Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 403528, 12 pages
http://dx.doi.org/10.1155/2014/403528

Research Article

Stochastic Nonlinear Thermoelastic System Coupled Sine-Gordon Equation Driven by Jump Noise

Shuilin Cheng,[1] Yantao Guo,[1,2] and Yanbin Tang[1]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China*
[2] *School of Mathematics and Statistics, Xuchang University, Xuchang, Henan 46100, China*

Correspondence should be addressed to Yanbin Tang; tangybhust@sina.com

Received 23 November 2013; Accepted 13 January 2014; Published 24 February 2014

Academic Editor: Ming Mei

This paper considers a stochastic nonlinear thermoelastic system coupled sine-Gordon equation driven by jump noise. We first prove the existence and uniqueness of strong probabilistic solution of an initial-boundary value problem with homogeneous Dirichlet boundary conditions. Then we give an asymptotic behavior of the solution.

1. Introduction

In this paper, we consider the following stochastic nonlinear thermoelastic coupled sine-Gordon system driven by Lévy noise:

$$\begin{gathered} du_t - \left(\alpha_1 u_{xxt} + u_{xx} + \beta \sin u + \alpha_2 \theta\right) dt = \int_Z \sigma_1 (t, u(t), z) \widetilde{\eta_1} (dz, dt), \\ d\theta - \left(\theta_{xx} - \alpha_2 u_t + g(t,\theta)\right) dt = \int_Z \sigma_2 (t, \theta(t), z) \widetilde{\eta_2} (dz, dt), \\ u(x,t) = 0, \quad \theta(x,t) = 0, \quad x = 0,\ x = L,\ t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \\ \theta(x,0) = \theta_0(x), \quad x \in [0, L], \end{gathered} \tag{1}$$

where $u_t = \partial u/\partial t$, $\widetilde{\eta}_i$ $(i = 1, 2)$ are the Lévy processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ (see Section 2 for the precise definition) [1–4], g and σ_i $(i = 1, 2)$ are given real-valued random functions that will be defined in later.

Recently, the study of high-temperature apparatus or heat resistant structures is becoming important and it is necessary to analyze not only the deterministic thermal stress but also the stochastic thermal stress. In high-temperature apparatus, it is very difficult to predict accurately the thermal environment and mechanical load on its components. Furthermore, many indeterminate factors must be considered, for example, the random high-cycle vibrations of the temperature of the upper core in fast breeder reactors and fluctuations in the heat transfer coefficients around the stationary blades of gas turbines. Therefore, the stochastic case of temperature and thermal stress is indispensable in considering these indeterminate factors of the thermal environment (see [5]).

All the time, a description of wave propagation phenomena in random media is usually based on the study of stochastically perturbed wave equations (see [6, 7]). In fact, lots of wave phenomena are temperature dependent or heat generating; then the wave equations are coupled with a stochastic heat equation. Caraballo et al. [8] studied the existence of invariant manifolds for coupled parabolic and hyperbolic stochastic partial differential equations. Bates et al. [9] proved the existence of random attractors for stochastic reaction-diffusion equations on unbounded domains, and Wang and Tang [10, 11] described the properties of the random attractors.

Meanwhile the sine-Gordon equation is an important model in physics; Fan [12] considered the random attractor for the stochastic sine-Gordon equation. What are the other properties of stochastic sine-Gordon equation? As we know, Coayla-Teran [13] and Liu et al. [3] studied the mild solution of stochastic fractional partial differential equation with fractional and jump noises and considered the strong probability solution driven by white or Lévy noise for the stochastic nonlinear nonlocal parabolic equation and 2D stochastic N-S equation. In deterministic coupled case, the well-posedness of the solution for the nonlinear thermoelastic coupled sine-Gordon system has been studied by many authors, and the global attractor was treated in [14]. Moreover, the more general thermoelastic system coupled model was considered. Gao and Muñoz Rivera [15] and Rivera [16] studied the well-posedness and energy decay rates. In deterministic case, nonlinear thermoelastic system coupled sine-Gordon equation is very weak coupling thermoelastic system; the more general model was investigated by Gao [17]; he considered the global attractor for the semilinear thermoelastic problem.

