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Research Article

Some Paranormed Double Difference Sequence Spaces for Orlicz Functions and Bounded-Regular Matrices

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The aim of this paper is to introduce some new double difference sequence spaces with the help of the Musielak-Orlicz function $\mathscr{F} = (F_{jk})$ and four-dimensional bounded-regular (shortly, RH-regular) matrices $A = (a_{nmjk})$. We also make an effort to study some topological properties and inclusion relations between these double difference sequence spaces.

1. Introduction, Notations, and Preliminaries

In [1], Hardy introduced the concept of regular convergence for double sequences. Some important work on double sequences is also found by Bromwich [2]. Later on, it was studied by various authors, for example, Móricz [3], Móricz and Rhoades [4], Başarır and Sonalcan [5], Mursaleen and Mohiuddine [6–8], and many others. Mursaleen [9] has defined and characterized the notion of almost strong regularity of four-dimensional matrices and applied these matrices to establish a core theorem (also see [10, 11]). Altay and Başar [12] have recently introduced the double sequence spaces $\mathscr{BS}$, $\mathscr{BS}(t)$, $\mathscr{CS}_p$, $\mathscr{CS}_{bp}$, $\mathscr{CS}_r$, and $\mathscr{BV}$ consisting of all double series whose sequence of partial sums are in the spaces $\mathscr{M}_u$, $\mathscr{M}_u(t)$, $\mathscr{C}_p$, $\mathscr{C}_{bp}$, $\mathscr{C}_r$, and $\mathscr{L}_u$, respectively. Başar and Sever [13] extended the well-known space ℓ_q from single sequence to double sequences, denoted by $\mathscr{L}_q$, and established its interesting properties. The authors of [14] defined some convex and paranormed sequences spaces and presented some interesting characterization. Most recently, Mohiuddine and Alotaibi [15] introduced some new double sequences spaces for σ-convergence of double sequences and invariant mean and also determined some inclusion results for these spaces. For more details on these concepts, one can be referred to [16–18].

The notion of difference sequence spaces was introduced by Kızmaz [19], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [20] by introducing the spaces $l_\infty(\Delta^r)$, $c(\Delta^r)$, and $c_0(\Delta^r)$.

Let w be the space of all complex or real sequences $x = (x_k)$ and let r and s be two nonnegative integers. Then for $Z = l_\infty, c, c_0$, we have the following sequence spaces:

$$Z(\Delta_s^r) = \{x = (x_k) \in w : (\Delta_s^r x_k) \in Z\}, \tag{1}$$

where $\Delta_s^r x = (\Delta_s^r x_k) = (\Delta_s^{r-1} x_k - \Delta_s^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_s^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+sv}. \tag{2}$$

We remark that for $s = 1$ and $r = s = 1$, we obtain the sequence spaces which were introduced and studied by Et and Çolak [20] and Kızmaz [19], respectively. For more details about sequence spaces see [21–27] and references therein.

An *Orlicz function* $F : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, and convex such that $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $F(x+y) \le F(x) + F(y)$, then this function is called *modulus function*. Lindenstrauss and Tzafriri [28] used the idea of Orlicz function to define the following sequence space:

$$\ell_F = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} F\left(\frac{|x_k|}{\rho}\right) < \infty,\ \rho > 0 \right\}, \quad (3)$$

which is known as an Orlicz sequence space. The space ℓ_F is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} F\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}. \quad (4)$$

Also it was shown in [28] that every Orlicz sequence space ℓ_F contains a subspace isomorphic to ℓ_p $(p \ge 1)$. An Orlicz function F can always be represented in the following integral form:

$$F(x) = \int_0^x \eta(t)\, dt, \quad (5)$$

where η is known as the kernel of F, is a right differentiable for $t \ge 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is nondecreasing, and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{F} = (F_k)$ of Orlicz functions is said to be a *Musielak-Orlicz function* (see [29, 30]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup \{ |v| u - F_k(u) : u \ge 0 \}, \quad k = 1, 2, \ldots, \quad (6)$$

which is called the complementary function of a Musielak-Orlicz function $\mathcal{F}$. For a given Musielak-Orlicz function $\mathcal{F}$, the Musielak-Orlicz sequence space $t_{\mathcal{F}}$ and its subspace $h_{\mathcal{F}}$ are defined as follows:

$$\begin{aligned} t_{\mathcal{F}} &= \{ x \in w : I_{\mathcal{F}}(cx) < \infty \text{ for some } c > 0 \}, \\ h_{\mathcal{F}} &= \{ x \in w : I_{\mathcal{F}}(cx) < \infty\ \forall c > 0 \}, \end{aligned} \quad (7)$$

where $I_{\mathcal{F}}$ is a convex modular defined by

$$I_{\mathcal{F}}(x) = \sum_{k=1}^{\infty} F_k(x_k), \quad x = (x_k) \in t_{\mathcal{F}}. \quad (8)$$

