Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 404521, 14 pages
http://dx.doi.org/10.1155/2014/404521

Research Article

On Pseudospherical Smarandache Curves in Minkowski 3-Space

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Received 3 June 2013; Accepted 21 November 2013; Published 24 February 2014

Academic Editor: Laurent Gosse

In this paper we define nonnull and null pseudospherical Smarandache curves according to the Sabban frame of a spacelike curve lying on pseudosphere in Minkowski 3-space. We obtain the geodesic curvature and the expressions for the Sabban frame's vectors of spacelike and timelike pseudospherical Smarandache curves. We also prove that if the pseudospherical null straight lines are the Smarandache curves of a spacelike pseudospherical curve α, then α has constant geodesic curvature. Finally, we give some examples of pseudospherical Smarandache curves.

1. Introduction

It is known that a Smarandache geometry is a geometry which has at least one Smarandachely denied axiom [1]. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes. Smarandache curves are the objects of Smarandache geometry. By definition, if the position vector of a curve β is composed by the Frenet frame's vectors of another curve α, then the curve β is called a Smarandache curve [2]. Special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors [3–7]. The curves lying on a pseudosphere S_1^2 in Minkowski 3-space E_1^3 are characterized in [8].

In this paper we define nonnull and null pseudospherical Smarandache curves according to the Sabban frame of a spacelike curve lying on pseudosphere in Minkowski 3-space. We obtain the geodesic curvature and the expressions for the Sabban frame's vectors of spacelike and timelike pseudospherical Smarandache curves. We also prove that if the pseudospherical null straight lines are the Smarandache curves of a spacelike pseudospherical curve α, then α has nonzero constant geodesic curvature. Finally, we give some examples of pseudospherical Smarandache curves in Minkowski 3-space.

2. Basic Concepts

The Minkowski 3-space $\mathbb{R}_1^3$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by

$$\langle\cdot,\cdot\rangle = -dx_1^2 + dx_2^2 + dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. Since g is an indefinite metric, recall that a nonzero vector $\vec{x} \in \mathbb{R}_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle\vec{x},\vec{x}\rangle > 0$, timelike if $\langle\vec{x},\vec{x}\rangle < 0$, and null (lightlike) if $\langle\vec{x},\vec{x}\rangle = 0$. In particular, the norm (length) of a vector $\vec{x} \in \mathbb{R}_1^3$ is given by $\|\vec{x}\| = \sqrt{|\langle\vec{x},\vec{x}\rangle|}$ and two vectors $\vec{x}$ and $\vec{y}$ are said to be orthogonal if $\langle\vec{x},\vec{y}\rangle = 0$. Next, recall that an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ can locally be *spacelike*, *timelike*, or *null* (*lightlike*) if all of its velocity vectors $\alpha'(s)$ are, respectively, *spacelike*, *timelike*, or *null* (*lightlike*) for all $s \in I$ [9]. A spacelike or a timelike curve α is parameterized by arclength parameter s if $\langle\alpha'(s),\alpha'(s)\rangle = 1$ or $\langle\alpha'(s),\alpha'(s)\rangle = -1$, respectively. For any two vectors $\vec{x} = (x_1, x_2, x_3)$ and

$\vec{y} = (y_1, y_2, y_3)$ in the space $\mathbb{R}^3_1$, the *pseudovector* product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(-x_2 y_3 + x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\right). \tag{2}$$

Lemma 1. *Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in $\mathbb{R}^3_1$. Then,*

(i) $\langle \vec{x} \times \vec{y}, \vec{z} \rangle = \det(\vec{x}, \vec{y}, \vec{z})$,

(ii) $\vec{x} \times (\vec{y} \times \vec{z}) = -\langle \vec{x}, \vec{z} \rangle \vec{y} + \langle \vec{x}, \vec{y} \rangle \vec{z}$,

(iii) $\langle \vec{x} \times \vec{y}, \vec{x} \times \vec{y} \rangle = -\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle^2$,

where x is the pseudovector product in $\mathbb{R}^3_1$.

Lemma 2. *In the Minkowski 3-space $\mathbb{R}^3_1$, the following properties are satisfied [9]:*

(i) *two timelike vectors are never orthogonal;*

(ii) *two null vectors are orthogonal if and only if they are linearly dependent;*

(iii) *timelike vector is never orthogonal to a null vector.*

The *pseudosphere* with center at the origin and of radius $r = 1$ in the Minkowski 3-space $\mathbb{R}^3_1$ is a quadric defined by

$$S^2_1 = \left\{ \vec{x} \in \mathbb{R}^3_1 \mid -x_1^2 + x_2^2 + x_3^2 = 1 \right\}. \tag{3}$$

Let $\alpha : I \subset \mathbb{R} \to S^2_1$ be a curve lying fully in pseudosphere S^2_1 in $\mathbb{R}^3_1$. Then its position vector α is a spacelike, which means that the tangent vector $T = \alpha'$ can be a spacelike, a timelike, or a null. Depending on the causal character of T, we distinguish the following three cases.

Case 1 (T is a unit spacelike vector). Then we have orthonormal Sabban frame $\{\alpha(s), T(s), \xi(s)\}$ along the curve α, where $\xi(s) = -\alpha(s) \times T(s)$ is the unit timelike vector. The corresponding Frenet formulae of α, according to the Sabban frame, read

$$\begin{bmatrix} \alpha' \\ T' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -k_g(s) \\ 0 & -k_g(s) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{4}$$

where $k_g(s) = \det(\alpha(s), T(s), T'(s))$ is the geodesic curvature of α and s is the arclength parameter of α. In particular, the following relations hold:

$$\alpha \times T = -\xi, \qquad T \times \xi = \alpha, \qquad \xi \times \alpha = T. \tag{5}$$

Case 2 (T is a unit timelike vector). Hence, we have orthonormal Sabban frame $\{\alpha(s), T(s), \xi(s)\}$ along the curve α, where $\xi(s) = \alpha(s) \times T(s)$ is the unit spacelike vector. The corresponding Frenet formulae of α, according to the Sabban frame, read

$$\begin{bmatrix} \alpha' \\ T' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g(s) \\ 0 & k_g(s) & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{6}$$

where $k_g(s) = \det(\alpha(s), T(s), T'(s))$ is the geodesic curvature of α and s is the arclength parameter of α. In particular, the following relations hold:

$$\alpha \times T = \xi, \qquad T \times \xi = \alpha, \qquad \xi \times \alpha = -T. \tag{7}$$

Case 3 (T is a null vector). It is known that the only null curves lying on pseudosphere S^2_1 are the null straight lines, which are the null geodesics.

3. Spacelike and Timelike Pseudospherical Smarandache Curves in Minkowski 3-Space

In this section, we consider spacelike pseudospherical curve α and define its spacelike and timelike pseudospherical Smarandache curves according to the Sabban frame of α in Mikowski 3-space. Let $\alpha = \alpha(s)$ be a unit speed spacelike curve with the Sabban frame $\{\alpha, T, \xi\}$, lying fully on pseudosphere S^2_1 in $\mathbb{R}^3_1$. Denote by $\beta = \beta(s^*)$ arbitrary nonnull curve lying on pseudosphere, where s^* is the arclength parameter of β. Then we have the following definitions of special pseudospherical Smarandache curves of α.

Definition 3. Let $\alpha : I \subset R \mapsto S^2_1$ be a unit speed spacelike curve lying fully on pseudosphere S^2_1. The nonnull pseudospherical $\alpha\xi$-Smarandache curve β of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}} \left(a\alpha(s) + b\xi(s)\right), \tag{8}$$

where s^* is arclength parameter of β, $a, b \in R_0$, and $a^2 - b^2 = 2$.

Definition 4. Let $\alpha : I \subset R \mapsto S^2_1$ be a unit speed spacelike curve lying fully on pseudosphere S^2_1. The nonnull pseudospherical αT-Smarandache curve β of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}} \left(a\alpha(s) + bT(s)\right), \tag{9}$$

where s^* is arclength parameter of β, $a, b \in R_0$, and $a^2 + b^2 = 2$.