However, as far as we know, no one refers to the strong solution for stochastic nonlinear thermoelastic coupled sine-Gordon system by jump noise.

This paper is organized as follows. In the next section, we recall some fundamental results related to the solution of the stochastic equation and Lévy noise. In Section 3, we use the Galerkin method to prove the existence and uniqueness of solution to the problem (1). In Section 4, we give an asymptotic behavior of the solution of the problem (1).

In this paper, C is a constant from line to line.

2. Preliminaries

In this section, we recall some fundamental results related to some basic function spaces and the property of Lévy process; for more information, one can see [1, 2, 9, 12]. Let $A = -\partial_{xx}$, with the domain $D(A) = H^2(0,L) \cap H_0^1(0,L)$, and $L^2(0,L), H_0^1(0,L)$ are separable Hilbert spaces with the norm $\|\cdot\|$ and $\|\cdot\|_1$, respectively; from Poincaré's inequality, $\|\nabla\cdot\|$ is equivalent to $\|\cdot\|_1$. Next, we recall some basic concepts related to Lévy process. The readers are referred to [1] for more details.

Let $(Z, \mathscr{Z})$ be a measurable space, and let ν be a σ-finite positive measure on it. If X is a topological space, then by $\mathscr{B}(X)$ we will denote the Borel σ-field on X, and λ is a Lebesgue measure on $(R, \mathscr{B}(R))$. Suppose that $(\Omega, \mathscr{F}, F, P)$ is a filtered probability space, where $F = (\mathscr{F}_t)_{t\geq 0}$ is a filtration and $\eta_i : \Omega\times\mathscr{B}(R^+)\times\mathscr{Z} \to \overline{N}$ is a time homogeneous Poisson random measure with the intensity measure ν defined over the filtered probability space $(\Omega, \mathscr{F}, F, P)$.

We will denote by $\widetilde{\eta_i} = \eta_i - \gamma_i$ the compensated Poisson random measure associated with η_i, where the compensator γ_i is given by

$$\mathscr{B}(R^+)\times\mathscr{Z} \ni (A, I) \longmapsto \gamma_i(A, I) = \nu(A)\lambda(I) \in R^+. \quad (2)$$

We assume that $(H, |\cdot|_H)$ is a Hilbert space. It is then known (see, for example, [1, 2]) that there exists a unique continuous linear operator $\mathscr{I}$ which associates with each progressively measurable process $\xi : R_+ \times Z \times \Omega \to H$ satisfying

$$E\int_0^T \int_Z |\xi(r,z)|_H^2 \nu(dz)\,dr < \infty, \quad T > 0. \quad (3)$$

Moreover, $\mathscr{I}(\xi)$ is an H-valued adapted and càdlàg process such that for any random step process $\xi(r,z)$ satisfying the condition (3) with a representation

$$\xi(r,z) = \sum_{j=1}^n 1_{(t_{j-1},t_j]}(r)\,\xi_j(z), \quad r \geq 0, \quad (4)$$

where $\{0 = t_0 < t_1 < \cdots < t_n < \infty\}$ is a partition of $[0,\infty)$, and for all j, ξ_j being an $\mathscr{F}_{t_{j-1}}$ measurable random variable, one has

$$\mathscr{I}(\xi)(t) = \sum_{j=1}^n \int_Z \xi_j(z)\,\widetilde{\eta}\left(dz, \left(t_{j-1}\wedge t, t_j \wedge t\right]\right), \quad t \geq 0. \quad (5)$$

In general case we write

$$\int_0^t \int_Z \xi(r,z)\,\widetilde{\eta}(dz,dr) := \mathscr{I}(\xi)(t), \quad t \geq 0. \quad (6)$$

The continuity (more precisely, isometry in Hilbert spaces) of the operator $\mathscr{I}$ mentioned above means that

$$E\left|\int_0^t \int_Z \xi(r,z)\,\widetilde{\eta}(dz,dr)\right|_H^2 = E\int_0^t \int_Z |\xi(r,z)|_H^2 \nu(dz)\,dr, \quad t \geq 0. \quad (7)$$

The class of all progressively measurable processes $\xi : R_+\times Z\times\Omega \to H$ satisfying the condition (3) will be denoted by $\mathscr{M}^2(R_+, L^2(Z,\nu,H))$. If $T > 0$, the class of all progressively measurable processes $\xi : [0,T]\times Z\times\Omega \to H$ satisfies the condition (3) just for this one T, which will be denoted by $\mathscr{M}^2(0,T,L^2(Z,\nu,H))$.

