Research Article

Operator Ideal of Cesaro Type Sequence Spaces Involving Lacunary Sequence

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The aim of this paper is to give the sufficient conditions on the sequence space Ces(𝜃, 𝑝) defined in Lim (1977) such that the class of all bounded linear operators between any arbitrary Banach spaces with 𝑛th approximation numbers of the bounded linear operators in Ces(𝜃, 𝑝) form an operator ideal.

1. Introduction

Most of the operator ideals in the class of Banach spaces or in the class of normed spaces in linear functional analysis are defined by different scalar sequence spaces. In [1], Pietsch studied the operator ideals generated by the approximation numbers and classical sequence space ℓ𝑝 (0 < 𝑝 < ∞). In [2], Faried and bakery [3] have studied the ideal of all bounded linear operators between any arbitrary Banach spaces whose sequence of approximation numbers belonged to the generalized Cesàro sequence space and Orlicz sequence space ℓ𝑀, when 𝑀(𝑡) = 𝑡𝑝, 0 < 𝑝 < ∞; these results coincide with that known for the classical sequence space ℓ𝑝. Bakery [4] has studied the operator ideals generated by the approximation numbers and generalized de La Vallée Poussin’s mean V(𝜆, 𝑝) defined by Şimşek et al. [5]; these results coincide with that known in [2] for the generalized Cesàro sequence space. By L(X, Y), we indicate the space of all bounded linear operators from a normed space X into a normed space Y. The set of nonnegative integers is denoted by ℕ = {0, 1, 2, …} and the real numbers by ℜ. By 𝜔, we denote the space of all real sequences. A map which assigns to every operator 𝑇 ∈ 𝐿(X, Y) a unique sequence (𝑠𝑛(𝑇))∞=0 is called an 𝑠-function and the number 𝑠𝑛(𝑇) is called the 𝑛th 𝑠-numbers of 𝑇 if the following conditions are satisfied:

(a) ‖𝑇‖ = 𝑠0(𝑇) ≥ 𝑠1(𝑇) ≥ ⋅⋅⋅ ≥ 0, for all 𝑇 ∈ 𝐿(X, Y),
(b) 𝑠𝑛(𝑇1 + 𝑇2) ≤ 𝑠𝑛(𝑇1) + ‖𝑇2‖, for all 𝑇1, 𝑇2 ∈ 𝐿(X, Y),
(c) 𝑠𝑛(RST) ≤ ‖R‖𝑠𝑛(S) ‖𝑇‖, for all 𝑇 ∈ 𝐿(X0, X), 𝑆 ∈ 𝐿(X, Y), and 𝑅 ∈ 𝐿(Y, 𝑌0), where 𝑋0 and 𝑌0 are normed spaces,
(d) 𝑠𝑛(𝜆𝑇) = |𝜆|𝑠𝑛(𝑇), for all 𝑇 ∈ 𝐿(X, Y), 𝜆 ∈ ℜ,
(e) rank(𝑇) ≤ 𝑛, if 𝑠𝑛(𝑇) = 0, for all 𝑇 ∈ 𝐿(X, Y),

\[ s_r(I_n) = \begin{cases} 1 & \text{for } r < n \\ 0 & \text{for } r \geq n \end{cases} \]  

where 𝐼𝑛 is the identity operator on the Euclidean space ℜ𝑛.

As examples of 𝑠-numbers, we mention that approximation numbers 𝛼𝑛(𝑇), Gelfand numbers 𝜖𝑛(𝑇), Kolmogorov numbers 𝑑𝑛(𝑇), and Tichomirov numbers 𝑑∗ 𝑛(𝑇) are defined by

\[ \alpha_n(T) = \inf \{\|T - A\| : A \in L(X, Y) \text{ and } \text{rank}(A) \leq n \} \]
All the numbers satisfy the following condition:

(2) If $\theta = 2^{n+1} - 1$ and $p_n = p$, for all $n \in \mathbb{N}$, then we obtain the sequences space $\text{Ces}_p$ studied in [14].

The idea of the paper is the following. We proceed in the following way: given a scalar sequence space $\text{Ces}(\theta, p)$, a pair of Banach spaces $X$ and $Y$, the space of bounded operators $L(X, Y)$, and the approximation $s$-numbers $a_n(T)$, $T \in L(X, Y)$, and $n \in \mathbb{N}$, we define the space $U_{\text{app}}^{\text{Ces}(\theta, p)}(X, Y)$. Then, we study the following two problems:

Problem A (a linear problem). When (for which $\text{Ces}(\theta, p)$) $U_{\text{app}}^{\text{Ces}(\theta, p)}$ is an operator ideal.