Definition 5. Let $\alpha : I \subset R \mapsto S^2_1$ be a unit speed spacelike curve lying fully on pseudosphere S^2_1. The nonnull pseudospherical $T\xi$-Smarandache curve β of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{2}} \left(aT(s) + b\xi(s)\right), \tag{10}$$

where s^* is arclength parameter of β, $a, b \in R_0$, and $a^2 - b^2 = 2$.

Definition 6. Let $\alpha : I \subset R \mapsto S^2_1$ be a unit speed spacelike curve lying fully on pseudosphere S^2_1. The nonnull pseudospherical $\alpha T\xi$-Smarandache curve β of α is defined by

$$\beta\left(s^*(s)\right) = \frac{1}{\sqrt{3}} \left(a\alpha(s) + bT(s) + c\xi(s)\right), \tag{11}$$

where s^* is arclength parameter of β, $a, b, c \in R_0$, and $a^2 + b^2 - c^2 = 3$.

Note that if α is a timelike pseudospherical curve, the corresponding nonnull pseudospherical Smarandache curves according to the Sabban frame of α can be defined in analogous way. In particular, if α is a null pseudospherical curve, then it is a null straight line, so the vectors $\alpha\times T$ and T are linearly dependent. Thus in this case we do not have the orthonormal Sabban frame of α.

Next we obtain the Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ and the geodesic curvature κ_g^β of some special spacelike and timelike pseudospherical Smarandache curves of α. We consider the following two cases: (i) β is a spacelike curve and (ii) β is a timelike curve.

Case 4 (β is a spacelike curve). Then, we have the following theorem.

Theorem 7. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a spacelike pseudospherical $\alpha\xi$-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} & 0 & \frac{b}{\sqrt{2}} \\ 0 & \epsilon & 0 \\ \epsilon\frac{b}{\sqrt{2}} & 0 & \epsilon\frac{a}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{12}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{b - ak_g}{\left|a - bk_g\right|}, \tag{13}$$

where $a, b \in R_0$, $a^2 - b^2 = 2$, and $\epsilon = \pm 1$.

Proof. Differentiating (8) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T, \tag{14}$$

and hence

$$T_\beta \frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T, \tag{15}$$

where

$$\frac{ds^*}{ds} = \sqrt{\langle \beta'(s), \beta'(s)\rangle} = \frac{\left|a - bk_g\right|}{\sqrt{2}}. \tag{16}$$

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \epsilon T, \tag{17}$$

where $\epsilon = +1$ if $a - bk_g > 0$ for all s and $\epsilon = -1$ if $a - bk_g < 0$ for all s.

Differentiating (17) with respect to s and using (4) we find

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \epsilon\left(-\alpha - k_g\xi\right). \tag{18}$$

In particular, from (16) and (18) we get

$$T'_\beta = \frac{\sqrt{2}\epsilon}{\left|a - bk_g\right|}\left(-\alpha - k_g\xi\right). \tag{19}$$

Since β and T_β are spacelike vectors, from (8) and (17) we obtain that the unit timelike vector ξ_β is given by

$$\begin{aligned} \xi_\beta &= -\beta \times T_\beta \\ &= \epsilon\frac{b}{\sqrt{2}}\alpha + \epsilon\frac{a}{\sqrt{2}}\xi. \end{aligned} \tag{20}$$

Consequently, the geodesic curvature k_g^β of β is given by

$$\begin{aligned} k_g^\beta &= \det\left(\beta, T_\beta, T'_\beta\right) \\ &= \frac{b - ak_g}{\left|a - bk_g\right|}. \end{aligned} \tag{21}$$

□

Theorem 8. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a spacelike pseudospherical αT-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 0 \\ \frac{-b}{\sqrt{2-(bk_g)^2}} & \frac{a}{\sqrt{2-(bk_g)^2}} & \frac{-bk_g}{\sqrt{2-(bk_g)^2}} \\ \frac{b^2k_g}{\sqrt{4-2(bk_g)^2}} & \frac{-abk_g}{\sqrt{4-2(bk_g)^2}} & \frac{2}{\sqrt{4-2(bk_g)^2}} \end{bmatrix} \times \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{22}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{-k_g b^2\varepsilon_1 + ak_g b\varepsilon_2 + 2\varepsilon_3}{\left(2 - b^2k_g^2\right)^{5/2}}, \tag{23}$$

where

$$\begin{aligned}\varepsilon_1 &= -b^3k_g'k_g + ab^2k_g^2 - 2a,\\ \varepsilon_2 &= ab^2k_g'k_g - b^3k_g^4 + 2bk_g^2 + b^3k_g^2 - 2b,\\ \varepsilon_3 &= -2bk_g' + ab^2k_g^3 - 2ak_g,\end{aligned} \tag{24}$$

$a, b \in R_0$, $a^2 + b^2 = 2$, and $b^2k_g^2 - 2 < 0$ for all s.

Proof. Differentiating (9) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-b\alpha + aT - bk_g\xi\right), \tag{25}$$

and consequently

$$T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-b\alpha + aT - bk_g\xi\right), \tag{26}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2 - \left(bk_g\right)^2}{2}}. \tag{27}$$

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \frac{1}{\sqrt{2 - \left(bk_g\right)^2}}\left(-b\alpha + aT - bk_g\xi\right), \tag{28}$$

where $2 - (bk_g)^2 > 0$ for all s.

Differentiating (28) with respect to s, it follows that

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left(2 - \left(bk_g\right)^2\right)^{3/2}}\left(\varepsilon_1\alpha + \varepsilon_2 T + \varepsilon_3\xi\right), \tag{29}$$

where

$$\begin{aligned}\varepsilon_1 &= -b^3k_g'k_g + ab^2k_g^2 - 2a,\\ \varepsilon_2 &= ab^2k_g'k_g - b^3k_g^4 + 2bk_g^2 + b^3k_g^2 - 2b,\\ \varepsilon_3 &= -2bk_g' + ab^2k_g^3 - 2ak_g.\end{aligned} \tag{30}$$

From (27) and (29) we get

$$T_\beta' = \frac{1}{\left(2 - \left(bk_g\right)^2\right)^2}\left(\varepsilon_1\alpha + \varepsilon_2 T + \varepsilon_3\xi\right). \tag{31}$$

On the other hand, from (9) and (28) it can be easily seen that the unit timelike vector ξ_β is given by

$$\begin{aligned}\xi_\beta &= -\beta \times T_\beta\\ &= \frac{b^2k_g}{\sqrt{4 - 2\left(bk_g\right)^2}}\alpha - \frac{abk_g}{\sqrt{4 - 2\left(bk_g\right)^2}}T + \frac{2}{\sqrt{4 - 2\left(bk_g\right)^2}}\xi.\end{aligned} \tag{32}$$

Therefore, the geodesic curvature k_g^β of β is given by

$$\begin{aligned}k_g^\beta &= \det\left(\beta, T_\beta, T_\beta'\right)\\ &= \frac{-k_gb^2\varepsilon_1 + ak_gb\varepsilon_2 + 2\varepsilon_3}{\left(2 - b^2k_g^2\right)^{5/2}}.\end{aligned} \tag{33}$$

□

Theorem 9. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a spacelike pseudospherical $T\xi$-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix}\beta\\ T_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix} 0 & \dfrac{a}{\sqrt{2}} & \dfrac{b}{\sqrt{2}}\\ \dfrac{-a}{\sqrt{a^2 - 2k_g^2}} & \dfrac{-bk_g}{\sqrt{a^2 - 2k_g^2}} & \dfrac{-ak_g}{\sqrt{a^2 - 2k_g^2}}\\ \dfrac{2k_g}{\sqrt{2a^2 - 4k_g^2}} & \dfrac{ab}{\sqrt{2a^2 - 4k_g^2}} & \dfrac{a^2}{\sqrt{2a^2 - 4k_g^2}}\end{bmatrix}\begin{bmatrix}\alpha\\ T\\ \xi\end{bmatrix}, \tag{34}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{-2k_g\varepsilon_1 - ba\varepsilon_2 + \varepsilon_3 a^2}{\left(a^2 - 2k_g^2\right)^{5/2}}, \tag{35}$$

where

$$\begin{aligned}\varepsilon_1 &= -2ak_g'k_g - 2bk_g^3 + a^2bk_g,\\ \varepsilon_2 &= -2ak_g^4 + \left(a^3 + 2a\right)k_g^2 - a^3 - a^2bk_g',\\ \varepsilon_3 &= -a^3k_g' - 2bk_g^4 + a^2bk_g^2,\end{aligned} \tag{36}$$

$a, b \in R_0$, $a^2 - b^2 = 2$, and $a^2 - 2k_g^2 > 0$ for all s.