We consider $t_{\mathcal{F}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{F}}\left(\frac{x}{k}\right) \le 1 \right\} \quad (9)$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k}\left(1 + I_{\mathcal{F}}(kx)\right) : k > 0 \right\}. \quad (10)$$

A Musielak-Orlicz function $\mathcal{F} = (F_k)$ is said to satisfy Δ_2-*condition* if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$F_k(2u) \le K F_k(u) + c_k \quad (11)$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $F_k(u) \le a$.

A double sequence $x = (x_{jk})$ is said to be *bounded* if $\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}| < \infty$. We denote by l_∞^2 the space of all bounded double sequences.

By the convergence of double sequence $x = (x_{jk})$ we mean the convergence in the Pringsheim sense; that is, a double sequence $x = (x_{jk})$ is said to *converge* to the limit L in Pringsheim sense (denoted by, P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ whenever $j, k > n$ (see [31]). We will write more briefly as P-convergent. If, in addition, $x \in l_\infty^2$, then x is said to be *boundedly P-convergent* to L. We will denote the space of all bounded convergent double sequences (or boundedly P-convergent) by c_∞^2.

Let $S \subseteq \mathbb{N} \times \mathbb{N}$ and let $\epsilon > 0$ be given. By $\chi_{S(x;\epsilon)}$, we denote the characteristic function of the set $S(x;\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk}| \ge \epsilon\}$.

Let $A = (a_{nmjk})$ be a four-dimensional infinite matrix of scalers. For all $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the sum

$$y_{nm} = \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} x_{jk} \quad (12)$$

is called the *A-means* of the double sequence (x_{jk}). A double sequence (x_{jk}) is said to be *A-summable* to the limit L if the A-means exist for all m, n in the sense of Pringsheim's convergence:

$$P\text{-}\lim_{p,q \to \infty} \sum_{j,k=0,0}^{p,q} a_{nmjk} x_{jk} = y_{nm}, \qquad P\text{-}\lim_{n,m \to \infty} y_{nm} = L. \quad (13)$$

A four-dimensional matrix A is said to be *bounded-regular* (or *RH*-regular) if every bounded P-convergent sequence is A-summable to the same limit and the A-means are also bounded.

The following is a four-dimensional analogue of the well-known Silverman-Toeplitz theorem [32].

Theorem 1 (Robison [33] and Hamilton [34]). *The four-dimensional matrix A is RH-regular if and only if*

(RH_1) P-$\lim_{n,m} a_{nmjk} = 0$ *for each* j *and* k,

(RH_2) P-$\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} |a_{nmjk}| = 1$,

(RH_3) P-$\lim_{n,m} \sum_{j=0}^{\infty} |a_{nmjk}| = 0$ *for each* k,

(RH_4) P-$\lim_{n,m} \sum_{k=0}^{\infty} |a_{nmjk}| = 0$ *for each* j,

(RH_5) $\sum_{j,k=0,0}^{\infty,\infty} |a_{nmjk}| < \infty$ *for all* $n, m \in \mathbb{N}_0$.

2. The Double Difference Sequence Spaces

In this section, we define some new paranormed double difference sequence spaces with the help of Musielak-Orlicz functions and four-dimensional bounded-regular matrices. Before proceeding further, first we recall the notion of paranormed space as follows.

A linear topological space X over the real field $\mathbb{R}$ (the set of real numbers) is said to be a *paranormed space* if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, and scalar multiplication is continuous; that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x| \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

The linear spaces $l_\infty(p)$, $c(p)$, and $c_0(p)$ were defined by Maddox [35] (also, see Simons [36]).

