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Research Article

A Hierarchy of Discrete Integrable Coupling System with Self-Consistent Sources

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Integrable coupling system of a lattice soliton equation hierarchy is deduced. The Hamiltonian structure of the integrable coupling is constructed by using the discrete quadratic-form identity. The Liouville integrability of the integrable coupling is demonstrated. Finally, the discrete integrable coupling system with self-consistent sources is deduced.

1. Introduction

Many physical problems may be modeled by soliton equation. The Hamiltonian structures of many systems have been obtained by the famous trace identity [1–6]. The study of integrable couplings of integrable systems has become the focus of common concern in recent years. It originates from the investigations on the symmetry problems and associated centerless Virasoro algebras [7]. Many integrable coupling systems have been constructed by using the methods of a direct method [8], perturbations [9], enlarging spectral problems [10, 11], creating new loop algebras [12, 13], and semidirect sums of Lie algebras [14, 15]. The Hamiltonian structures of the integrable couplings of lattice equations can be constructed by means of the discrete quadratic-form identity [16, 17].

Since Mel'Nikov proposed a new kind of integrable model which was called soliton equations with self-consistent sources [18] in 1983, many soliton equations with self-consistent sources [19–23] have been presented in recent years. For applications, these kinds of systems are usually used to describe interactions between different solitary waves. In this paper, we deduce a hierarchy of discrete integrable coupling system with self-consistent sources which are few compared with the continuous ones.

The paper will be organized as follows. We first get a hierarchy of integrable lattice soliton equation with self-consistent sources in Section 2. In Section 3, a hierarchy of discrete integrable coupling system is derived by making use of the discrete zero curvature representation. By means of the discrete quadratic-form identity we establish the Hamiltonian structures of the hierarchy. Further, the resulting Hamiltonian equations are all proved to be integrable in Liouville sense. Finally, we give the integrable coupling systems with self-consistent sources.

2. A Hierarchy of Integrable Lattice Soliton Equations with Self-Consistent Sources

We first briefly describe our notations. Assume $f_n = f(n)$ is a lattice function; the shift operator E and the inverse of E are defined by

$$\begin{gathered} Ef_n = f(n+1), \quad E^{-1}f_n = f(n-1), \quad n \in Z, \\ E^k f_n = f(n+k), \quad n, k \in Z. \end{gathered} \tag{1}$$

A system of discrete equations

$$\partial_{t_m} u_n = K_m(u_n) \tag{2}$$

is said to have a discrete Lax pair

$$E\varphi_n = U_n(u_n, \lambda)\varphi_n,$$
$$\partial_{t_m}\varphi_n = V_n^{[m]}(u_n, \lambda)\varphi_n, \tag{3}$$

if it is equivalent to the compatibility condition

$$\partial_{t_m}U_n(u_n,\lambda) = \left(EV_n^{[m]}(u_n,\lambda)\right)U_n(u_n,\lambda) - U_n(u_n,\lambda)V_n^{[m]}(u_n,\lambda). \tag{4}$$

In [16], a Lie algebra is presented as

$$G = \operatorname{span}\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8\}, \tag{5}$$

where

$$\omega_1 = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \qquad \omega_2 = \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},$$
$$\omega_3 = \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}, \qquad \omega_4 = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},$$
$$\omega_5 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}, \qquad \omega_6 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}, \tag{6}$$
$$\omega_7 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}, \qquad \omega_8 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}.$$

Set $G_1 = \operatorname{span}\{\omega_1,\omega_2,\omega_3,\omega_4\}$ and $G_2 = \operatorname{span}\{\omega_5,\omega_6,\omega_7,\omega_8\}$; it is easy to see that G, G_1, and G_2 construct three Lie algebra, and

$$G = G_1 \oplus G_2, \quad [G_1, G_2] \equiv G_1G_2 - G_2G_1 \subseteq G_2. \tag{7}$$

So G_2 is an Abelian ideal of the Lie algebra G. The corresponding loop algebra $\widetilde{G}$ is defined by

$$\widetilde{G} = \operatorname{span}\{\omega_i(m), i = 1,2,\ldots,8\}, \quad \omega_i(m) = \omega_i\lambda^m. \tag{8}$$