Proof. Differentiating (10) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-a\alpha - bk_gT - ak_g\xi\right), \tag{37}$$

and consequently

$$T_\beta\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-a\alpha - bk_gT - ak_g\xi\right), \tag{38}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{a^2 - 2k_g^2}{2}}. \quad (39)$$

It follows that the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \frac{1}{\sqrt{a^2 - 2k_g^2}} \left(-a\alpha - bk_g T - ak_g \xi\right), \quad (40)$$

where $a^2 - 2k_g^2 > 0$ for all s. Differentiating (40) with respect to s, we find

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\left(a^2 - 2k_g^2\right)^{3/2}} \left(\varepsilon_1\alpha + \varepsilon_2 T + \varepsilon_3\xi\right), \quad (41)$$

where

$$\begin{aligned} \varepsilon_1 &= -2ak_g'k_g - 2bk_g^3 + a^2bk_g, \\ \varepsilon_2 &= -2ak_g^4 + \left(a^3 + 2a\right)k_g^2 - a^3 - a^2bk_g', \\ \varepsilon_3 &= -a^3k_g' - 2bk_g^4 + a^2bk_g^2. \end{aligned} \quad (42)$$

From (39) and (41) we get

$$T_\beta' = \frac{1}{\left(a^2 - 2k_g^2\right)^2} \left(\varepsilon_1\alpha + \varepsilon_2 T + \varepsilon_3\xi\right). \quad (43)$$

Equations (10) and (40) imply

$$\begin{aligned} \xi_\beta &= -\beta \times T_\beta \\ &= \frac{2k_g}{\sqrt{2a^2 - 4k_g^2}}\alpha + \frac{ab}{\sqrt{2a^2 - 4k_g^2}}T + \frac{a^2}{\sqrt{2a^2 - 4k_g^2}}\xi. \end{aligned} \quad (44)$$

Finally, the geodesic curvature k_g^β of the curve β is given by

$$\begin{aligned} k_g^\beta &= \det\left(\beta, T_\beta, T_\beta'\right) \\ &= \frac{-2k_g\varepsilon_1 - ba\varepsilon_2 + \varepsilon_3 a^2}{\left(a^2 - 2k_g^2\right)^{5/2}}. \end{aligned} \quad (45)$$

□

Theorem 10. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a spacelike pseudospherical $\alpha T\xi$-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{3}} & \frac{b}{\sqrt{3}} & \frac{c}{\sqrt{3}} \\ \frac{-b}{\sqrt{\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)}} & \frac{a - ck}{\sqrt{\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)}} & \frac{-bk_g}{\sqrt{\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)}} \\ \frac{c\left(a - ck_g\right) + b^2k_g}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}} & \frac{-abk_g + bc}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}} & \frac{a\left(a - ck_g\right) + b^2}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix} \quad (46)$$

and the corresponding geodesic curvature k_g^β reads

$$\begin{aligned} k_g^\beta = &\left(\left(\left(a^2 - 3\right)k_g - ac\right)\varepsilon_1 + \left(abk_g - bc\right)\varepsilon_2 \right. \\ &\left. + \left(-ack_g + c^2 + 3\right)\varepsilon_3\right) \\ &\times \left(\left(\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)\right)^{5/2}\right)^{-1}, \end{aligned} \quad (47)$$

where

$$\begin{aligned} \varepsilon_1 = &\left(a^2 - 3\right)bk_g'k_g - abck_g' + \left(a^2 - 3\right)ck_g^3 \\ &+ \left(ab^2 - 3ac^2\right)k_g^2 + \left(3a^2c + b^2c\right)k_g - a\left(c^2 + 3\right), \\ \varepsilon_2 = &\left(a^2 - 3\right)ck_g'k_g^2 + \left(ab^2 - 2ac^2\right)k_g'k_g + a^2ck_g' \\ &+ b\left(a^2 - 3\right)k_g^4 - 2abck_g^3 + \left(b^3 + 3b\right)k_g^2 \\ &+ 2abck_g - b\left(c^2 + 3\right), \\ \varepsilon_3 = &\,2b\left(a^2 - 3\right)k_g'k_g^2 - 3abck_g'k_g + b\left(c^2 + 3\right)k_g' \\ &+ c\left(a^2 - 3\right)k_g^4 + \left(ab^2 - 3ac^2\right)k_g^3 + \left(3a^2c + b^2c\right)k_g^2 \\ &- a\left(c^2 + 3\right)k_g, \end{aligned} \quad (48)$$

$a, b, c \in R_0$, $a^2 + b^2 - c^2 = 3$, and $(a - ck_g)^2 + b^2(1 - k_g^2) > 0$ for all s.

Proof. Differentiating (11) with respect to s and by using (4) we find

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(-b\alpha + \left(a - ck_g\right)T - bk_g\xi\right), \quad (49)$$

and thus

$$T_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left(-b\alpha + \left(a - ck_g\right)T - bk_g\xi\right), \quad (50)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)}{3}}. \quad (51)$$

Therefore, the unit spacelike tangent vector of the curve β is given by

$$T_\beta = \frac{1}{\sqrt{\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)}} \times \left(-b\alpha + \left(a - ck_g\right)T - bk_g\xi\right), \quad (52)$$

where $(a - ck_g)^2 + b^2(1 - k_g^2) > 0$ for all s.

Differentiating (52) with respect to s and using (51), it follows that

$$T'_\beta = \frac{1}{\left(\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)\right)^2}\left(\varepsilon_1\alpha + \varepsilon_2 T + \varepsilon_3\xi\right), \quad (53)$$

where

$$\begin{aligned}
\varepsilon_1 = {} & \left(a^2 - 3\right)bk'_g k_g - abck'_g + \left(a^2 - 3\right)ck_g^3 \\
& + \left(ab^2 - 3ac^2\right)k_g^2 + \left(3a^2c + b^2c\right)k_g - a\left(c^2 + 3\right), \\
\varepsilon_2 = {} & \left(a^2 - 3\right)ck'_g k_g^2 + \left(ab^2 - 2ac^2\right)k'_g k_g \\
& + a^2ck'_g + b\left(a^2 - 3\right)k_g^4 - 2abck_g^3 \\
& + \left(b^3 + 3b\right)k_g^2 + 2abck_g - b\left(c^2 + 3\right), \\
\varepsilon_3 = {} & 2b\left(a^2 - 3\right)k'_g k_g^2 - 3abck'_g k_g + b\left(c^2 + 3\right)k'_g \\
& + c\left(a^2 - 3\right)k_g^4 + \left(ab^2 - 3ac^2\right)k_g^3 \\
& + \left(3a^2c + b^2c\right)k_g^2 - a\left(c^2 + 3\right)k_g.
\end{aligned} \quad (54)$$

From (11) and (52) we get

$$\begin{aligned}
\xi_\beta &= -\beta \times T_\beta \\
&= \frac{c\left(a - ck_g\right) + b^2k_g}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}}\alpha \\
&\quad + \frac{-abk_g + bc}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}}T \\
&\quad + \frac{a\left(a - ck_g\right) + b^2}{\sqrt{3\left(a - ck_g\right)^2 + 3b^2\left(1 - k_g^2\right)}}\xi.
\end{aligned} \quad (55)$$

Consequently, the geodesic curvature k_g^β of β reads

$$\begin{aligned}
k_g^\beta &= \det\left(\beta, T_\beta, T'_\beta\right) \\
&= \left(\left(\left(a^2 - 3\right)k_g - ac\right)\varepsilon_1 + \left(abk_g - bc\right)\varepsilon_2 \right. \\
&\quad \left. + \left(-ack_g + c^2 + 3\right)\varepsilon_3\right) \\
&\quad \times \left(\left(\left(a - ck_g\right)^2 + b^2\left(1 - k_g^2\right)\right)^{5/2}\right)^{-1}.
\end{aligned} \quad (56)$$

□

Case 5 (β is a timelike curve). Then, we have the following theorem.