Let $\mathcal{F} = (F_{jk})$ be a Musielak-Orlicz function; that is, $\mathcal{F}$ is a sequence of Orlicz functions and let $A = (a_{nmjk})$ be a nonnegative four-dimensional bounded-regular matrix. Then, we define the following:

$$W_0^2(A,\mathcal{F},u,\Delta_s^r,p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}} \right] = 0 \right\},$$

$$W^2(A,\mathcal{F},u,\Delta_s^r,p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0 \text{ for some } L \in \mathbb{C} \right\}, \quad (14)$$

where $p = (p_{jk})$ is a double sequence of real numbers such that $p_{jk} > 0$ for j,k, $\sup_{j,k} p_{jk} = H < \infty$, and $u = (u_{jk})$ is a double sequence of strictly positive real numbers.

Remark 2. If we take $\mathcal{F}(x) = x$ in $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)$ and $W^2(A,\mathcal{F},u,\Delta_s^r,p)$, then we have the following spaces:

$$W_0^2(A,u,\Delta_s^r,p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}} \right] = 0 \right\},$$

$$W^2(A,u,\Delta_s^r,p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[\left(u_{jk}\left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0 \text{ for some } L \in \mathbb{C} \right\}. \quad (15)$$

Remark 3. Let $p = (p_{jk}) = 1$ for all j,k. Then $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)$ and $W^2(A,\mathcal{F},u,\Delta_s^r,p)$ are reduced to

$$W_0^2(A,\mathcal{F},u,\Delta_s^r) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right) \right] = 0 \right\},$$

$$W^2(A,\mathcal{F},u,\Delta_s^r) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk} - L\right|\right) \right] = 0 \text{ for some } L \in \mathbb{C} \right\}, \quad (16)$$

respectively.

Remark 4. Let $u = (u_{jk}) = 1$ for all j, k. Then, the spaces $W_0^2(A, \mathscr{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathscr{F}, u, \Delta_s^r, p)$ are reduced to

$$W_0^2(A, \mathscr{F}, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}} \right] = 0 \right\},$$

$$W^2(A, \mathscr{F}, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(\left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0 \text{ for some } L \in \mathbb{C} \right\}, \quad (17)$$

respectively.

Remark 5. If we take $A = (C, 1, 1)$ in $W_0^2(A, \mathscr{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathscr{F}, u, \Delta_s^r, p)$, then we have the following spaces:

$$W_0^2(\mathscr{F}, u, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{m-1,n-1} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}} \right] = 0 \right\},$$

$$W^2(\mathscr{F}, u, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{m-1,n-1} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0 \text{ for some } L \in \mathbb{C} \right\}. \quad (18)$$

Remark 6. If we take $A = (C, 1, 1)$ and $\mathscr{F}(x) = x$ in $W_0^2(A, \mathscr{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathscr{F}, u, \Delta_s^r, p)$, then we have the following spaces:

$$W_0^2(u, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{m-1,n-1} \left[\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}} \right] = 0 \right\},$$

$$W^2(u, \Delta_s^r, p) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k=0,0}^{m-1,n-1} \left[\left(u_{jk}\left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0 \text{ for some } L \in \mathbb{C} \right\}. \quad (19)$$

Remark 7. Let $p_{jk} = u_{jk} = 1$ for all j, k. If, in addition, $\mathscr{F}(x) = F(x)$ and $r = 0$, then the spaces $W_0^2(A, \mathscr{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathscr{F}, u, \Delta_s^r, p)$ are reduced to $W_0^2(A, F)$ and $W^2(A, F)$ which were introduced and studied by Yurdakadim and Tas [37] as below:

$$W_0^2(A, F) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k} a_{nmjk} F\left(\left|x_{jk}\right|\right) = 0 \right\},$$

$$W^2(A, F) = \left\{ x = (x_{jk}) : P\text{-}\lim_{n,m} \sum_{j,k} a_{nmjk} F\left(\left|x_{jk} - L\right|\right) = 0 \text{ for some } L \in \mathbb{C} \right\}. \quad (20)$$

Throughout the paper, we will use the following inequality: let (a_{jk}) and (b_{jk}) be two double sequences. Then

$$\left|a_{jk} + b_{jk}\right|^{p_{jk}} \le K\left(\left|a_{jk}\right|^{p_{jk}} + \left|b_{jk}\right|^{p_{jk}}\right), \quad (21)$$

where $K = \max(1, 2^{H-1})$ and $\sup_{j,k} p_{jk} = H$ (see [15]). We will also assume throughout this paper that the symbol $\mathscr{F}$ will denote the sublinear Musielak-Orlicz function.