In [15], a new discrete matrix spectral problem has been proposed:

$$E\phi_n = \widetilde{U}_n(r_n,\lambda)\phi_n, \quad \widetilde{U}_n(r_n,\lambda) = r_n\omega_2(1) + \omega_3(0), \tag{9}$$

by solving the stationary discrete zero curvature equation

$$(E\Gamma_n)U_n - U_n\Gamma_n = 0, \tag{10}$$

where

$$\Gamma_n = a_n\omega_1(0) - a_n\omega_2(0) + b_n\omega_3(0) + c_n\omega_4(1),$$
$$a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-m}, \qquad b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-m}, \tag{11}$$
$$c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-m},$$

and introducing the auxiliary spectral problems associated with the spectral problem (9)

$$\partial_{t_m}\phi_n = \widetilde{V}_n^{[m]}\phi_n, \quad m \geq 0,$$
$$\widetilde{V}_n^{[m]} = \sum_{i=0}^{m}\left[a_n^{(i)}\omega_1(m-i) - a_n^{(i)}\omega_2(m-i) + b_n^{(i)}\omega_3(m-i) + c_n^{(i)}\omega_4(m-i+1)\right] - a_n^{(m)}\omega_1(0) + a_n^{(m)}\omega_2(0), \tag{12}$$

a hierarchy of integrable lattice soliton equations with a potential r_n has been presented:

$$\partial_{t_m}r_n = r_n\left(a_{n+1}^{(m)} - a_n^{(m)}\right), \quad m \geq 0, \tag{13}$$

where

$$a_n^{(0)} = -\frac{1}{2}, \qquad a_n^{(1)} = \frac{1}{r_nr_{n-1}},$$
$$a_n^{(2)} = -\frac{1}{r_nr_{n-1}}\left(\frac{1}{r_nr_{n-1}} + \frac{1}{r_{n-2}r_{n-1}} + \frac{1}{r_{n+1}r_n}\right), \ldots. \tag{14}$$

Equation (13) possesses the following Hamiltonian forms [15]:

$$\partial_{t_m}r_n = \widetilde{J}\frac{\delta\widetilde{F}_n^{(m)}}{\delta r_n} = \widetilde{M}\frac{\delta\widetilde{F}_n^{(m-1)}}{\delta r_n}, \quad m \geq 1, \tag{15}$$

where

$$\widetilde{J} = r_n(1-E)(1+E)^{-1}r_n, \qquad \widetilde{M} = E - E^{-1},$$
$$\widetilde{F}_n^{(m)} = \sum_{n\in z}F_n^{(m)}, \quad F_n^{(m)} = -\frac{a_n^{(m)}}{m}, \quad m \geq 1. \tag{16}$$

Next, we will construct a hierarchy of integrable lattice soliton equations (13) with self-consistent sources. For n distinct real λ_j, consider the auxiliary linear problem

$$E\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix} = \widetilde{U}_n(r_n,\lambda_j)\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix},$$
$$\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix}_{t_m} = \widetilde{V}_n(r_n,\lambda_j)\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix}. \tag{17}$$

Based on the results in [24], we show the following equation:

$$\frac{\delta\widetilde{F}_n^{(m)}}{\delta r_n} + \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta r_n} = 0, \tag{18}$$

where

$$\frac{\delta\lambda_j}{\delta r_n} = \frac{1}{2}\operatorname{Tr}\left(\psi_j \frac{\partial\widetilde{U}\left(r_n,\lambda_j\right)}{\partial r_n}\right),$$
$$\psi_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{3j}\phi_{4j} & -\phi_{3j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{4j}^2 & -\phi_{3j}\phi_{4j} \\ 0 & 0 & \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ 0 & 0 & \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}, \quad j = 1,2,\ldots,N. \tag{19}$$