Theorem 11. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. Then the timelike pseudospherical $\alpha\xi$-Smarandache curve of α does not exist.*

Proof. Assume that there exists a timelike pseudospherical $\alpha\xi$-Smarandache curve β of α. Differentiating (8) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T, \quad (57)$$

where s^* is the acrlength parameter of β. The previous equation implies

$$T_\beta \frac{ds^*}{ds} = \frac{a - bk_g}{\sqrt{2}}T. \quad (58)$$

This means that a timelike vector T_β is collinear with a spacelike vector T, which is a contradiction. □

In the theorems which follow, in a similar way as in the Case 4, we obtain the Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ and geodesic curvature κ_g^β of a timelike pseudospherical Smarandache curve β. We omit the proofs of Theorems 11, 12, and 13, since they are analogous to the proofs of Theorems 8, 9, and 10.

Theorem 12. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a timelike pseudospherical αT-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \dfrac{a}{\sqrt{2}} & \dfrac{b}{\sqrt{2}} & 0 \\ \dfrac{-b}{\sqrt{(bk_g)^2-2}} & \dfrac{a}{\sqrt{(bk_g)^2-2}} & \dfrac{-bk_g}{\sqrt{(bk_g)^2-2}} \\ \dfrac{-b^2k_g}{\sqrt{2(bk_g)^2-4}} & \dfrac{abk_g}{\sqrt{2(bk_g)^2-4}} & \dfrac{-2}{\sqrt{2(bk_g)^2-4}} \end{bmatrix} \times \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{59}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{-k_g b^2 \varepsilon_1 + ak_g b\varepsilon_2 + 2\varepsilon_3}{\left((bk_g)^2-2\right)^{5/2}}, \tag{60}$$

where

$$\begin{aligned} \varepsilon_1 &= b^3 k_g' k_g - ab^2 k_g^2 + 2a, \\ \varepsilon_2 &= -ab^2 k_g' k_g + b^3 k_g^4 - \left(b^3+2b\right)k_g^2 + 2b, \\ \varepsilon_3 &= 2bk_g' - ab^2 k_g^3 + 2ak_g, \end{aligned} \tag{61}$$

$a, b \in R_0$, $a^2+b^2=2$, and $(bk_g)^2-2>0$ for all s.

Theorem 13. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a timelike pseudospherical $T\xi$-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{a}{\sqrt{2}} & \dfrac{b}{\sqrt{2}} \\ \dfrac{-a}{\sqrt{2k_g^2-a^2}} & \dfrac{-bk_g}{\sqrt{2k_g^2-a^2}} & \dfrac{-ak_g}{\sqrt{2k_g^2-a^2}} \\ \dfrac{-2k_g}{\sqrt{4k_g^2-2a^2}} & \dfrac{-ab}{\sqrt{4k_g^2-2a^2}} & \dfrac{-a^2}{\sqrt{4k_g^2-2a^2}} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{62}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{-2k_g\varepsilon_1 - ba\varepsilon_2 + a^2\varepsilon_3}{\left(2k_g^2-a^2\right)^{5/2}}, \tag{63}$$

where

$$\begin{aligned} \varepsilon_1 &= 2ak_g' k_g + 2bk_g^3 - a^2 bk_g, \\ \varepsilon_2 &= 2ak_g^4 - \left(a^3+2a\right)k_g^2 + a^3 + a^2 bk_g', \\ \varepsilon_3 &= a^3 k_g' + 2bk_g^4 - a^2 bk_g^2, \end{aligned} \tag{64}$$

$a, b \in R_0$, $a^2-b^2=2$, and $2k_g^2-a^2>0$ for all s.

Theorem 14. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If $\beta : I \subset R \mapsto S_1^2$ is a timelike pseudospherical $\alpha T\xi$-Smarandache curve of α, then its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \dfrac{a}{\sqrt{3}} & \dfrac{b}{\sqrt{3}} & \dfrac{c}{\sqrt{3}} \\ \dfrac{-b}{\sqrt{b^2\left(k_g^2-1\right)-\left(a-ck_g\right)^2}} & \dfrac{a-ck_g}{\sqrt{b^2\left(k_g^2-1\right)-\left(a-ck_g\right)^2}} & \dfrac{-bk_g}{\sqrt{b^2\left(k_g^2-1\right)-\left(a-ck_g\right)^2}} \\ \dfrac{-b^2k_g-\left(a-ck_g\right)c}{\sqrt{3b^2\left(k_g^2-1\right)-3\left(a-ck_g\right)^2}} & \dfrac{abk_g-bc}{\sqrt{3b^2\left(k_g^2-1\right)-3\left(a-ck_g\right)^2}} & \dfrac{-\left(a-ck_g\right)a-b^2}{\sqrt{3b^2\left(k_g^2-1\right)-3\left(a-ck_g\right)^2}} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix} \tag{65}$$

and the corresponding geodesic curvature k_g^{β} reads

$$k_g^{\beta} = \left(\left(\left(a^2 - 3\right) k_g - ac\right) \varepsilon_1 + \left(-bc + abk_g\right) \varepsilon_2 + \left(c^2 + 3 - ack_g\right) \varepsilon_3\right) \times \left(\left(b^2 \left(k_g^2 - 1\right) - \left(a - ck_g\right)^2\right)^{5/2}\right)^{-1}, \tag{66}$$

where

$$\begin{aligned}
\varepsilon_1 &= \left(3 - a^2\right) bk_g' k_g + abck_g' + \left(a^2 - 3\right) ck_g^3 \\
&\quad + \left(ab^2 - 3ac^2\right) k_g^2 + \left(b^2 c - 3a^2 c\right) k_g + a\left(c^2 - 3\right), \\
\varepsilon_2 &= \left(3 - a^2\right) ck_g' k_g^2 + \left(2ac^2 - ab^2\right) k_g' k_g \\
&\quad + a^2 ck_g' + \left(a^2 - 3\right) bk_g^4 \\
&\quad - 2abck_g^3 + \left(b^3 - 3b\right) k_g^2 - 2abck_g + b\left(c^2 - 3\right), \\
\varepsilon_3 &= -abck_g' k_g + \left(c^2 - 3\right) bk_g' + \left(a^2 - 3\right) ck_g^4 \\
&\quad + \left(ab^2 - 3ac^2\right) k_g^3 \\
&\quad + \left(b^2 c - 3a^2 c\right) k_g^2 + \left(c^2 - 3\right) ak_g,
\end{aligned} \tag{67}$$

$a, b, c \in R_0$, $a^2 + b^2 - c^2 = 3$, and $b^2(k_g^2 - 1) - (a - ck_g)^2 > 0$ for all s.

Corollary 15. *If α is a spacelike geodesic curve on pseudosphere S_1^2 in Minkowski 3-space E_1^3, then*

(1) *the spacelike and timelike pseudospherical αT-Smarandache curves are also geodesic on S_1^2;*

(2) *the spacelike and timelike pseudospherical $T\xi$ and $\alpha T\xi$-Smarandache curves have constant geodesic curvatures on S_1^2;*

(3) *the spacelike pseudospherical $\alpha\xi$-Smarandache curve has constant geodesic curvature on S_1^2.*

4. Null Pseudospherical Smarandache Curves in Minkowski 3-Space

In this section, we give definitions of null pseudospherical Smarandache curves which are analogous to the definitions of nonnull pseudospherical Smarandache curves of α, given in Section 3.

Definition 16. Let $\alpha : I \subset R \mapsto S_1^2$ and $\beta : I \subset R \mapsto S_1^2$ be a unit speed spacelike and a null curve, respectively, lying fully in pseudosphere S_1^2. The curve β is pseudospherical $\alpha\xi$-Smarandache curve of α, if it is given by

$$\beta(s) = \frac{1}{\sqrt{2}} (a\alpha(s) + b\xi(s)), \tag{68}$$

where $a, b \in R_0$, and $a^2 - b^2 = 2$.