The main technical tool in our paper is the Itô formula. Let us consider the Hilbert spaces $V \subseteq H \cong H' \subseteq V'$ and a V'-valued càdlàg process of the form

$$X(t) = X_0 + \int_0^t Y(s)\,ds + \int_0^t \int_Z G(s,z)\,\widetilde{\eta}(dz,ds), \quad t \in [0,T], \quad (8)$$

where Y is a V'-valued process and G is an H-valued process; we have the following.

Theorem 1 (see [1, 2]). *Suppose that $X_0 \in L^2(\Omega, \mathscr{F}; V')$ and $G \in L^2(\Omega\times[0,T]; L^2(Z,\nu;H))$ are progressively measurable processes. Suppose that X is a V'-valued process given by (8) and there exists a V-valued process $\overline{X} \in L^2(\Omega\times[0,T];V)$ such that $\overline{X} = X, dP\otimes dt$ in V. Then X is an H-valued càdlàg $\mathscr{F}_t$-adapted process (up to distinguishable) and*

$$\begin{aligned}\|X(t)\|^2 &= \|X_0\|^2 + 2\int_0^t \left\langle \overline{X}, Y(s)\right\rangle ds \\ &+ 2\int_0^t \int_Z (X(s-), G(s,z))\,\widetilde{\eta}(dz,ds) \\ &+ \int_0^t \int_Z \|G(s,z)\|^2 \eta(dz,ds).\end{aligned} \quad (9)$$

3. Existence and Uniqueness of Solution

In this section, we use the Galerkin method to prove the local existence and uniqueness of solution; then making use of a priori estimates, we prove that there exists a convergence subsequence such that the solution is global.

As is well known, system (1) is equivalent to the following Itô system:

$$\begin{aligned} du &= v dt, \\ dv &= \left(\alpha_1 v_{xx} + u_{xx} + \beta \sin u + \alpha_2 \theta\right) dt \\ &\quad + \int_Z \sigma_1 (t, u(t), z) \widetilde{\eta_1}(dz, dt), \\ d\theta &= \left(\theta_{xx} - \alpha_2 v + g(t, \theta)\right) dt \\ &\quad + \int_Z \sigma_2 (t, \theta(t), z) \widetilde{\eta_2}(dz, dt), \\ u(x,t) &= 0, \qquad \theta(x,t) = 0, \\ x &= 0, \quad x = L, \quad t \geq 0, \\ u(x,0) &= u_0(x), \qquad v(x,0) = u_1(x), \\ \theta(x,0) &= \theta_0(x), \quad x \in [0, L]. \end{aligned} \tag{10}$$

For simplicity, denote that $g(\theta) := g(t,\theta)$, $\sigma_i(u) := \sigma_i(t,u,z)$ $(i = 1,2)$, and $L^2(0,L) = L^2$, $H^k(0,L) = H^k$, $k \in N^+$. To obtain the existence of solution to (10), we suppose that the functions

$$\begin{aligned} \sigma_i &: [0,\infty) \times L^2(0,L) \longrightarrow L^2\left(Z, \nu; L^2(0,L)\right), \\ g &: [0,\infty) \times L^2(0,L) \longrightarrow L^2(0,L) \end{aligned} \tag{11}$$

satisfy the following conditions:

$$\begin{aligned} C_1 &: \|\sigma_i(u)\|^2_{L^2(Z,\nu;L^2(0,L))} \leq k_1 \|u\|^2, \qquad \|g(u)\|^2 \leq K_1 \|u\|^2, \\ C_2 &: \|\sigma_i(u) - \sigma_i(v)\|^2_{L^2(Z,\nu;L^2(0,L))} \leq k_2 \|u - v\|^2, \\ &\quad \|g(u) - g(v)\|^2 \leq K_2 \|u - v\|^2, \end{aligned} \tag{12}$$

where $k_i, K_i > 0$, $i = 1,2$ and $u, v \in L^2(0,L)$.