According to the approach proposed in [24–26], through a direct computation, we obtain the discrete integrable hierarchy with self-consistent sources as follows:

$$\begin{aligned} \partial_{t_m} r_n &= \widetilde{J}\left(\frac{\delta\widetilde{F}_n^{(m)}}{\delta r_n} + \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta r_n}\right) \\ &= \widetilde{J}\left(\frac{\delta\widetilde{F}_n^{(m)}}{\delta r_n} - \sum_{j=1}^{N}\lambda_j\phi_{1j}\phi_{2j}\right), \quad m \geq 1. \end{aligned} \tag{20}$$

Taking $m = 1$ in the above system, under $t_1 \rightarrow t$, we can obtain the following equation with self-consistent sources:

$$\partial_t r_n = \frac{1}{r_{n+1}} - \frac{1}{r_{n-1}} - r_n (1-E)(1+E)^{-1} r_n \sum_{j=1}^{N}\lambda_j\phi_{1j}\phi_{2j}. \tag{21}$$

3. A Hierarchy of Discrete Integrable Coupling System with Self-Consistent Sources

First, we will give out the integrable couplings of the hierarchy (13). Consider the discrete isospectral problem

$$\begin{aligned} E\phi_n &= U_n\left(u_n,\lambda\right)\phi_n, \\ U_n\left(u_n,\lambda\right) &= r_n\omega_2(1) + \omega_3(0) + \omega_4(1) + s_n\omega_6(1), \end{aligned} \tag{22}$$

in which $u_n = (r_n, s_n)^T$ is the potential, $r_n = r(n,t)$ and $s_n = s(n,t)$ are real functions defined over $Z \times R$, λ is a spectral parameter, $\lambda_t = 0$, and $\phi_n = (\phi_1(n),\phi_2(n),\phi_3(n),\phi_4(n))^T$ is the eigenfunction vector.

We solve the stationary discrete zero curvature equation

$$\left(E\Gamma_n\right)U_n - U_n\Gamma_n = 0, \tag{23}$$

where

$$\begin{aligned} \Gamma_n &= a_n\omega_1(0) - a_n\omega_2(0) + b_n\omega_3(0) + c_n\omega_4(1) \\ &\quad + e_n\omega_5(0) - e_n\omega_6(0) + g_n\omega_7(1) + f_n\omega_8(0). \end{aligned} \tag{24}$$

Equation (23) gives

$$\begin{gathered} b_{n+1} = c_n, \\ a_n + \lambda r_n b_{n+1} + a_{n+1} = 0, \\ a_n + \lambda r_n c_n + a_{n+1} = 0, \\ c_{n+1} - b_n + r_n\left(a_n - a_{n+1}\right) = 0, \\ f_{n+1} = g_n, \\ e_{n+1} + e_n + \lambda r_n f_{n+1} + \lambda s_n b_{n+1} = 0, \\ e_{n+1} + e_n + \lambda r_n g_n + \lambda s_n c_n = 0, \\ -f_n + g_{n+1} + r_n\left(e_n - e_{n+1}\right) + s_n\left(a_n - a_{n+1}\right) = 0. \end{gathered} \tag{25}$$

Substituting the expansions

$$\begin{gathered} a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-m}, \qquad b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-m}, \\ c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-m}, \qquad e_n = \sum_{m=0}^{\infty} e_n^{(m)}\lambda^{-m}, \\ f_n = \sum_{m=0}^{\infty} f_n^{(m)}\lambda^{-m}, \qquad g_n = \sum_{m=0}^{\infty} g_n^{(m)}\lambda^{-m} \end{gathered} \tag{26}$$

into (25), we can get the recursion relation

$$\begin{gathered} a_{n+1}^{(m)} + a_n^{(m)} = -r_n b_{n+1}^{(m+1)}, \\ a_{n+1}^{(m)} + a_n^{(m)} = -r_n c_n^{(m+1)}, \\ r_n\left(a_n^m - a_{n+1}^{(m)}\right) + c_{n+1}^{(m)} - b_n^m = 0, \\ e_{n+1}^{(m)} + e_n^{(m)} = -s_n b_{n+1}^{(m+1)} - r_n f_{n+1}^{(m+1)}, \\ e_{n+1}^{(m)} + e_n^{(m)} = -s_n c_n^{(m+1)} - r_n g_n^{(m+1)}, \\ s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) + r_n\left(e_n^{(m)} - e_{n+1}^{(m)}\right) - f_n^{(m)} + g_{n+1}^{(m)} = 0. \end{gathered} \tag{27}$$