Definition 17. Let $\alpha : I \subset R \mapsto S_1^2$ and $\beta : I \subset R \mapsto S_1^2$ be a unit speed spacelike and null curve, respectively, lying fully in pseudosphere S_1^2. The curve β is pseudospherical αT-Smarandache curve of α, if it is given by

$$\beta(s) = \frac{1}{\sqrt{2}} (a\alpha(s) + bT(s)), \tag{69}$$

where $a, b \in R_0$, and $a^2 + b^2 = 2$.

Definition 18. Let $\alpha : I \subset R \mapsto S_1^2$ and $\beta : I \subset R \mapsto S_1^2$ be a unit speed spacelike and null curve, respectively, lying fully in pseudosphere S_1^2. The curve β is pseudospherical $T\xi$-Smarandache curve of α, if it is given by

$$\beta(s) = \frac{1}{\sqrt{2}} (aT(s) + b\xi(s)), \tag{70}$$

where $a, b \in R_0$, and $a^2 - b^2 = 2$.

Definition 19. Let $\alpha : I \subset R \mapsto S_1^2$ and $\beta : I \subset R \mapsto S_1^2$ be a unit speed spacelike and null curve, respectively, lying fully in pseudosphere S_1^2. The curve β is pseudospherical $\alpha T\xi$-Smarandache curve of α, if it is given by

$$\beta(s) = \frac{1}{\sqrt{3}} (a\alpha(s) + bT(s) + c\xi(s)), \tag{71}$$

where $a, b, c \in R_0$, and $a^2 + b^2 - c^2 = 3$.

Theorem 20. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. Then the null pseudospherical $\alpha\xi$-Smarandache curve of α does not exist.*

Proof. Assume that there exists null pseudospherical $\alpha\xi$-Smarandache curve of α. Differentiating (68) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{a - bk_g}{\sqrt{2}} T. \tag{72}$$

This means that a null vector $T_\beta = \beta'$ is collinear with a spacelike vector T, which is a contradiction. □

Theorem 21. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If β is a null pseudospherical αT-Smarandache curve of α, then α has constant geodesic curvature given by*

$$k_g^2 = \frac{2}{2 - a^2}, \quad a \in R_0, \ a^2 < 2. \tag{73}$$

Proof. Differentiating (69) with respect to s and using (4) we obtain

$$\beta'(s) = \frac{1}{\sqrt{2}}\left(-b\alpha + aT - bk_g\xi\right), \tag{74}$$

and consequently

$$T_\beta = \frac{1}{\sqrt{2}}\left(-b\alpha + aT - bk_g\xi\right), \tag{75}$$

where $a, b \in R_0$, and $a^2 + b^2 = 2$. The condition $\langle T_\beta, T_\beta\rangle = 0$ implies

$$k_g^2 = \frac{2}{2-a^2}, \quad a \in R_0,\ a^2 < 2, \tag{76}$$

which proves the theorem. □

The next two theorems can be proved in a similar way, so we omit their proofs.

Theorem 22. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If β is a null pseudospherical $T\xi$-Smarandache curve of α, then α has constant geodesic curvature given by*

$$k_g^2 = \frac{a^2}{2}, \quad a \in R_0,\ a^2 > 2. \tag{77}$$

Theorem 23. *Let $\alpha : I \subset R \mapsto S_1^2$ be a unit speed spacelike curve lying fully in S_1^2 with the Sabban frame $\{\alpha, T, \xi\}$ and the geodesic curvature k_g. If β is a null pseudospherical $\alpha T\xi$-Smarandache curve of α, then α has constant geodesic curvature given by*

$$k_g = \frac{2ac \pm \sqrt{4a^2c^2 - 4\left(c^2-b^2\right)\left(a^2+b^2\right)}}{2\left(a^2-3\right)}, \tag{78}$$

where $a, b, c \in R_0$, $a^2 + b^2 - c^2 = 3$, and $a^2 \neq 3$.

Corollary 24. *There are no spacelike pseudospherical geodesic curves whose pseudospherical αT, $T\xi$, and $\alpha T\xi$-Smarandache curves are the null curves.*

5. Examples

Example 1. Let α be a unit speed spacelike curve lying on pseudosphere S_1^2 in the Minkowski 3-space $\mathbb{E}_1^3$ with parameter equation (see Figure 1)

$$\alpha(s) = \left(\frac{s^2}{2}, s, 1 - \frac{s^2}{2}\right). \tag{79}$$

The orthonormal Sabban frame $\{\alpha, T, \xi\}$ along the curve α is given by

$$\begin{aligned}\alpha(s) &= \left(\frac{s^2}{2}, s, 1 - \frac{s^2}{2}\right),\\ T(s) &= \alpha'(s) = (s, 1, -s),\\ \xi(s) &= -\alpha(s) \times T(s) = \left(-1 - \frac{s^2}{2}, -s, \frac{s^2}{2}\right).\end{aligned} \tag{80}$$

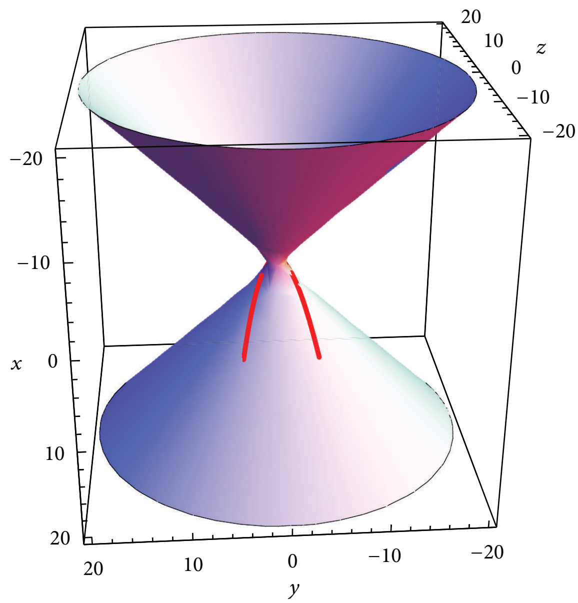

Figure 1: The curve α on S_1^2.

In particular, the geodesic curvature k_g of the curve α has the form

$$k_g(s) = \det\left(\alpha(s), T(s), T'(s)\right) = -1. \tag{81}$$

Case 1. Taking $a = \sqrt{3}$ and $b = 1$ and using (8), we obtain that pseudospherical $\alpha\xi$-Smarandache curve β is given by (see Figure 2)

$$\beta(s) = \frac{\sqrt{2}}{2}\left(\frac{\left(\sqrt{3}-1\right)}{2}s^2 - 1, \left(\sqrt{3}-1\right)s, \sqrt{3} - \frac{\left(\sqrt{3}-1\right)}{2}s^2\right). \tag{82}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle > 0$, which means that β is a spacelike curve. According to Theorem 7, its Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ T_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{6}}{2} & 0 & \frac{\sqrt{2}}{2}\\ 0 & 1 & 0\\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{6}}{2}\end{bmatrix}\begin{bmatrix}\alpha\\ T\\ \xi\end{bmatrix}, \tag{83}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{b - ak_g}{\left|a - bk_g\right|} = 1. \tag{84}$$

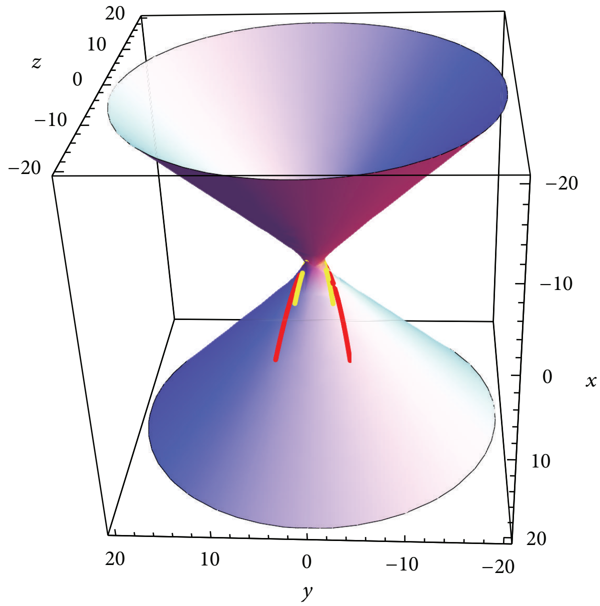

FIGURE 2: The spacelike $\alpha\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