Definition 2. An $\mathscr{F}_t$-adapted stochastic process $\{(u(t), v(t), \theta(t))\}_{t \geq 0}$ is said to be a strong probabilistic solution of stochastic nonlinear thermoelastic coupled system driven by Lévy noise (10) if it satisfies the following:

(1) $(u(t), v(t), \theta(t)) \in L^2(\Omega; C([0,T]; H^1 \times L^2 \times L^2))$ a.s. for any $T > 0$;

(2) the identities

$$\begin{aligned} (u(t), \varphi_1) &= (u_0, \varphi_1) + \int_0^t (v(s), \varphi_1) ds, \\ (v(t), \varphi_2) &= (v_0, \varphi_2) \\ &\quad + \int_0^t \left(\alpha_1 v_{xx} + u_{xx} + \beta \sin u + \alpha_2 \theta, \varphi_2\right) ds \\ &\quad + \int_0^t \int_Z (\sigma_1 (s, u(s), z), \varphi_2) \widetilde{\eta_1}(dz, ds), \\ (\theta(t), \varphi_3) &= (\theta_0, \varphi_3) + \int_0^t (\theta_{xx} - \alpha_2 v + g(\theta), \varphi_3) ds \\ &\quad + \int_0^t \int_Z (\sigma_2 (s, \theta(s), z), \varphi_3) \widetilde{\eta_2}(dz, ds), \end{aligned} \tag{13}$$

hold P−a.s. for all $(\varphi_1, \varphi_2, \varphi_3) \in L^2 \times H_0^1 \times H_0^1$.

We now give our main result.

Theorem 3. *Assume that $(u_0, v_0, \theta_0) \in H^1 \times L^2 \times L^2$ is $\mathscr{F}_0$-adapted. The conditions C_1 and C_2 are satisfied. $\widetilde{\eta_i} = \eta_i - \gamma_i$ $(i = 1,2)$ are the compensated Poisson random measure associated with η_i, where the compensator γ_i is defined in Section 2. Then for any $T > 0$ the stochastic nonlinear thermoelastic coupled system driven by Lévy noise (10) has a unique solution*

$$(u(t), v(t), \theta(t)) \in L^2\left(\Omega; C\left([0,T]; H^1 \times L^2 \times L^2\right)\right) \tag{14}$$

such that

$$E \sup_{0 \leq t \leq T} \left(\|u(t)\|_1^2 + \|v(t)\|^2 + \|\theta(t)\|^2\right) \leq C, \tag{15}$$

where C is a positive constant.

Proof. Existence. We use the Galerkin approximation and some useful a priori estimates to prove the existence of solution. Set

$$\begin{aligned} u_n(t) &= \sum_{i=1}^n a_i \phi_i, \qquad v_n(t) = \sum_{i=1}^n b_i \phi_i, \\ \theta_n(t) &= \sum_{i=1}^n c_i \phi_i, \end{aligned} \tag{16}$$

where $\{\phi_i\}_{i=1}^\infty$ is the set of eigenfunctions of $-\partial_{xx}$ with domain $H^2(0,L) \cap H_0^1(0,L)$); it is an orthogonal set of $H = L^2$ and orthonormal one in H_0^1, $a_i = (u, \phi_i)$, $b_i = (v, \phi_i)$, and $c_i = (\theta, \phi_i)$. Denote by $P_n : H \rightarrow H_n$ the orthogonal projector, where H_n =Span$\{\phi_1, \ldots, \phi_n\}$.

Hence, we can rewrite (10) as

$$\begin{aligned} du_n &= v_n dt, \\ dv_n &= \left(\alpha_1 v_{nxx} + u_{nxx} + \beta P_n \sin u_n + \alpha_2 \theta_n\right) dt \\ &\quad + \int_Z P_n \sigma_1 (u_n) \widetilde{\eta_1}(dz, dt), \\ d\theta_n &= \left(\theta_{nxx} - \alpha_2 v_n + P_n g(\theta_n)\right) dt \\ &\quad + \int_Z P_n \sigma_2 (\theta_n) \widetilde{\eta_2}(dz, dt), \end{aligned}$$

$$u_n(x,t)=0,\quad \theta_n(x,t)=0,\quad x=0,\ x=L,\ t\ge 0,$$
$$u_n(x,0)=u_{0n}(x),$$
$$v_n(x,0)=u_{1n}(x)=v_{0n}(x),$$
$$\theta_n(x,0)=\theta_{0n}(x),$$
$$x\in[0,L], \tag{17}$$

where, for each $n\in N^+$, $(u_{0n},v_{0n},\theta_{0n})\to(u_0,v_0,\theta_0)$ in $H^1\times L^2\times L^2$.