Problem B (topological problems). When the ideal of the finite range operators in the class of Banach spaces is dense in $U_{\text{app}}^{\text{Ces}(\theta, p)}$ and completeness of the components of the ideal.

Throughout this paper, the sequence $(p_n)$ is a bounded sequence of positive real numbers with the following:

(a1) the sequence $(p_n)$ of positive real numbers is increasing and bounded with $\lim_{n \to \infty} p_n < \infty$ and $\lim_{n \to \infty} \inf p_n > 1$,

(a2) the sequence $(h_n)$ is a nondecreasing sequence of positive real numbers tending to $\infty$, with $\sum_{n=0}^{\infty} (1/h_n) p_n < \infty$.

Also, we define $\delta_i = (0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 appears at the $i$th place for all $i \in \mathbb{N}$.

Recently different classes of paranormed sequence spaces have been introduced and their different properties have been investigated by Et et al. [15], Tripathy and Dutta [16, 17], and Tripathy and Borgohain [18], and see also [19–23].

The following well-known inequality will be used throughout the paper. For any bounded sequence of positive numbers $(p_n), |a_n + b_n|^{p_n} \leq 2^{p_n-1}(|a_n|^{p_n} + |b_n|^{p_n}), \text{H} = sup_n p_n$ and $p_n \geq 1$ for all $n \in \mathbb{N}$. See [24].

2. Preliminary and Notation

Definition 1. A class of linear sequence spaces $E$ is called a special space of sequences (sss) having three properties:

(1) $E$ is a linear space and $e_n \in E$ for each $n \in \mathbb{N}$;

(2) if $x \in \omega$, $y \in E$, and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x \in E$; "that is, $E$ is solid;"

(3) if $(x_{[n/2]}^\alpha)_{n=0}^\infty \in E$, then $(x_{[n/2]}^\alpha)_{n=0}^\infty = (x_0, x_0, x_1, x_1, x_2, x_2, \ldots) \in E$, where $[n/2]$ denotes the integral part of $n/2$.

Example 2. $\ell_p$ is a special space of sequences for $0 < p < \infty$.

Example 3. $\text{Ces}_p$ defined in [14] is a special space of sequences for $1 < p < \infty$.

Example 4. Let $M$ be an Orlicz function satisfying $\Delta_2$-condition; then $\ell_M$ is a special space of sequences.
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Example 5. Ces(?p) studied in [3] is a special space of sequences, if (?p) is an increasing sequence of positive real numbers, limn→∞ sup pn < ∞ and limn→∞ inf pn > 1.

Example 6. V(λ, p) is a special space of sequences, if the following conditions are satisfied:

1. the sequence (pn) of positive real numbers is increasing and bounded with limn→∞ sup pn < ∞ and limn→∞ inf pn > 1;
2. the sequence (λn) is a nondecreasing sequence of positive real numbers tending to ∞, λ0 = 1 and λn+1 ≤ λn + 1 with ∑n=0 (1/λn)pn < ∞.

Definition 7. Uapp E := {Uapp ∈ L(𝑋,𝑌) : (𝛼n(T))n=0 ∈ E}.

We state the following result without proof.

Theorem 8. Uapp E is an operator ideal if E is a special space of sequences (ss).

We study here the operator ideals generated by the approximation numbers and the sequence space Ces(θ, p) which are involving Lacunary sequence.

3. Main Results

Theorem 9. Uapp Ces(θ, p) is an operator ideal, if conditions (a1) and (a2) are satisfied.

Proof. (1-i) Let x, y ∈ Ces(θ, p); since ∑n=0 ((1/hn)∑k∈Ie |xk + yk|)pn ≤ 2H−1 ∑n=0 ((1/hn)∑k∈Ie |xk|)pn + ∑n=0 (∑k∈Ie |yk|)pn, H = suppn, then x + y ∈ Ces(θ, p).

(1-ii) Let λ ∈ R, x ∈ Ces(θ, p); then ∑n=0 ((1/hn)∑k∈Ie |λxk|)pn ≤ supn|λ|pn ∑n=0 ((1/hn)∑k∈Ie |xk|)pn < ∞; we get λx ∈ Ces(θ, p), from (1-i) and (1-ii), and Ces(θ, p) is a linear space.