3. Main Results

Theorem 8. *Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function, $A = (a_{nmjk})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_{jk}), y = (y_{jk}) \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist integers M_α and N_β such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq N_\beta$.

Since $\mathcal{F} = (F_{jk})$ is a nondecreasing function, so by inequality (21), we have

$$\begin{aligned}
&\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(\alpha x_{jk}+\beta y_{jk}\right)\right|\right)^{p_{jk}}\right]\\
&\quad\leq \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\alpha\Delta_s^r x_{jk}+\beta\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\\
&\quad\leq K\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}M_\alpha\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]\\
&\qquad+K\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}N_\beta\left(u_{jk}\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\\
&\quad\leq KM_\alpha^H\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]\\
&\qquad+KN_\beta^H\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\longrightarrow 0.
\end{aligned} \tag{22}$$

Thus $\alpha x + \beta y \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$. This proves that $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is a linear space. Similarly we can prove that $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is also a linear space. □

Theorem 9. *Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function, $A = (a_{nmjk})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ are paranormed spaces with the paranorm*

$$g(x) = \sup_{n,m}\sum_{j,k=0,0}^{\infty,\infty}\left\{a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]\right\}^{1/M}, \tag{23}$$

where $0 < p_{jk} \leq \sup p_{jk} = H < \infty$ *and* $M = \max(1, H)$.

Proof. We will prove the result for $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$. Let $x = (x_{jk}) \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$. Then for each $x = (x_{jk}) \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$, $g(x)$ exists. Also it is clear that $g(0) = 0$, $g(-x) = g(x)$, and $g(x + y) \leq g(x) + g(y)$.

We now show that the scalar multiplication is continuous. First observe the following:

$$\begin{aligned}
g(\lambda x) &= \sup_{nm}\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]\\
&\leq (1+[|\lambda|])\,g(x),
\end{aligned} \tag{24}$$

where $[|\lambda|]$ denotes the integer part of $|\lambda|$. It is also clear that if $x \to 0$ and $\lambda \to 0$ implies $g(\lambda x) \to 0$. For fixed λ, if $x \to 0$, then $g(\lambda x) \to 0$. We need to show that for fixed $x, \lambda \to 0$ implies $g(\lambda x) \to 0$. Let $x \in W^2(A, \mathcal{F}, u, \Delta_s^r, p)$. Thus

$$P\text{-}\lim_{n,m}\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]=0. \tag{25}$$

Then, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\frac{\epsilon}{4} \tag{26}$$

for $m, n > N$. Also, for each m, n with $1 \leq m, n \leq N$, since

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\infty, \tag{27}$$

there exists an integer $M_{m,n}$ such that

$$\sum_{j,k>M_{m,n}} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\frac{\epsilon}{4}. \tag{28}$$

Let $M = \max_{1\leq(m,n)\leq N}\{M_{m,n}\}$. We have for each m, n with $1 \leq m, n \leq N$

$$\sum_{j,k>M} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\frac{\epsilon}{4}. \tag{29}$$

Also from (26), for $m, n > N$, we have

$$\sum_{j,k>M} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\frac{\epsilon}{4}. \tag{30}$$

Thus M is an integer independent of m, n such that

$$\sum_{j,k>M} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right]<\frac{\epsilon}{4}. \tag{31}$$

Since $|\lambda|^{p_{jk}} \leq \max(1, |\lambda|^H)$, therefore

$$\begin{aligned}
&\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] \\
&= \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L + \lambda L\right|\right)^{p_{jk}}\right] \\
&\leq \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] + \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda L\right|\right)^{p_{jk}}\right] \\
&\leq \sum_{j,k>M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] \\
&\quad + \sum_{j,k\leq M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] \\
&\quad + \sum_{j\geq M,k<M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] \\
&\quad + \sum_{j<M,k\geq M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] \\
&\quad + \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda L\right|\right)^{p_{jk}}\right].
\end{aligned} \tag{32}$$

For each m, n and by the continuity of F as $\lambda \to 0$, we have the following:

$$\sum_{j,k\leq M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] + \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda L\right|\right)^{p_{jk}}\right] \longrightarrow 0 \tag{33}$$

in Pringsheim's sense. Now choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\sum_{j,k\leq M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] + \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda L\right|\right)^{p_{jk}}\right] < \frac{\epsilon}{4}. \tag{34}$$