Applying Itô formula to the process $\|v_n(t)\|^2$, we obtain

$$\begin{aligned}\|v_n(t)\|^2&=\|v_{0n}\|^2+\int_0^t\int_Z\|P_n\sigma_1(u_n)\|^2\eta_1(dz,ds)\\&\quad+2\int_0^t\int_Z(v_n(s-),P_n\sigma_1(u_n))\widetilde{\eta_1}(dz,ds)\\&\quad+2\int_0^t(v_n,\alpha_1v_{nxx}+u_{nxx}+\beta P_n\sin u_n+\alpha_2\theta_n)\,ds\\&=\|v_{0n}\|^2+\|u_{0n}\|_1^2-\|u_n(t)\|_1^2\\&\quad-2\alpha_1\int_0^t\|v_n(t)\|_1^2ds\\&\quad+2\int_0^t(v_n,\beta P_n\sin u_n+\alpha_2\theta_n)\,ds\\&\quad+2\int_0^t\int_Z(v_n(s-),P_n\sigma_1(u_n))\widetilde{\eta_1}(dz,ds)\\&\quad+\int_0^t\int_Z\|P_n\sigma_1(u_n)\|^2\eta_1(dz,ds).\end{aligned}\tag{18}$$

Denote that

$$E(t):=\int_0^t\int_Z(u_n(s-),P_n\sigma_1(u_n))\widetilde{\eta_1}(dz,ds).\tag{19}$$

Since $\{u_n(t)\}_{t\in[0,T]}$ is an adapted and càdlàg process, the process $E(t)$ is a martingale. Applying the Burkholder-Davis-Gundy inequality, condition C_1, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}&E\sup_{0\le s\le t}\int_0^s\int_Z(v_n(r-),P_n\sigma_1(u_n))\widetilde{\eta_1}(dz,dr)\\&\quad\le 3E\left[\int_0^t\int_Z\|v_n(s-)\|^2\|P_n\sigma_1(u_n)\|^2\nu(dz)\,ds\right]^{1/2}\\&\quad\le 3\left[E\sup_{0\le s\le t}\|v_n(s)\|^2\right]^{1/2}\left[k_1\int_0^t\|u_n(s)\|^2ds\right]^{1/2}\\&\quad\le\frac14E\sup_{0\le s\le t}\|v_n(s)\|^2+9k_1E\int_0^t\|u_n(s)\|^2ds.\end{aligned}\tag{20}$$

Taking into account that the process

$$t\longmapsto\int_0^t\int_Z\|P_n\sigma_1(u_n)\|^2\eta_1(dz,ds)\tag{21}$$

has only positive jumps, we obtain

$$\begin{aligned}&E\sup_{0\le s\le t}\int_0^s\int_Z\|P_n\sigma_1(u_n)\|^2\eta_1(dz,dr)\\&\quad\le E\int_0^t\int_Z\|\sigma_1(u_n)\|^2\nu(dz)\,ds\\&\quad\le k_1E\int_0^t\|u_n(s)\|^2ds.\end{aligned}\tag{22}$$

From the Hölder inequality, we have

$$\left|\int_0^t(v_n,\beta P_n\sin u_n)\,ds\right|\le\int_0^t\|v_n(s)\|^2ds+Ct.\tag{23}$$