The initial values are taken as

$$\begin{gathered} a_n^{(0)} = -\frac{1}{2}, \qquad b_n^{(0)} = 0, \qquad c_n^{(0)} = 0, \\ e_n^{(0)} = -\frac{1}{2}, \qquad f_n^{(0)} = 0, \qquad g_n^{(0)} = 0. \end{gathered} \tag{28}$$

Note that the definition of the inverse operator of $D = (E - 1)$ does not yield any arbitrary constant in computing $a_n^{(m)}$ and $e_n^{(m)}$, $m \geq 1$. Thus, the recursion relation (27) uniquely determines

$$a_n^{(m)}, b_n^{(m)}, c_n^{(m)}, e_n^{(m)}, f_n^{(m)}, g_n^{(m)}, \quad m \geq 1, \tag{29}$$

and the first few quantities are given by

$$\begin{gathered}
a_n^{(1)} = \frac{1}{r_n r_{n-1}}, \qquad b_n^{(1)} = \frac{1}{r_{n-1}}, \qquad c_n^{(1)} = \frac{1}{r_n}, \\
e_n^{(1)} = \frac{1}{r_{n-1} r_n} - \frac{s_{n-1}}{r_{n-1}^2 r_n} - \frac{s_n}{r_n^2 r_{n-1}}, \qquad f_n^{(1)} = \frac{1}{r_{n-1}} - \frac{s_{n-1}}{r_{n-1}^2}, \\
g_n^{(1)} = \frac{1}{r_n} - \frac{s_n}{r_n^2}, \\
a_n^{(2)} = -\frac{1}{r_n r_{n-1}}\left(\frac{1}{r_n r_{n-1}} + \frac{1}{r_{n-2} r_{n-1}} + \frac{1}{r_{n+1} r_n}\right), \\
b_n^{(2)} = -\frac{1}{r_{n-1}^2}\left(\frac{1}{r_{n-2}} + \frac{1}{r_n}\right), \qquad c_n^{(2)} = -\frac{1}{r_n^2}\left(\frac{1}{r_{n-1}} + \frac{1}{r_{n+1}}\right), \\
e_n^{(2)} = -\frac{1}{r_n r_{n-1}}\left(\frac{1}{r_n r_{n-1}} + \frac{1}{r_n r_{n+1}} + \frac{1}{r_{n-2} r_{n-1}}\right) \\
+ \frac{2s_n}{r_{n-1} r_n^3}\left(\frac{1}{r_{n+1}} + \frac{1}{r_{n-1}}\right) + \frac{2s_{n-1}}{r_n r_{n-1}^3}\left(\frac{1}{r_{n-2}} + \frac{1}{r_n}\right) \\
+ \frac{1}{r_n^2 r_{n+1}^2}\left(\frac{s_{n-1}}{r_{n+2}} + \frac{s_{n+1}}{r_{n-1}}\right) + \frac{1}{r_{n-1}^2 r_n r_{n-2}}\left(\frac{s_{n-2}}{r_{n-2}} + \frac{s_n}{r_n}\right), \\
f_n^{(2)} = \frac{2s_{n-1}}{r_{n-1}^2}\left(\frac{1}{r_n} + \frac{1}{r_{n-2}}\right) + \frac{1}{r_{n-1}^2 r_{n-2}}\left(\frac{s_{n-2}}{r_{n-2}} - 1\right) \\
+ \frac{1}{r_{n-1}^2 r_n}\left(\frac{s_n}{r_n} - 1\right), \\
g_n^{(2)} = \frac{2s_n}{r_n^2}\left(\frac{1}{r_{n+1}} + \frac{1}{r_{n-1}}\right) + \frac{1}{r_n^2 r_{n-1}}\left(\frac{s_{n-1}}{r_{n-1}} - 1\right) \\
+ \frac{1}{r_n^2 r_{n+1}}\left(\frac{s_{n+1}}{r_{n+1}} - 1\right).
\end{gathered} \tag{30}$$