In the same manner, we have

$$\sum_{j\geq M,k<M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] < \frac{\epsilon}{4}, \tag{35}$$

$$\sum_{j<M,k\geq M} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk} - \lambda L\right|\right)^{p_{jk}}\right] < \frac{\epsilon}{4}. \tag{36}$$

It follows from (31), (34), (35), and (36) that

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\lambda\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] < \epsilon \quad \forall m, n. \tag{37}$$

Thus $g(\lambda x) \to 0$ as $\lambda \to 0$. Therefore $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is a paranormed space. Similarly, we can prove that $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is a paranormed space. This completes the proof. □

Theorem 10. *Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function, $A = (a_{nmjk})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ and $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ are complete topological linear spaces.*

Proof. Let (x_{jk}^q) be a Cauchy sequence in $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$; that is, $g(x^q - x^t) \to 0$ as $q, t \to \infty$. Then, we have

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}^t\right|\right)^{p_{jk}}\right] \longrightarrow 0. \tag{38}$$

Thus for each fixed j and k as $q, t \to \infty$, since $A = (a_{nmjk})$ is nonnegative, we are granted that

$$F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}^t\right|\right) \longrightarrow 0, \tag{39}$$

and by continuity of $\mathcal{F} = (F_{jk})$, (x_{jk}^q) is a Cauchy sequence in $\mathbb{C}$ for each fixed j and k.

Since $\mathbb{C}$ is complete as $t \to \infty$, we have $x_{jk}^q \to x_{jk}$ for each (j, k). Now from (36), we have that, for $\epsilon > 0$, there exists a natural number N such that

$$\sum_{j,k=0,0\; q,t>N}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}^t\right|\right)^{p_{jk}}\right] < \epsilon \quad \forall m, n. \tag{40}$$

Since for any fixed natural number M, from (38) we have

$$\sum_{j,k\leq M\; q,t>N}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}^t\right|\right)^{p_{jk}}\right] < \epsilon \quad \forall m, n. \tag{41}$$

By letting $t \to \infty$ in the above expression we obtain

$$\sum_{j,k\leq M\; q>N}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] < \epsilon. \tag{42}$$

Since M is arbitrary, by letting $M \to \infty$ we obtain

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^q - \Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] < \epsilon \quad \forall m, n. \tag{43}$$

Thus $g(x^q - x) \to 0$ as $q \to \infty$. This proves that $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is a complete topological linear space.

Now we will show that $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ is a complete topological linear space. For this, since (x^q) is also a sequence in $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ by definition of $W^2(A, \mathcal{F}, u, \Delta_s^r, p)$, for each q, there exists L^q with

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk}^q - \Delta_s^r L^q\right|\right)^{p_{jk}} \right] \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty; \tag{44}$$

whence from the fact that $\sup_{nm} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} < \infty$ and from the definition of Musielak-Orlicz function, we have $F_{jk}|\Delta_s^r L^q - \Delta_s^r L| \to 0$ as $q \to \infty$ and so L^q converges to L. Thus

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty. \tag{45}$$

Hence $x \in W^2(A, \mathcal{F}, u, \Delta_s^r, p)$ and this completes the proof. □

Theorem 11. *Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function which satisfies the Δ_2-condition. Then $W^2(A, u, \Delta_s^r, p) \subseteq W^2(A, \mathcal{F}, u, \Delta_s^r, p)$.*

Proof. Let $x = (x_k) \in W^2(A, u, \Delta_s^r, p)$; that is,

$$\lim_{n,m} \sum_{j,k} a_{nmjk} \left[\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] = 0. \tag{46}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $F_{jk}(t) < \epsilon$ for $0 \le t \le \delta$. Write $y_{jk} = (u_{jk}|\Delta_s^r x_{jk} - L|)$ and consider

$$\begin{aligned}\sum_{j,k} a_{nmjk} \left[F_{jk}\left(y_{jk}\right)^{p_{jk}} \right] &= \sum_{j,k:|y_{jk}|\le\delta} a_{nmjk} \left[F_{jk}\left(y_{jk}\right)^{p_{jk}} \right] + \sum_{j,k:|y_{jk}|>\delta} a_{nmjk} \left[F_{jk}\left(y_{jk}\right)^{p_{jk}} \right] \\ &= \epsilon \sum_{j,k:|y_{jk}|\le\delta} a_{nmjk} + \sum_{j,k:|y_{jk}|>\delta} a_{nmjk} \left[F_{jk}\left(y_{jk}\right)^{p_{jk}} \right].\end{aligned} \tag{47}$$