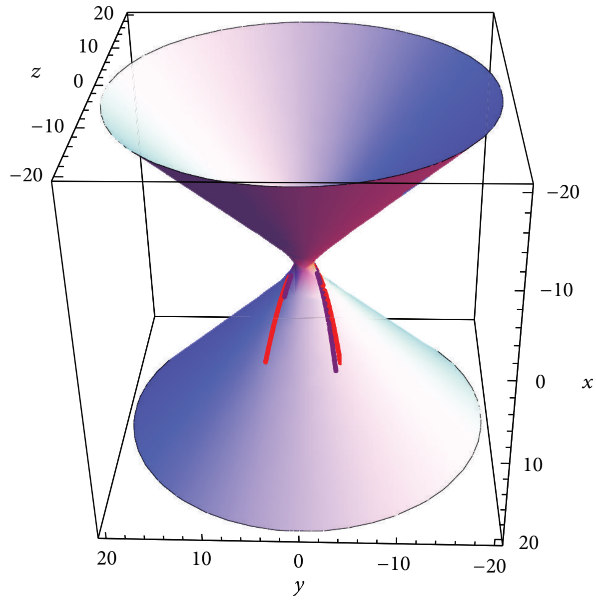

FIGURE 3: The spacelike αT-pseudospherical Smarandache curve β and the curve α on S_1^2.

Case 2. Taking $a = 1$ and $b = -1$ and using (9) we get that pseudospherical αT-Smarandache curve is given by (see Figure 3)

$$\beta(s^*(s)) = \frac{\sqrt{2}}{2}\left(\frac{s^2-2s}{2}, s-1, -\frac{s^2}{2}+s\right). \quad (85)$$

It can be easily checked that $\langle\beta'(s), \beta'(s)\rangle > 0$, which means that β is a spacelike curve. According to Theorem 8, its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta \\ T_\beta \\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 1 & 1 & 1 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\sqrt{2}\end{bmatrix}\begin{bmatrix}\alpha \\ T \\ \xi\end{bmatrix}, \quad (86)$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = -\sqrt{2}. \quad (87)$$

Case 3. Taking $a = 2$ and $b = \sqrt{2}$, from (10) we find that the pseudospherical $T\xi$-Smarandache curve is given by (see Figure 4)

$$\beta(s^*(s)) = \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}s^2 + 2s - \sqrt{2}, 2-\sqrt{2}s, \frac{\sqrt{2}}{2}s^2 - 2s\right). \quad (88)$$

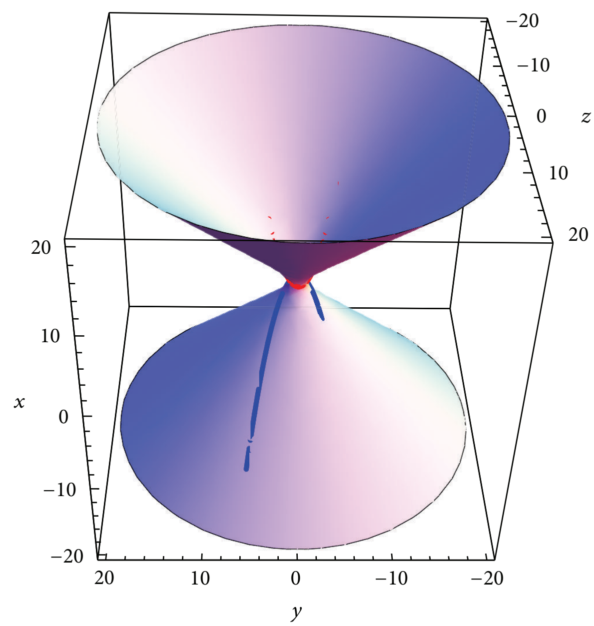

FIGURE 4: The spacelike $T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

It can be easily checked that $\langle\beta'(s), \beta'(s)\rangle > 0$, which means that β is a spacelike curve. According to Theorem 9, its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta \\ T_\beta \\ \xi_\beta\end{bmatrix} = \begin{bmatrix}0 & \sqrt{2} & 1 \\ -\sqrt{2} & -1 & -\sqrt{2} \\ -1 & -\sqrt{2} & -2\end{bmatrix}\begin{bmatrix}\alpha \\ T \\ \xi\end{bmatrix}, \quad (89)$$

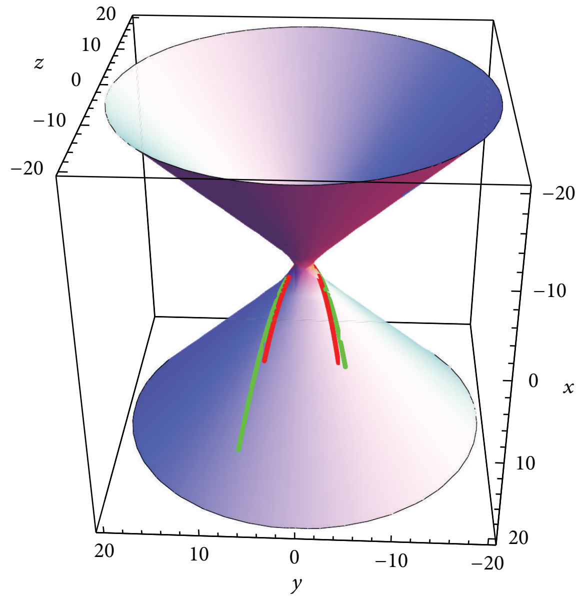

Figure 5: The spacelike $\alpha T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

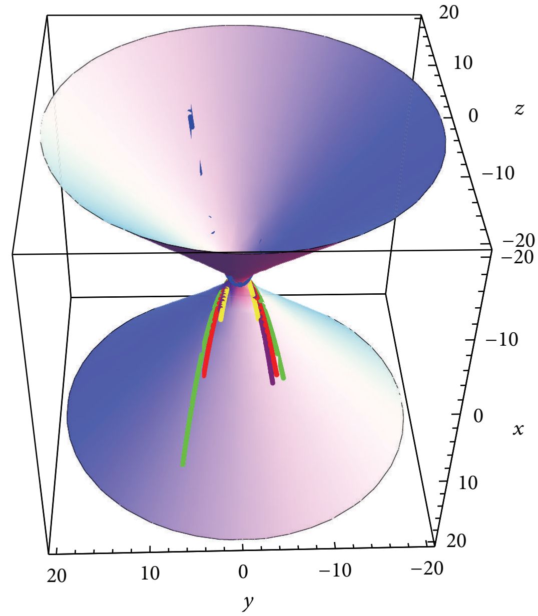

Figure 6: Special spacelike pseudospherical Smarandache curves of α and the curve α on S_1^2.

