies that if the following equations are satisfied

$$\overline{G}(0,0) = \overline{G}(0,1) = \overline{G}(1,0) = \overline{G}(1,1) = 0 \tag{71}$$

then we have $\overline{G}(m'_1, m'_2) = 0$, for all $m'_1, m'_2 \in Z$, and thus the function (67) is an exact solution of (47). By introducing the notations as

$$\begin{aligned}
&A = \left(a_{jl}\right)_{4\times 3}, \qquad b = \left(b_1, b_2, b_3, b_4\right)^T, \\
&a_{j1} = -4\pi^2\sum_{n_1,n_2\in Z^2}\left\langle 2n-s^j, k\right\rangle\left(2n_1-s_1^j\right)\varepsilon_j(n), \\
&a_{j2} = -4\pi^2\sum_{n_1,n_2\in Z^2}\left\langle 2n-s^j, k\right\rangle\left(2n_2-s_2^j\right)\varepsilon_j(n), \\
&a_{j3} = \sum_{n_1,n_2\in Z^2}\varepsilon_j(n), \\
&b_j = -l(t)(2\pi i)^6\left\langle 2n-s^j, k\right\rangle^6 \\
&\qquad -n(t)(2\pi i)^2\times\left\langle 2n-s^j, k\right\rangle^2, \\
&\varepsilon_j(n) = \lambda_1^{n_1^2+(n_1-s_1^j)^2}\lambda_2^{n_2^2+(n_2-s_2^j)^2}\lambda_1^{n_1n_2+(n_1-s_1^j)(n_2-s_2^j)}, \\
&\lambda_1 = e^{-\pi\tau_{11}}, \qquad \lambda_2 = e^{-\pi\tau_{22}}, \qquad \lambda_3 = e^{-2\pi\tau_{12}}, \\
&s^j = \left(s_1^j, s_2^j\right), \quad j = 1,2,3,4, \\
&s^1 = (0,0), \qquad s^2 = (1,0), \\
&s^3 = (0,1), \qquad s^4 = (1,1),
\end{aligned} \tag{72}$$

(71) can be written as a linear system:

$$A(\omega_1, \omega_2, c)^T = b. \tag{73}$$

Notice that both A and b depend on k, so the solution of (73) also depends on k. And if

$$\operatorname{rank}(A(k)) = \operatorname{rank}(A(k), b(k)) = 3, \tag{74}$$

then there is a unique nonzero solution of (ω_1, ω_2, c) to the system (73). Solving this system, we can get a two-periodic wave solution of (1):

$$u = 2\rho e^{-\int m(t)dt}(\log\vartheta(\xi_1, \xi_2, \tau))_{xx}, \tag{75}$$

with $\vartheta(\xi_1, \xi_2)$ and ω_1, ω_2, c given by (67) and (73), respectively, while $k = (k_1, k_2)$ is determined by (74).

4.4. Asymptotic Property of Two-Periodic Waves. In this subsection, we consider the asymptotic properties of the two-periodic solution (75). In a similar way to Theorem 3, we can establish the relation between the two-periodic solution (75) and the two-soliton solution (21) as follows.

Theorem 4. *Suppose that the vector $(\omega_1, \omega_2, c)^T$ is a solution of the system* (73)*, and for the quasi-periodic wave solution* (75)*, one lets*

$$\xi_j = \frac{\xi'_j}{2\pi i} + \frac{\tau_{jj}}{2i}, \qquad \xi'_j = k'_j x + \int \omega'_j dt + \xi_j^{(0)'}, \tag{76}$$

where $k'_j = 2\pi i k_j$, $\omega'_j = 2\pi i\omega_j$, $\xi_j^{(0)'} = 2\pi i\xi_j^{(0)} - \pi\tau_{jj}$, and $\tau_{12} = A_{12}/2\pi$, $j = 1, 2$, and A_{12} is given in (21)*. Then the two-periodic solution* (75) *tends to the two-soliton solution* (21) *under a small amplitude limit; that is,*

$$u \longrightarrow u_2, \quad \text{as } \lambda_1, \lambda_2 \longrightarrow 0. \tag{77}$$

Proof. According to formula (67), we expand the function $\vartheta(\xi_1, \xi_2)$ in the following form:

$$\begin{aligned}
\vartheta(\xi_1, \xi_2, \tau) = 1 &+ \left(e^{2\pi i\xi_1} + e^{-2\pi i\xi_1}\right)e^{-\pi\tau_{11}} \\
&+ \left(e^{2\pi i\xi_2} + e^{-2\pi i\xi_2}\right)e^{-\pi\tau_{22}}, \\
&+ \left(e^{2\pi i(\xi_1+\xi_2)} + e^{-2\pi i(\xi_1+\xi_2)}\right) \\
&\times e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots.
\end{aligned} \tag{78}$$

By using (76), we get

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau) &= 1+e^{\xi_1'}+e^{\xi_2'}+e^{\xi_1'+\xi_2'-2\pi\tau_{12}} \\ &\quad +\lambda_1^2 e^{-\xi_1'}+\lambda_2^2 e^{-\xi_2'}, \\ &\quad +\lambda_1^2\lambda_2^2 e^{-\xi_1'-\xi_2'-2\pi\tau_{12}}+\cdots \\ &\longrightarrow 1+e^{\xi_1'}+e^{\xi_2'}+e^{\xi_1'+\xi_2'+A_{12}}, \\ &\qquad \text{as } \lambda_1,\lambda_2\longrightarrow 0,\end{aligned} \tag{79}$$

where $\xi_j' = k_j'x+\int\omega_j'dt+\xi_j^{(0)'}$, $j=1,2$. Thus the two-periodic wave solutions can be reduced to two-soliton solutions (21) under the limit $\lambda_1,\lambda_2 \to 0$; we only need to prove that

$$\omega_j' \longrightarrow -l(t)\,k_j'^5 - n(t)\,k_j', \quad j=1,2. \tag{80}$$

□

And the proof of (80) is similar to the proof of formula (64).

5. Conclusions

In this paper, the generalized variable-coefficient fifth-order Korteweg-de Vries equation is investigated. By virtue of the Bell-polynomial approach, bilinear form of (1) has been derived under condition (2) and bilinear Bäcklund transformation, Lax pairs, and infinite conservation laws of the equation are constructed. Furthermore, the Riemann theta functions have been used to generate one-periodic and two-periodic wave solutions of the equation, and the relations between the periodic wave solutions and soliton solutions are also established. The quasi-periodic solutions play an important role in understanding the diversity and integrability of nonlinear differential equations. We think that there are still many deep relations between the quasi-periodic solutions and other kinds of solutions which still remain open and worth studying.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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