Putting (20)–(23) into (18), for $t\in[0,T]$, we have

$$\begin{aligned}&E\sup_{0\le s\le t}\left(\|u_n\|_1^2+\frac12\|v_n\|^2\right)\\&\quad\le\|u_{0n}\|_1^2+\|v_{0n}\|^2-2\alpha_1E\int_0^t\|v_n\|_1^2ds\\&\qquad+2E\int_0^t\|v_n\|^2ds+19k_1E\int_0^t\|u_n(s)\|^2ds\\&\qquad+2\alpha_2E\int_0^t(v_n,\theta_n)\,ds+Ct.\end{aligned}\tag{24}$$

Similarly, using the Itô formula to the process $\|\theta_n(t)\|^2$, we obtain

$$\begin{aligned}\|\theta_n(t)\|^2&=\|\theta_{0n}\|^2+2\int_0^t(\theta_n,\theta_{nxx}-\alpha_2v_n+P_ng(\theta_n))\,ds\\&\quad+\int_0^t\int_Z\|P_n\sigma_2(\theta_n(s))\|^2\eta_2(dz,ds)\\&\quad+2\int_0^t\int_Z(\theta_n(s-),P_n\sigma_2(\theta_n(s)))\widetilde{\eta_2}(dz,ds).\end{aligned}\tag{25}$$

Due to the Burkholder-Davis-Gundy inequality, condition C_1, the Hölder inequality, and the Young inequality,

$$\begin{aligned}&E\sup_{0\le s\le t}\int_0^s\int_Z(\theta_n(r-),P_n\sigma_2(\theta_n))\widetilde{\eta_2}(dz,dr)\\&\quad\le\frac14E\sup_{0\le s\le t}\|\theta_n(s)\|^2+9k_1E\int_0^t\|\theta_n(s)\|^2ds,\\&E\sup_{0\le s\le t}\int_0^s\int_Z\|P_n\sigma_2(\theta_n)\|^2\eta_2(dz,dr)\le k_1E\int_0^t\|\theta_n(s)\|^2ds,\\&2\int_0^t(\theta_n,P_ng(\theta_n))\,ds\le(1+K_1)\int_0^t\|\theta_n\|^2ds.\end{aligned}\tag{26}$$

Hence, we get

$$\frac{1}{2}E\sup_{0\le s\le t}\|\theta_n(t)\|^2 \le \|\theta_{0n}\|^2 + (1+K_1)E \times \int_0^t \|\theta_n\|^2 ds - 2\alpha_2 E\int_0^t (v_n,\theta_n)\,ds - 2E\int_0^t \|\theta_n\|^2 ds + 19k_1 E \times \int_0^t \|\theta_n(s)\|^2 ds + Ct. \quad (27)$$

For every $t \in [0,T]$, (24) and (27) imply that

$$E\sup_{0\le s\le t}\left(\|u_n\|_1^2 + \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|\theta_n\|^2\right) + 2\alpha_1 E\int_0^t \|v_n\|_1^2 ds + 2E\int_0^t \|\theta_n\|_1^2 ds \le \|u_{0n}\|_1^2 + \|v_{0n}\|^2 + \|\theta_{0n}\|^2 + \beta E\int_0^t \left(\|u_n\|_1^2 + \|v_n\|^2 + \|\theta_n\|^2\right) ds + Ct, \quad (28)$$

where $\beta = \max\{4, 2(1+K_1+19k_1)\}$.

Then the Gronwall lemma implies that

$$E\sup_{0\le s\le t}\left(\|u_n(t)\|_1^2 + \|v_n(t)\|^2 + \|\theta_n(t)\|^2\right) \le Ct. \quad (29)$$

Substituting (29) into (28), we get

$$E\int_0^t \left(\|v_n(s)\|_1^2 + \|\theta_n(s)\|_1^2\right) ds \le Ct. \quad (30)$$

To complete the proof of the existence of solution we need to pass the limits in the Galerkin approximation. Owing to (29) and (30), there exists a subsequence of $\{(u_n, v_n, \theta_n)\}$, not relabeled, such that

$$\begin{aligned} &u_n \rightharpoonup \hat{u} \text{ in } L^2\left(\Omega; C\left([0,t]; H^1(0,L)\right)\right),\\ &v_n \rightharpoonup \hat{v} \text{ in } L^2\left(\Omega; C\left([0,t]; L^2(0,L)\right)\right) \cap L^2\left(\Omega\times[0,t]; H^1(0,L)\right),\\ &\theta_n \rightharpoonup \hat{\theta} \text{ in } L^2\left(\Omega; C\left([0,t]; L^2(0,L)\right)\right) \cap L^2\left(\Omega\times[0,t]; H^1(0,L)\right). \end{aligned} \quad (31)$$