To prove that e_m ∈ Ces(θ, p) for each m ∈ N, since ∑n=0 (1/hn)pn < ∞. So, we get

ρ(e_m) = ∑n=0 (1/hn ∑k∈Ie |e_m(k)|)pn = ∑n=0 (1/hn)pn < ∞.

Hence, e_m ∈ Ces(θ, p).

(2) Let |x_k| ≤ |y_k| for each n ∈ N; then ∑n=0 ((1/hn)∑k∈Ie |x_k|)pn ≤ ∑n=0 ((1/hn)∑k∈Ie |y_k|)pn, since y ∈ Ces(θ, p). Thus, x ∈ Ces(θ, p).

(3) Let (x_n) ∈ Ces(θ, p); then we have

∪n=0 (1/hn ∑k∈Ie |x_{k[n]|})pn

≤ ∑n=0 (1/hn ∑k∈Ie |x_{k[n]|})pn + ∑n=0 (1/hn ∑k∈Ie |x_{k[n]|})pn

3. Main Results

Theorem 10. Uapp Ces(θ, p) is an operator ideal if (p_n) is an increasing sequence of positive real numbers, limn→∞ sup pn < ∞ and limn→∞ inf pn > 1.

Corollary 11. Uapp Ces(p) is an operator ideal if 1 < p < ∞.

Theorem 12. The linear space F(𝑋,𝑌) is dense in Uapp Ces(θ, p) (𝑋, 𝑌) if conditions (a1) and (a2) are satisfied.

Proof. First, we show that every finite mapping 𝑇 ∈ F(𝑋,𝑌) belongs to Uapp Ces(θ, p) (𝑋, 𝑌). Since e_m ∈ Ces(θ, p) for each m ∈ N and Ces(θ, p) is a linear space, then for every finite mapping 𝑇 ∈ F(𝑋,𝑌), that is, the sequence (𝛼_n(T))n=0 contains only finitely many numbers different from zero. Now, we prove that Uapp Ces(θ, p) (𝑋, 𝑌) ⊂ F(𝑋,𝑌). On taking 𝑇 ∈ Uapp Ces(θ, p) (𝑋, 𝑌), we obtain (𝛼_n(T))n=0 ∈ Ces(θ, p), and since ρ((𝛼_n(T))n=0) < ∞, let ε ∈ (0, 1); then there exists a natural number s > 0 such that ρ((𝛼_n(T))n=0) < ε/2H+2Δc for some c ≥ 1, where δ = max{1, ∑n=0 (1/hn)pn}. Since 𝛼_n(T) is decreasing for each n ∈ N, we get

∪n=0 (1/hn ∑k∈Ie |𝛼_k(T)|)pn ≤ ∑n=0 (1/hn ∑k∈Ie |𝛼_k(T)|)pn ≤ ∑n=0 (1/hn ∑k∈Ie |𝛼_k(T)|)pn < ε/2H+2Δc.

(7)
then there exists \( A \in F_2(X,Y) \) and \( \text{rank}(A) \leq 2s \) with
\[
\sum_{n=2s+1}^{3s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} \leq \sum_{n=2s+1}^{2s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} < \frac{\varepsilon}{2^{H+2} \delta c},
\]
and since \((p_n)\) is a bounded sequence of positive real numbers, so on considering
\[
\sup_{n=1}^{\infty} \left( \sum_{k \in I_n} \|T - A\| \right)^{p_n} < \frac{\varepsilon}{2^{H+2} \delta c},
\]
also \( \alpha_n(T) = \inf \{ \|T - A\| : A \in L(X,Y) \text{ and } \text{rank}(A) \leq n \} \).
Then, there exists a natural number \( N > 0 \), \( A_N \) with rank \( \text{rank}(A_N) \leq N \) and \( \|T - A_N\| \leq 2\alpha_N(T) \). Since \( \alpha_n(T) \xrightarrow{\text{as } n \to \infty} 0 \), then \( \|T - A_N\| \xrightarrow{\text{as } N \to \infty} 0 \) so we can take
\[
\sum_{n=0}^{s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} < \frac{\varepsilon}{2^{H+2} \delta c}.
\]
(10)