and the corresponding geodesic curvature k_g^β has the following form:

$$k_g^\beta = 1. \tag{90}$$

Case 4. Taking $a = \sqrt{2}$, $b = \sqrt{2}$, and $c = 1$ and using (11), we find that the pseudospherical $\alpha T\xi$-Smarandache curve β has parameter equation (see Figure 5):

$$\beta(s^*(s)) = \frac{1}{\sqrt{3}}\left(\frac{(\sqrt{2}+1)}{2}s^2 + \sqrt{2}s + 1, (1+\sqrt{2})s + \sqrt{2}, -\frac{(1+\sqrt{2})}{2}s^2 - \sqrt{2}s + \sqrt{2}\right). \tag{91}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle > 0$, which means that β is a spacelike curve. By Theorem 10, its frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ -\sqrt{2}-2 & 1 & -\sqrt{2}-2 \\ -\frac{2\sqrt{6}+3\sqrt{3}}{3} & \frac{\sqrt{6}}{3} & -\frac{3\sqrt{6}+2\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{92}$$

while the corresponding geodesic curvature k_g^β has the following form:

$$k_g^\beta = -1200\sqrt{2} - 1697. \tag{93}$$

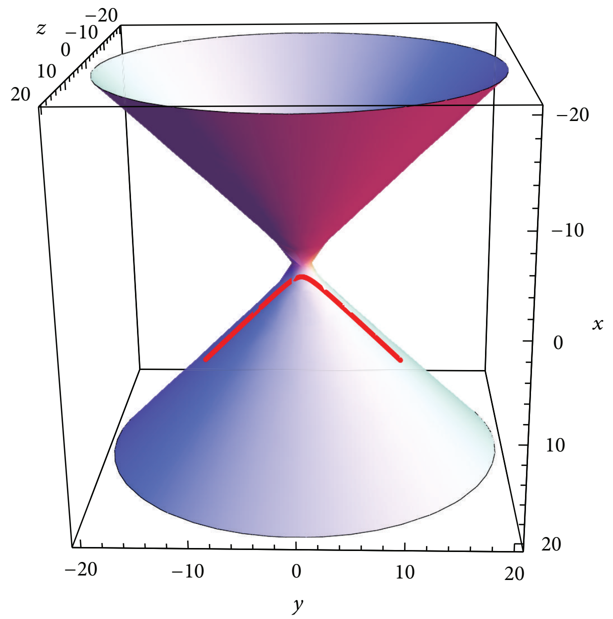

Figure 7: The spacelike circle α on S_1^2.

Special spacelike Smarandache curves of α and the curve α on S_1^2 are shown in Figure 6.

Example 2. Let us consider a unit speed spacelike circle α lying on pseudosphere S_1^2 in the Minkowski 3-space $\mathbb{E}_1^3$ with parameter equation (see Figure 7):

$$\alpha(s) = (\cosh s, \sinh s, \sqrt{2}). \tag{94}$$

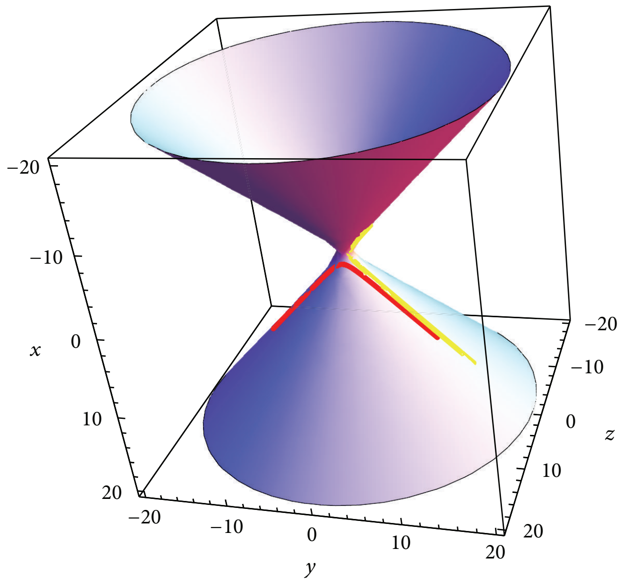

Figure 8: The timelike αT-pseudospherical Smarandache curve β and the curve α on S_1^2.

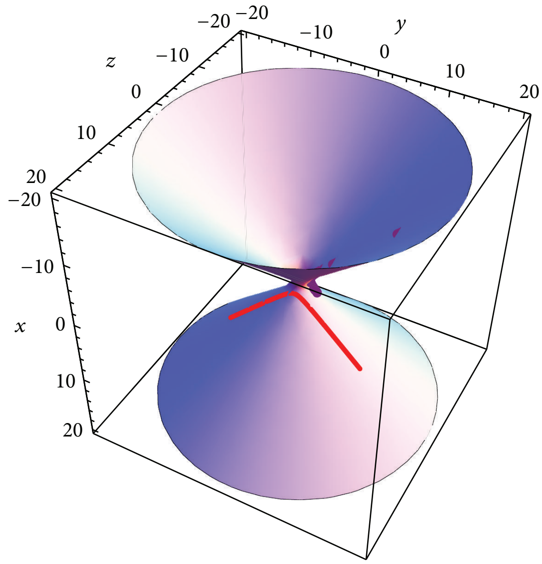

Figure 9: The timelike $T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

The orthonormal Sabban frame $\{\alpha, T, \xi\}$ along the curve α is given by

$$\begin{aligned}\alpha(s) &= \left(\cosh s, \sinh s, \sqrt{2}\right),\\ T(s) &= \alpha'(s) = (\sinh s, \cosh s, 0),\\ \xi(s) &= -\alpha(s) \times T(s) = \left(-\sqrt{2}\cosh s, -\sqrt{2}\sinh s, -1\right).\end{aligned} \tag{95}$$

In particular, the geodesic curvature k_g of α reads

$$k_g(s) = \det\left(\alpha(s), T(s), T'(s)\right) = -\sqrt{2}. \tag{96}$$

Case 1. Taking $a = (\sqrt{2}/2)$ and $b = (\sqrt{6}/2)$ and using (9), we obtain that pseudospherical αT-Smarandache curve β is given by (see Figure 8)

$$\beta(s) = \left(\frac{1}{2}\cosh s + \frac{\sqrt{3}}{2}\sinh s, \frac{1}{2}\sinh s + \frac{\sqrt{3}}{2}\cosh s, \frac{\sqrt{2}}{2}\right). \tag{97}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle < 0$, which means that β is a timelike curve. According to Theorem 12, its Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ T_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} & \sqrt{3}\\ \frac{3}{2} & \frac{\sqrt{3}}{2} & \sqrt{2}\end{bmatrix}\begin{bmatrix}\alpha\\ T\\ \xi\end{bmatrix}, \tag{98}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = \frac{-k_g b^2 \varepsilon_1 + a k_g b \varepsilon_2 + 2\varepsilon_3}{\left(\left(bk_g\right)^2 - 2\right)^{5/2}} = -10. \tag{99}$$

Case 2. Taking $a = \sqrt{3}$ and $b = 1$ and using (10), we obtain that pseudospherical $T\xi$-Smarandache curve β is given by (see Figure 9)

$$\beta(s) = \left(\sqrt{3}\sinh s - \sqrt{2}\cosh s, \sqrt{3}\cosh s - \sqrt{2}\sinh s, -1\right). \tag{100}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle < 0$, which means that β is a timelike curve. According to Theorem 13, its Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix}\beta\\ T_\beta\\ \xi_\beta\end{bmatrix} = \begin{bmatrix}0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2}\\ -\sqrt{3} & \sqrt{2} & \sqrt{6}\\ 2\sqrt{2} & -\frac{\sqrt{6}}{2} & -\frac{3\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\alpha\\ T\\ \xi\end{bmatrix}, \tag{101}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = 11. \tag{102}$$

Case 3. Taking $a = -\sqrt{2}$, $b = \sqrt{2}$, and $c = 1$ and using (11), we obtain that pseudospherical $\alpha T\xi$-Smarandache curve β is given by (see Figure 10)

$$\beta(s) = \left(\sqrt{2}\sinh s - 2\sqrt{2}\cosh s, \sqrt{2}\cosh s - 2\sqrt{2}\sinh s, -3\right). \tag{103}$$

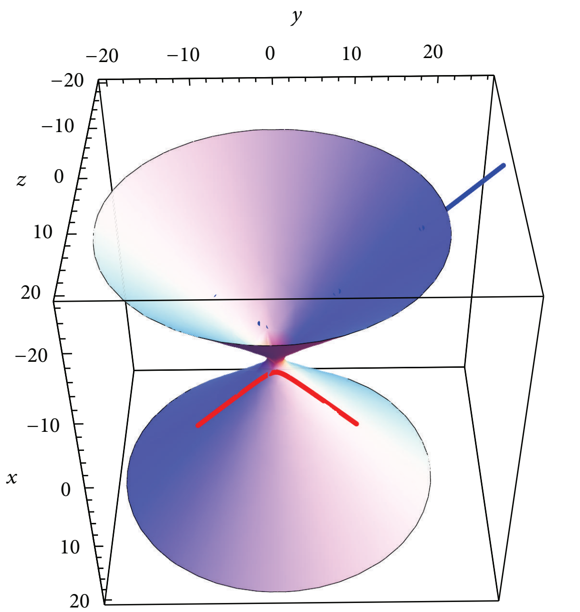

Figure 10: The timelike $\alpha T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