Set

$$\begin{aligned}
V_n^{(m)} = \sum_{i=0}^{m} \Big[& a_n^{(i)} \omega_1 (m-i) - a_n^{(i)} \omega_2 (m-i) + b_n^{(i)} \omega_3 (m-i) \\
&+ c_n^{(i)} \omega_4 (m-i+1) + e_n^{(i)} \omega_5 (m-i) - e_n^{(i)} \omega_6 \\
&\times (m-i) + g_n^{(i)} \omega_7 (m-i+1) + f_n^{(i)} \omega_8 (m-i)\Big],
\end{aligned} \tag{31}$$

so

$$\begin{aligned}
&E\left(V_n^{(m)}\right) U_n - U_n V_n^{(m)} \\
&\quad = -r_n c_n^{(m+1)} \omega_3 (0) + r_n c_n^{(m+1)} \omega_4 (1) - \left(e_n^{(m)} + e_{n+1}^{(m)}\right) \omega_7 (1) \\
&\qquad + \left(e_n^{(m)} + e_{n+1}^{(m)}\right) \omega_8 (0).
\end{aligned} \tag{32}$$

Take $\eta_n^{(m)} = -a_n^{(m)} \omega_1(0) + a_n^{(m)} \omega_2(0) - e_n^{(m)} \omega_5(0) + e_n^{(m)} \omega_6(0)$, $m \geq 0$, and let

$$V_n^{[m]} = V_n^{(m)} + \eta_n^{(m)}. \tag{33}$$

We introduce the auxiliary spectral problems associated with the spectral problem (22):

$$\partial_{t_m} \phi_n = V_n^{[m]} \phi_n, \quad m \geq 0. \tag{34}$$

The compatibility conditions of (22) and (34) are

$$\partial_{t_m} U_n = \left(E V_n^{[m]}\right) U_n - U_n V_n^{[m]}, \quad m \geq 0, \tag{35}$$

which give rise to the following hierarchy of integrable lattice equations:

$$\begin{gathered}
\partial_{t_m} r_n = r_n \left(a_{n+1}^{(m)} - a_n^{(m)}\right), \quad m \geq 0, \\
\partial_{t_m} s_n = s_n \left(a_{n+1}^{(m)} - a_n^{(m)}\right) - r_n \left(e_n^{(m)} - e_{n+1}^{(m)}\right), \quad m \geq 0.
\end{gathered} \tag{36}$$

So (35) is the discrete zero curvature representation of (36); the discrete spectral problems (22) and (34) constitute the Lax pairs of (36), and (36) are a hierarchy of Lax integrable nonlinear lattice equations. It is easy to verify that the first nonlinear lattice equation in (36), when $m = 1$, under $t_1 \rightarrow t$, is

$$\begin{gathered}
\partial_t r_n = \left(E - E^{-1}\right) \frac{1}{r_n}, \\
\partial_t s_n = \left(E^{-1} - E\right) \frac{s_n}{r_n^2} + \left(E - E^{-1}\right) \frac{1}{r_n}.
\end{gathered} \tag{37}$$

In (36) the first lattice equations

$$\partial_{t_m} r_n = r_n \left(a_{n+1}^{(m)} - a_n^{(m)}\right), \quad m \geq 0, \tag{38}$$

constitute a hierarchy of integrable lattice soliton equations with a potential r_n; in the view of integrable coupling theory [7, 13, 17], (36) are integrable coupling systems of (13) or (15).

In what follows, we would like to establish the Hamiltonian structures for the integrable coupling systems (36).

Set $a = \sum_{i=1}^{8} a_i \omega_i$, $b = \sum_{i=1}^{8} b_i \omega_i$, and $c = \sum_{i=1}^{8} c_i \omega_i \in G$. We define a map

$$\sigma : G \longrightarrow R^8, \quad a \longmapsto \left(a_1, a_2, \ldots, a_8\right)^T, \quad a \in G. \tag{39}$$