Since \((p_n)\) is an increasing sequence, by using (7), (8), (9), and (10), we acquire
\[
d(T, A) = \rho(\alpha_n(T - A))^{\infty}_{n=0}
= \sum_{n=0}^{3s-1} \left( \frac{1}{h_n} \sum_{k \in I_n} \alpha_k(T - A) \right)^{p_n} + \sum_{n=3s}^{\infty} \left( \frac{1}{h_n} \sum_{k \in I_n} \alpha_k(T - A) \right)^{p_n}
\leq \sum_{n=0}^{3s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} + \sum_{n=3s}^{\infty} \left( \frac{1}{h_n} \sum_{k \in I_n} \alpha_k(T - A) \right)^{p_n}
\leq 3 \sum_{n=0}^{s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} + \sum_{n=3s}^{\infty} \left( \frac{1}{h_n} \sum_{k \in I_n} \alpha_k(T - A) \right)^{p_n}
\leq 3 \sum_{n=0}^{s} \left( \frac{1}{h_n} \sum_{k \in I_n} \|T - A\| \right)^{p_n} + 2^{-H-1} \left( \sum_{n=3s}^{\infty} \left( \frac{1}{h_n} \sum_{k \in I_n} \alpha_k(T - A) \right)^{p_n} \right)
\]

This completes the proof.

Definition 13. A subclass of the special space of sequences called premodular special space of sequences characterized for the existence of a function \( \rho : E \rightarrow [0, \infty) \), closely connected with the notion of modular but without assumption of the convexity, which satisfies the following:

(i) \( \rho(x) \geq 0 \) for all \( x \in E \) and \( \rho(x) = 0 \) if and only if \( x = 0 \), where \( 0 \) is the zero element of \( E \);

(ii) there exists a constant \( N \geq 1 \) such that \( \rho(\lambda x) \leq N|\lambda| \rho(x) \) for all values of \( x \in E \) and for any scalar \( \lambda \);

(iii) for some numbers \( K \geq 1 \), we have the inequality \( \rho(x + y) \leq K(\rho(x) + \rho(y)) \) for all \( x, y \in E \);

(iv) if \( |x_n| \leq |y_n| \) for all \( n \in N \), then \( \rho((x_n)) \leq \rho((y_n)) \);

(v) for some numbers \( K_0 \geq 1 \), we have the inequality \( \rho((x_n)) \leq \rho((x_{n|z|})) \leq K_0 \rho((x_n)) \);

(vi) for each \( x = (x(i))_{i=0}^{\infty} \in E \), there exists \( s \in N \) such that \( \rho(x(i))_{i=s}^{\infty} < \infty \); this means the set of all finite sequences is \( \rho \)-dense in \( E \);

(vii) for any \( \lambda > 0 \), there exists a constant \( \zeta > 0 \) such that \( \rho(\lambda x, 0, 0, 0, \ldots) \geq \zeta \rho(1, 0, 0, 0, \ldots) \).

It is obvious from condition (ii) that \( \rho \) is continuous at the zero element of \( E \). The function \( \rho \) defines a metrizable topology in \( E \) endowed with this topology which is denoted by \( E_{\rho} \).
Example 14. $\ell_p$ is a premodular special space of sequences for $0 < p < \infty$ with $\rho(x) = \sum_{n=0}^{\infty} |x_n|^p$.

Example 15. Ces$_p$ is a premodular special space of sequences for $1 < p < \infty$ with $\rho(x) = \sum_{n=0}^{\infty} ((1/(n+1)) \sum_{k=n}^{\infty} |x_k|)^p$.

Example 16. Let $M$ be an Orlicz function satisfying $\Delta_2$-condition; then $\ell_M$ is a pre-modular special space of sequences with $\rho(x) = \sum_{n=0}^{\infty} M(|x_n|)$.

Example 17. If $(p_n)$ is an increasing sequence of positive real numbers, $\lim_{n \to \infty} \sup p_n < \infty$ and $\lim_{n \to \infty} \inf p_n > 1$, then Ces$(p)$ is a premodular special space of sequences for $1 < p < \infty$, with $\rho(x) = \sum_{n=0}^{\infty} ((1/(n+1)) \sum_{k=n}^{\infty} |x_k|)^{p_k}$.

Example 18. If the following conditions are satisfied:

(i) the sequence $(p_n)$ of positive real numbers is increasing and bounded with $\limsup p_n < \infty$ and $\liminf p_n > 1$;

(ii) the sequence $(\lambda_n)$ is a nondecreasing sequence of positive real numbers tending to $\infty$, $\lambda_0 = 1$, and $\lambda_{n+1} \leq \lambda_n + 1$ with $\sum_{n=0}^{\infty} (1/\lambda_n)^{p_k} < \infty$; then $V(\lambda, p)$ is a premodular special space of sequences.