For $y_{jk} > \delta$, we use the fact that $y_{jk} < y_{jk}/\delta < 1 + y_{jk}/\delta$. Hence

$$F_{jk}\left(y_{jk}\right) < F_{jk}\left(1 + \frac{y_{jk}}{\delta}\right) < \frac{F_{jk}(2)}{2} + \frac{1}{2} F_{jk}\left(2\frac{y_{jk}}{\delta}\right). \tag{48}$$

Since $\mathcal{F}$ satisfies the Δ_2-condition, we have

$$F_{jk}\left(y_{jk}\right) < K \frac{y_{jk}}{2\delta} F_{jk}(2) + K \frac{y_{jk}}{2\delta} F_{jk}(2) = K \frac{y_{jk}}{\delta} F_{jk}(2), \tag{49}$$

and hence

$$\sum_{j,k:|y_{jk}|>\delta} a_{nmjk} \left[F_{jk}\left(y_{jk}\right)^{p_{jk}} \right] \le K \frac{F_{jk}}{\delta}(2) \sum_{j,k} a_{nmjk} \left[\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right]. \tag{50}$$

Since A is RH-regular and $x \in W^2(A, u, \Delta_s^r, p)$, we get $x \in W^2(A, \mathcal{F}, u, \Delta_s^r, p)$. □

Theorem 12. *Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function and let $A = (a_{nmjk})$ be a nonnegative four-dimensional RH-regular matrix. Suppose that $\beta = \lim_{t\to\infty}(F_{jk}(t)/t) < \infty$. Then*

$$W^2\left(A, u, \Delta_s^r, p\right) = W^2\left(A, \mathcal{F}, u, \Delta_s^r, p\right). \tag{51}$$

Proof. In order to prove that $W^2(A, u, \Delta_s^r, p) = W^2(A, \mathcal{F}, u, \Delta_s^r, p)$, it is sufficient to show that $W^2(A, \mathcal{F}, u, \Delta_s^r, p) \subset W^2(A, u, \Delta_s^r, p)$. Now, let $\beta > 0$. By definition of β, we have $F_{jk}(t) \ge \beta t$ for all $t \ge 0$. Since $\beta > 0$, we have $t \le (1/\beta)F_{jk}(t)$ for all $t \ge 0$. Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta_s^r, p)$. Thus, we have

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right] \le \frac{1}{\beta} \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right], \tag{52}$$

which implies that $x = (x_{jk}) \in W^2(A, u, \Delta_s^r, p)$. This completes the proof. □

Theorem 13. (i) *Let $0 < \inf\ p_{jk} < p_{jk} \le 1$. Then*

$$W^2\left(A, \mathcal{F}, u, \Delta_s^r, p\right) \subseteq W^2\left(A, \mathcal{F}, u, \Delta_s^r\right). \tag{53}$$

(ii) *Let $1 \le p_{jk} \le \sup p_{jk} < \infty$. Then*

$$W^2\left(A, \mathcal{F}, u, \Delta_s^r\right) \subseteq W^2\left(A, \mathcal{F}, u, \Delta_s^r, p\right). \tag{54}$$

Proof. (i) Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta_s^r, p)$. Then since $0 < \inf p_{jk} < p_{jk} \le 1$, we obtain the following:

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right) \right] \le \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right)^{p_{jk}} \right]. \tag{55}$$

Thus $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta_s^r)$.