Following [16], we introduce the matrix

$$F = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{40}$$

It is easy to verify that F meets $F^T = F$. Under the definition of the quadratic-form function

$$\{a, b\} = a^T F b, \tag{41}$$

we have $\{ab,c\} = \{a,bc\}$ and $a,b,c \in G$. Set $R_n = \Gamma_n U_n^{-1}$; through a direct calculation, we get

$$\begin{gathered}\left\{R_n, \frac{\partial U_n}{\partial \lambda}\right\} = \frac{e_n + a_n}{\lambda} + r_n\left(c_n + g_n\right) + s_n c_n, \\ \left\{R_n, \frac{\partial U_n}{\partial r_n}\right\} = \lambda\left(c_n + g_n\right), \quad \left\{R_n, \frac{\partial U_n}{\partial s_n}\right\} = \lambda c_n.\end{gathered} \tag{42}$$

By the discrete quadratic-form identity [16]

$$\begin{gathered}\frac{\delta}{\delta r_n}\sum_{n\in Z}\left\{R_n, \frac{\partial U_n}{\partial \lambda}\right\} = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\right)\left\{R_n, \frac{\partial U_n}{\partial r_n}\right\}, \\ \frac{\delta}{\delta s_n}\sum_{n\in Z}\left\{R_n, \frac{\partial U_n}{\partial \lambda}\right\} = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\right)\left\{R_n, \frac{\partial U_n}{\partial s_n}\right\},\end{gathered} \tag{43}$$

with γ being a constant to be determined, we have

$$\begin{aligned}&\frac{\delta}{\delta r_n}\sum_{n\in Z}\left[\frac{e_n + a_n}{\lambda} + r_n\left(c_n + g_n\right) + s_n c_n\right] \\ &\quad = \lambda^{-\lambda}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\left[\lambda\left(c_n + g_n\right)\right], \\ &\frac{\delta}{\delta s_n}\sum_{n\in Z}\left[\frac{e_n + a_n}{\lambda} + r_n\left(c_n + g_n\right) + s_n c_n\right] \\ &\quad = \lambda^{-\lambda}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\gamma}\left(\lambda c_n\right).\end{aligned} \tag{44}$$

By the substitution of

$$\begin{gathered}a_n = \sum_{m=0}^{\infty} a_n^{(m)}\lambda^{-m}, \qquad b_n = \sum_{m=0}^{\infty} b_n^{(m)}\lambda^{-m}, \\ c_n = \sum_{m=0}^{\infty} c_n^{(m)}\lambda^{-m}, \qquad e_n = \sum_{m=0}^{\infty} e_n^{(m)}\lambda^{-m}, \\ f_n = \sum_{m=0}^{\infty} f_n^{(m)}\lambda^{-m}, \qquad g_n = \sum_{m=0}^{\infty} g_n^{(m)}\lambda^{-m}\end{gathered} \tag{45}$$

into (44) and comparing the coefficients of λ^{-m-1} in (44), we get

$$\begin{aligned}&\begin{pmatrix}\dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n}\end{pmatrix}\sum_{n\in Z}\left[e_n^{(m)} + a_n^{(m)} + r_n\left(c_n^{(m+1)} + g_n^{(m+1)}\right) + s_n c_n^{(m+1)}\right] \\ &= \left(-m+\gamma\right)\begin{pmatrix}c_n^{(m+1)} + g_n^{(m+1)} \\ c_n^{(m+1)}\end{pmatrix}.\end{aligned} \tag{46}$$

When $m = 0$ in (46), a direct calculation shows that $\gamma = 0$. So we have

$$\begin{aligned}&\begin{pmatrix}\dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n}\end{pmatrix} \\ &\quad\times\sum_{n\in Z}\left(\left[e_n^{(m)} + a_n^{(m)} + r_n\left(c_n^{(m+1)} + g_n^{(m+1)}\right)\right.\right. \\ &\qquad\left.\left. + s_n c_n^{(m+1)}\right]\left(-m\right)^{-1}\right) \\ &= \begin{pmatrix}c_n^{(m+1)} + g_n^{(m+1)} \\ c_n^{(m+1)}\end{pmatrix}.\end{aligned} \tag{47}$$