Theorem 19. Ces$(\theta, p)$ with $\rho(x) = \sum_{n=0}^{\infty} (1/h_n) \sum_{k \in E} |x_k|^{p_k}$ is a premodular special space of sequences, if conditions (a1) and (a2) are contented.

Proof. (i) Clearly, $\rho(x) \geq 0$ and $\rho(x) = 0 \iff x = 0$.

(ii) Since $(p_n)$ is bounded, then there exists a constant $N \geq 1$ such that $\rho(\lambda, x) \leq N\rho(x)$ for all values of $x \in E$ and for any scalar $\lambda$.

(iii) For some numbers $K = \max(1, 2^{H-1}) \geq 1$, we have the inequality $\rho(x + y) \leq K(\rho(x) + \rho(y))$ for all $x, y \in \text{Ces}(\theta, p)$.

(iv) Let $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$; then $\sum_{n=0}^{\infty} (1/h_n) \sum_{k \in E} |x_k|^p \leq \sum_{n=0}^{\infty} (1/h_n) \sum_{k \in E} |y_k|^p$.

(v) There exist some numbers $K_0 = 2^{2H-1} - 2^{2H+1} - 2^{H} \geq 1$; by using (iv), we have the inequality $\rho((x_n)) \leq \rho((x_{[n/2]})) \leq K_0(\rho(x_i))$.

(vi) It is clear that the set of all finite sequences is $\rho$-dense in Ces$(\theta, p)$.

(vii) For any $\lambda > 0$, there exists a constant $0 < \zeta < \lambda^{H^{-1}}$ such that $\rho(\lambda, 0, 0, 0, \ldots) \leq \zeta \lambda(1, 0, 0, 0, \ldots)$.

Theorem 20. Let $X$ be a normed space, let $Y$ be a Banach space, and let conditions (a1) and (a2) be satisfied; then $U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y)$ is complete.

Proof. Let $(T_m)$ be a Cauchy sequence in $U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y)$. Since Ces$(\theta, p)$ with $\rho(x) = \sum_{n=0}^{\infty} (1/h_n) \sum_{k \in E} |x_k|^{p_k}$ is a premodular special space of sequences, then, by using condition (vii) and since $U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y) \subseteq L(X, Y)$, we have $\rho((\alpha_n(T_m - T_n))_{n=0}^{\infty}) \geq \rho((\alpha_n(T_m - T_n), 0, 0, 0, \ldots) = \rho(\|T_m - T_n\|, 0, 0, 0, \ldots) \geq \|T_m - T_n\| \rho(0, 0, 0, 0, \ldots)$, then $(T_m)$ is also a Cauchy sequence in $L(X, Y)$. Since the space $L(X, Y)$ is a Banach space, then there exists $T \in L(X, Y)$ such that

$$\|T_m - T\| \to 0 \text{ as } m \to \infty$$

and since $(\alpha_n(T_m))_{n=0}^{\infty} \in E$ for all $m \in \mathbb{N}$, $\rho$ is continuous at 0 and, using (iii), we have

$$\rho(\alpha_n(T))_{n=0}^{\infty} = \rho(\alpha_n(T - T_m + T_m))_{n=0}^{\infty} \leq K\rho(\alpha_n(T_m - T))_{n=0}^{\infty} + K\rho(\alpha_n(T_m))_{n=0}^{\infty} < \varepsilon$$

for some $K \geq 1$.

Hence, $(\alpha_n(T))_{n=0}^{\infty} \in \text{Ces}(\theta, p)$ as such $T \in U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y)$.

Corollary 21. Let $X$ be a normed space, let $Y$ be a Banach space, and let $(p_n)$ be an increasing sequence of positive real numbers with $\lim_{n \to \infty} \sup p_n < \infty$ and $\lim_{n \to \infty} \inf p_n > 1$; then $U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y)$ is complete.

Corollary 22. Let $X$ be a normed space, let $Y$ be a Banach space, and let $(p_n)$ be an increasing sequence of positive real numbers with $1 < p < \infty$; then $U^{\text{app}}_{\text{Ces}(\theta, p)}(X, Y)$ is complete.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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