(ii) Let $p_{jk} \ge 1$ for each j and k and $\sup p_{jk} < \infty$. Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta_s^r)$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk} \left[F_{jk}\left(u_{jk} \left|\Delta_s^r x_{jk} - L\right|\right) \right] \le \epsilon < 1 \quad \forall m,n \ge N. \tag{56}$$

This implies that

$$\sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)^{p_{jk}}\right] \leq \sum_{j,k=0,0}^{\infty,\infty} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}-L\right|\right)\right]. \quad (57)$$

Therefore $x=(x_{jk})\in W^2(A,\mathcal{F},u,\Delta_s^r,p)$. This completes the proof. □

Lemma 14. *Let $\mathcal{F}=(F_{jk})$ be a sublinear Musielak-Orlicz function which satisfies the Δ_2-condition and let $A=(a_{nmjk})$ be a nonnegative four-dimensional RH-regular matrix. Then $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ is an ideal in l_∞^2.*

Proof. Let $x\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ and $y\in l_\infty^2$. We need to show that $xy\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$. Since $y\in l_\infty^2$, there exists $T_1>1$ such that $\|y\|<T_1$. In this case $|x_{jk}y_{jk}|<T_1|x_{jk}|$ for all j,k. Since $\mathcal{F}$ is nondecreasing and satisfies Δ_2-condition, we have

$$\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(x_{jk}y_{jk}\right)\right|\right)^{p_{jk}}\right] < \left[F_{jk}\left(u_{jk}T_1\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] \leq T\left(T_1\right)\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right], \quad (58)$$

for all j,k and $T>0$. Therefore $\lim_{n,m}\sum_{j,k}a_{nmjk}[F_{jk}(u_{jk}|\Delta_s^r(x_{jk}y_{jk})|)^{p_{jk}}]=0$. Thus $xy\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$. This completes the proof. □

Lemma 15. *Let G be an ideal in l_∞^2 and let $x=(x_{jk})\in l_\infty^2$. Then x is in the closure of G in l_∞^2 if and only if $\chi_{S(x;\epsilon)}\in G$ for all $\epsilon>0$.*

Proof. Let x be in the closure of G and let $\epsilon>0$ be given. Suppose that $z=(z_{jk})\in G$ such that $\|x-z\|<\epsilon/2$ and observe that $S(x;\epsilon)\subseteq S(z;\epsilon/2)$. Define a double sequence $y=(y_{jk})\in l_\infty^2$ by

$$y_{jk}=\begin{cases}\dfrac{1}{z_{jk}}, & \text{if } \left|z_{jk}\right|\geqq\dfrac{\epsilon}{2},\\ 0, & \text{otherwise.}\end{cases} \quad (59)$$

Clearly $yz=\chi_{S(z;\epsilon/2)}\in G$. Since $S(x;\epsilon)\subseteq S(z;\epsilon/2)$ and $\chi_{S(x;\epsilon)}\in l_\infty^2$, hence $\chi_{S(x;\epsilon)}\chi_{S(z;\epsilon/2)}=\chi_{S(x;\epsilon)}\in G$.

Conversely, if $x\in l_\infty^2$ then $\|x-x\chi_{S(x;\epsilon)}\|<\epsilon$. It follows that $\chi_{S(x;\epsilon)}\in G$ for all $\epsilon>0$; then x is in the closure of G. □

Lemma 16. *If A is a nonnegative four-dimensional RH-regular matrix, then $W_0^2(A,u,\Delta_s^r,p)\cap l_\infty^2$ is a closed ideal in l_∞^2.*

Proof. We have $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2\subset l_\infty^2$ and it is clear that $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2\neq 0$. For $x,y\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$, we get $|x_{jk}+y_{jk}|<|x_{jk}|+|y_{jk}|$. Now, we have

$$\begin{aligned}&\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(x_{jk}+y_{jk}\right)\right|\right)^{p_{jk}}\right]\\ &\leq\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|+\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\\ &<\frac{1}{2}\left[F_{jk}\left(u_{jk}2\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]+\frac{1}{2}\left[F_{jk}\left(u_{jk}2\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\\ &<\frac{1}{2}K_1\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]+\frac{1}{2}K_2\left[F_{jk}\left(u_{jk}\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right]\end{aligned} \quad (60)$$

by the Δ_2-condition and the convexity of F. Since

$$\begin{aligned}&\sum_{j,k}a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(x_{jk}+y_{jk}\right)\right|\right)^{p_{jk}}\right]\\ &\leq\frac{1}{2}K\sum_{j,k}a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]\\ &\quad+\frac{1}{2}K\sum_{j,k}a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r y_{jk}\right|\right)^{p_{jk}}\right],\end{aligned} \quad (61)$$

where $K=\max\{K_1,K_2\}$, so $x+y,x-y\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$.