Set

$$\begin{aligned}&\widetilde{H}_n^{(m)} \\ &\quad = \sum_{n\in Z}\frac{\left[-e_n^{(m)} - a_n^{(m)} - r_n\left(c_n^{(m+1)} + g_n^{(m+1)}\right) - s_n c_n^{(m+1)}\right]}{m}, \\ &\qquad\qquad m \geq 1.\end{aligned} \tag{48}$$

Now we can rewrite those lattice equations in (36) as

$$\begin{pmatrix}r_n \\ s_n\end{pmatrix}_{t_m} = J\begin{pmatrix}\dfrac{\delta \widetilde{H}_n^{(m)}}{\delta r_n} \\ \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta s_n}\end{pmatrix}, \tag{49}$$

where J is a local difference operator defined by

$$J = \begin{pmatrix}J_{11} & J_{12} \\ J_{21} & J_{22}\end{pmatrix}, \tag{50}$$

where

$$\begin{gathered}J_{11} = 0, \\ J_{12} = J_{21} = r_n(1+E)^{-1}\left(1-E\right)r_n, \\ J_{22} = s_n(1+E)^{-1}\left(1-E\right)r_n + r_n(1+E)^{-1}\left(1-E\right)s_n \\ + r_n(1+E)^{-1}\left(E-1\right)r_n.\end{gathered} \tag{51}$$

Obviously, the operator J is a skew-symmetric operator; that is, $J^* = -J$. Moreover, we can prove that the operator J satisfies the Jacobi identity

$$\left\langle J'\left(u_n\right)\left[Jf_n\right]g_n, h_n\right\rangle + \text{Cycle}\left(f_n, g_n, h_n\right) = 0. \tag{52}$$

So we have the following facts.

Proposition 1. *J is a discrete Hamiltonian operator.*

Set

$$\frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n} = \zeta_n \frac{\delta \widetilde{H}_n^{(m-1)}}{\delta u_n}. \tag{53}$$

From the recursion relation (27) we can get the recursion operator ζ_n in (53).

Therefore, we have

$$\begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n} = J\zeta_n \frac{\delta \widetilde{H}_n^{(m-1)}}{\delta u_n} = J\zeta_n^m \frac{\delta \widetilde{H}_n^{(0)}}{\delta u_n}, \quad m \geq 0. \tag{54}$$

So (49) are a family of Hamiltonian systems. The hierarchy of lattice equations (36) possesses Hamiltonian structures (54). Furthermore, a direct calculation shows that

$$M = J\zeta_n = \begin{pmatrix} 0 & E - E^{-1} \\ E - E^{-1} & E^{-1} - E \end{pmatrix}. \tag{55}$$

It is easy to verify that the operator M is a skew-symmetric operator; that is, $M^* = -M$. So we have the following.

Proposition 2. *$\{\widetilde{H}_n^{(m)}\}_{m\geq 1}$ defined by* (48) *forms an infinite set of conserved functionals of the hierarchy* (36), *and $\widetilde{H}_n^{(m)}$, $m \geq 1$, are involution in pairs with respect to the Poisson bracket.*

Proof. We can find that $M^* = -M$. Namely, $(J\zeta_n)^* = -J\zeta_n$, and then $\zeta_n^* J = J\zeta_n$. Hence

$$\begin{aligned}
\left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J &= \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, J\frac{\delta \widetilde{H}_n^{(l)}}{\delta u_n} \right\rangle \\
&= \left\langle \zeta_n^{m-1} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n}, J\zeta_n^{l-1} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\langle \zeta_n^{m-1} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n}, \zeta_n^* J\zeta_n^{l-2} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\langle \zeta_n^{m} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n}, J\zeta_n^{l-2} \frac{\delta \widetilde{H}_n^{(1)}}{\delta u_n} \right\rangle \\
&= \left\{\widetilde{H}_n^{(m+1)}, \widetilde{H}_n^{(l-1)}\right\}_J = \cdots = \left\{\widetilde{H}_n^{(m+l-1)}, \widetilde{H}_n^{(1)}\right\}_J.
\end{aligned} \tag{56}$$

Similarly, we get

$$\left\{\widetilde{H}_n^{(l)}, \widetilde{H}_n^{(m)}\right\}_J = \left\{\widetilde{H}_n^{(m+l-1)}, \widetilde{H}_n^{(1)}\right\}_J. \tag{57}$$