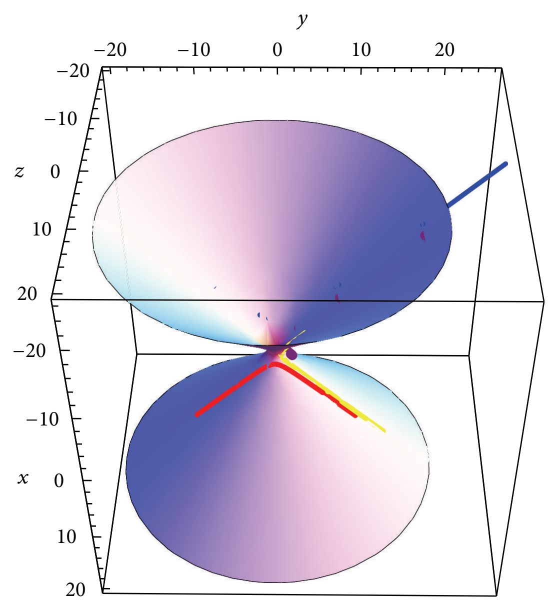

Figure 11: Special timelike pseudospherical Smarandache curves of α and the curve α on S_1^2.

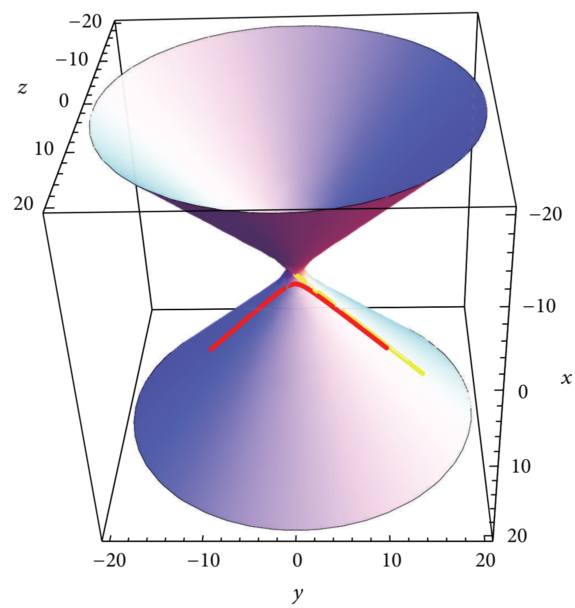

Figure 12: The null $T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle < 0$, which means that β is a timelike curve. According to Theorem 14, its Sabban frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ -1 & 0 & -\sqrt{2} \\ \frac{2\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}, \tag{104}$$

and the corresponding geodesic curvature k_g^β reads

$$k_g^\beta = -5\sqrt{2}. \tag{105}$$

Special timelike Smarandache curves of α and the curve α on S_1^2 are shown in Figure 11.

Example 3. Let us consider a unit speed spacelike circle α lying on pseudosphere S_1^2 in the Minkowski 3-space $\mathbb{E}_1^3$ with parameter (94). Then its geodesic curvature is given by (96).

Case 1. If β is null pseudospherical αT-Smarandache curve of α, then according to Theorem 21 the curve α has nonzero constant geodesic curvature given by

$$k_g^2 = \frac{2}{2-a^2}, \quad a \in R_0, \ a^2 < 2. \tag{106}$$

The last relation together with (96) implies $a = 1$. By Definition 17 there holds $a^2 + b^2 = 2$ and therefore we can take $b = 1$. Finally, by using (69) we obtain that pseudospherical αT-Smarandache curve β of α is given by (see Figure 12)

$$\beta(s) = \frac{\sqrt{2}}{2}\left(\sinh s + \cosh s, \cosh s + \sinh s, \sqrt{2}\right). \tag{107}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle = 0$, which means that β is a null straight line.

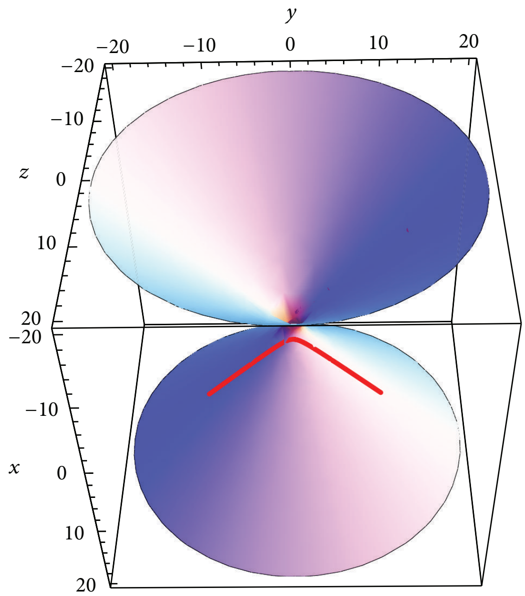

FIGURE 13: The null $T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

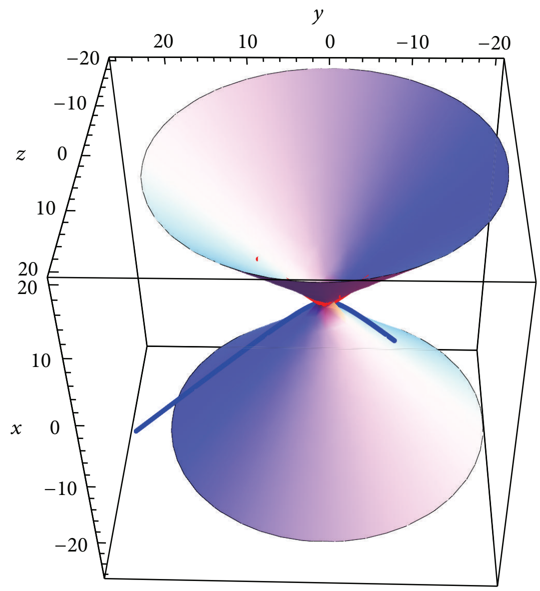

FIGURE 14: The null $\alpha T\xi$-pseudospherical Smarandache curve β and the curve α on S_1^2.

Case 2. If β is null pseudospherical $T\xi$-Smarandache curve of α, then according to Theorem 22 the curve α has nonzero constant geodesic curvature given by

$$k_g^2 = \frac{a^2}{2}, \quad a \in R_0, \ a^2 > 2. \tag{108}$$

The last relation together with (96) implies $a = 2$. By Definition 18 there holds $a^2 - b^2 = 2$ and therefore we can take $b = \sqrt{2}$. Finally, by using (70) we obtain that pseudospherical $T\xi$-Smarandache curve β of α is given by (see Figure 13)

$$\beta(s) = \left(2\sinh s - 2\cosh s, 2\cosh s - 2\sinh s, -\sqrt{2}\right). \tag{109}$$

It can be easily checked that $\langle \beta'(s), \beta'(s) \rangle = 0$, which means that β is a null straight line.

Case 3. If β is null pseudospherical $\alpha T\xi$-Smarandache curve of α, then according to Theorem 23 the curve α has nonzero constant geodesic curvature given by

$$k_g = \frac{2ac \pm \sqrt{4a^2c^2 - 4\left(c^2 - b^2\right)\left(a^2 + b^2\right)}}{2\left(a^2 - 3\right)}. \tag{110}$$

The last relation together with (96) implies $a = 2$, $b = \sqrt{3}$. By Definition 19 there holds $a^2 + b^2 - c^2 = 3$ and therefore we can take $c = -2$. Finally, by using (71) we obtain that pseudospherical $\alpha T\xi$-Smarandache curve β of α is given by (see Figure 14)

$$\beta(s) = \left(\left(2 + 2\sqrt{2}\right)\cosh s + \sqrt{3}\sinh s, \left(2 + 2\sqrt{2}\right)\sinh s + \sqrt{3}\cosh s, 2\sqrt{2} + 2\right). \tag{111}$$

It can be easily checked that $\langle \beta'(s), \beta'(s) \rangle = 0$, which means that β is a null straight line.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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