Let $x\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ and $y\in l_\infty^2$. Thus, there exists a positive integer K, so that, for every j,k, we have $|x_{jk}y_{jk}|\leq K|x_{jk}|$. Therefore

$$\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(x_{jk}y_{jk}\right)\right|\right)^{p_{jk}}\right]\leq\left[F_{jk}\left(u_{jk}K\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] \leq T\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right], \quad (62)$$

and so

$$\sum_{j,k}a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r\left(x_{jk}y_{jk}\right)\right|\right)^{p_{jk}}\right] \leq T\sum_{j,k}a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right]. \quad (63)$$

Hence $xy\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$. So $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ is an ideal in l_∞^2 for a Musielak-Orlicz function which satisfies the Δ_2-condition.

Now, we have to show that $W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ is closed. Let $x\in\overline{W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2}$; there exists $x^{cd}=x_{jk}^{cd}\in W_0^2(A,\mathcal{F},u,\Delta_s^r,p)\cap l_\infty^2$ such that $x^{cd}\to x\in l_\infty^2$.

For every $\epsilon > 0$ there exists $N_1(\epsilon) \in \mathbb{N}$ such that, for all $c, d > N_1(\epsilon)$, $|\Delta_s^r x^{cd} - \Delta_s^r x| < \epsilon$. Now, for $\epsilon > 0$, we have

$$\begin{aligned}
&\sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] \\
&\quad = \sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk} - \Delta_s^r x_{jk}^{cd} + \Delta_s^r x_{jk}^{cd}\right|\right)^{p_{jk}}\right] \\
&\quad \le \sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk} - \Delta_s^r x_{jk}^{cd}\right| + \left|\Delta_s^r x_{jk}^{cd}\right|\right)^{p_{jk}}\right] \\
&\quad \le \frac{1}{2}\sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}2\left|\Delta_s^r x_{jk} - \Delta_s^r x_{jk}^{cd}\right|\right)^{p_{jk}}\right] \\
&\qquad + \frac{1}{2}\sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}2\left|\Delta_s^r x_{jk}^{cd}\right|\right)^{p_{jk}}\right] \\
&\quad \le \frac{1}{2}KF_{jk}(\epsilon)\sum_{j,k} a_{nmjk} + \frac{1}{2}K\sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}^{cd}\right|\right)^{p_{jk}}\right].
\end{aligned} \tag{64}$$

Since $x^{cd} \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2$ and A is RH-regular, we get

$$\lim_{n,m}\sum_{j,k} a_{nmjk}\left[F_{jk}\left(u_{jk}\left|\Delta_s^r x_{jk}\right|\right)^{p_{jk}}\right] = 0; \tag{65}$$

so $x \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2$. This completes the proof. □

Theorem 17. *Let $x = (x_{jk})$ be a bounded sequence, $\mathcal{F} = (F_{jk})$ a sublinear Musielak-Orlicz function which satisfies the Δ_2-condition, and A a nonnegative four-dimensional RH-regular matrix. Then $W^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2 = W^2(A, u, \Delta_s^r, p) \cap l_\infty^2$.*

Proof. Without loss of generality we may take $L = 0$ and establish

$$W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2 = W_0^2(A, u, \Delta_s^r, p) \cap l_\infty^2. \tag{66}$$

Since $W_0^2(A, u, \Delta_s^r, p) \subseteq W_0^2(A, \mathcal{F}, u, \Delta_s^r, p)$, therefore $W_0^2(A, u, \Delta_s^r, p) \cap l_\infty^2 \subseteq W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2$. We need to show that $W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2 \subseteq W_0^2(A, u, \Delta_s^r, p) \cap l_\infty^2$. Notice that if $S \subset \mathbb{N} \times \mathbb{N}$, then

$$\sum_{j,k} a_{nmjk}\left[F_{jk}(\chi_S(j,k))^{p_{jk}}\right] = F_{jk}(1)\sum_{j,k} a_{nmjk}(\chi_S(j,k))^{p_{jk}}, \tag{67}$$

for all n, m. Observe that $\chi_S(j,k) \in W_0^2(A, u, \Delta_s^r, p) \cap l_\infty^2$ whenever $x \in W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2$ by Lemmas 14 and 15, so

$$W_0^2(A, \mathcal{F}, u, \Delta_s^r, p) \cap l_\infty^2 \subseteq W_0^2(A, u, \Delta_s^r, p) \cap l_\infty^2. \tag{68}$$

The proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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