This implies that

$$\left\{\widetilde{H}_n^{(l)}, \widetilde{H}_n^{(m)}\right\}_J = -\left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J. \tag{58}$$

Thus

$$\begin{aligned}
&\left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J = 0, \quad m, l \geq 1, \\
&\left(\widetilde{H}_n^{(m)}\right)_{t_l} = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, u_{nt_l} \right\rangle = \left\langle \frac{\delta \widetilde{H}_n^{(m)}}{\delta u_n}, J\frac{\delta \widetilde{H}_n^{(l)}}{\delta u_n} \right\rangle \\
&= \left\{\widetilde{H}_n^{(m)}, \widetilde{H}_n^{(l)}\right\}_J = 0, \quad m, l \geq 1.
\end{aligned} \tag{59}$$

□

In summary, we obtain the following theorem.

Theorem 3. *The lattice equations in* (36) *or the discrete Hamiltonian equations in* (49) *are all discrete Liouville integrable Hamiltonian systems.*

Now we search for the integrable coupling systems with self-consistent sources. For n distinct real λ_j, consider the auxiliary linear problem

$$\begin{aligned}
E\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix} &= U_n\left(u_n, \lambda_j\right) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}, \\
\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}_{t_m} &= V_n^{[m]}\left(u_n, \lambda_j\right) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}.
\end{aligned} \tag{60}$$

Based on the results in [24], we show the following equation:

$$\frac{\delta H_n^{(m)}}{\delta u_n} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u_n} = 0, \tag{61}$$

where

$$\begin{aligned}
\frac{\delta \lambda_j}{\delta u_n} &= \frac{1}{2} \operatorname{Tr}\left(\psi_j \frac{\partial U\left(u_n, \lambda_j\right)}{\partial u_n}\right), \\
\psi_j &= \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{3j}\phi_{4j} & -\phi_{3j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{4j}^2 & -\phi_{3j}\phi_{4j} \\ 0 & 0 & \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ 0 & 0 & \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}, \\
& j = 1, 2, \ldots, N.
\end{aligned} \tag{62}$$

According to the approach proposed in [24–26], through a direct computation, we get the discrete integrable hierarchy with self-consistent sources as follows:

$$\begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J \begin{pmatrix} \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta r_n} + \displaystyle\sum_{j=1}^{N} \dfrac{\delta \lambda_j}{\delta r_n} \\ \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta s_n} + \displaystyle\sum_{j=1}^{N} \dfrac{\delta \lambda_j}{\delta s_n} \end{pmatrix}$$
$$= J \begin{pmatrix} \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta r_n} - \displaystyle\sum_{j=1}^{N} \lambda_j \phi_{1j} \phi_{2j} \\ \dfrac{\delta \widetilde{H}_n^{(m)}}{\delta s_n} - \displaystyle\sum_{j=1}^{N} \lambda_j \phi_{3j} \phi_{4j} \end{pmatrix}, \quad m \geq 0. \tag{63}$$

When $m = 1$ in the above system, under $t_1 \rightarrow t$, we can obtain the following coupling equations with self-consistent sources:

$$\partial_t r_n = \left(E - E^{-1}\right) \frac{1}{r_n} - r_n (1 - E)(1 + E)^{-1} r_n \sum_{j=1}^{N} \lambda_j \phi_{1j} \phi_{2j},$$
$$\partial_t s_n = \left(E^{-1} - E\right) \frac{s_n}{r_n^2} + \left(E - E^{-1}\right) \frac{1}{r_n} - r_n (1 - E) \times (1 + E)^{-1} r_n \left(\sum_{j=1}^{N} \lambda_j \phi_{1j} \phi_{2j} + \sum_{j=1}^{N} \lambda_j \phi_{3j} \phi_{4j} \right) - \left[s_n (1 + E)^{-1} (1 - E) r_n + r_n (1 + E)^{-1} (1 - E) s_n \right] \times \sum_{j=1}^{N} \lambda_j \phi_{3j} \phi_{4j}. \tag{64}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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