From the conditions in g, σ_i $(i=1,2)$, and (29)-(30), $\forall t \in [0,T]$, we have

$$\begin{aligned} &E\int_0^t \|P_n \sin(u_n)\|^2 ds < \infty,\\ &E\int_0^t \|P_n g(s,\theta_n)\|^2 ds < \infty,\\ &E\int_0^t\int_Z \|P_n\sigma_1(u_n)\|^2 \nu(dz)\,ds \le k_1 E\int_0^t \|u_n(s)\|^2 ds < \infty,\\ &E\int_0^t\int_Z \|P_n\sigma_2(\theta_n)\|^2 \nu(dz)\,ds \le k_1 E\int_0^t \|\theta_n(s)\|^2 ds < \infty. \end{aligned} \quad (32)$$

Hence, there exist the functions

$$\begin{aligned} &f^*, g^* \in L^2\left(\Omega\times[0,t]; L^2(0,L)\right),\\ &\sigma_i^* \in L^2\left(\Omega\times[0,t]; L^2\left(Z,\nu; L^2(0,L)\right)\right), \quad (i=1,2) \end{aligned} \quad (33)$$

such that

$$\begin{aligned} &P_n \sin(u_n) \rightharpoonup f^* \text{ in } L^2\left(\Omega\times[0,t]; L^2(0,L)\right),\\ &P_n g(\cdot,\theta_n) \rightharpoonup g^* \text{ in } L^2\left(\Omega\times[0,t]; L^2(0,L)\right),\\ &P_n\sigma_1(\cdot,u_n) \rightharpoonup \sigma_1^* \text{ in } L^2\left(\Omega\times[0,t]; L^2\left(Z,\nu; L^2(0,L)\right)\right),\\ &P_n\sigma_2(\cdot,\theta_n) \rightharpoonup \sigma_2^* \text{ in } L^2\left(\Omega\times[0,t]; L^2\left(Z,\nu; L^2(0,L)\right)\right). \end{aligned} \quad (34)$$

Combining (31) and (34) and letting $n \to \infty$ in (17), since the linear map

$$f \longmapsto \int_0^t\int_Z f(s,z)\,\widetilde{\eta_i}(dz,ds) \quad (35)$$

is continuous from $L^2(\Omega; L^2(Z,\nu; L^2(0,L)))$ to $L^2(\Omega; L^2(0,L))$ (in fact an isometry), it is continuous with respect to the weak topologies. Therefore, in view of the weak convergence, we have

$$\begin{aligned} (\hat{u}(t),\varphi_1) &= (u_0,\varphi_1) + \int_0^t (\hat{v}(s),\varphi_1)\,ds,\\ (\hat{v}(t),\varphi_2) &= (v_0,\varphi_2) + \int_0^t \left(\alpha_1\hat{v}_{xx} + \hat{u}_{xx} + \beta f^* + \alpha_2\hat{\theta}, \varphi_2\right) ds + \int_0^t\int_Z (\sigma_1^*(s,z),\varphi_2)\,\widetilde{\eta_1}(dz,ds),\\ \left(\hat{\theta}(t),\varphi_3\right) &= (\theta_0,\varphi_3) + \int_0^t \left(\hat{\theta}_{xx} - \alpha_2\hat{v} + g^*, \varphi_3\right) ds + \int_0^t\int_Z (\sigma_2^*(s,z),\varphi_3)\,\widetilde{\eta_2}(dz,ds), \end{aligned} \quad (36)$$

for almost everywhere $(\omega,t) \in \Omega\times[0,T]$ and $(\varphi_1,\varphi_2,\varphi_3) \in L^2\times H_0^1\times H_0^1$.

Denote by $\{(u(t), v(t), \theta(t))\}_{t\in[0,T]}$ the process which has a.s. sample paths being continuous in $(H^1 \times L^2 \times L^2)$, is $\mathscr{F}_t$-adapted, and equals to $\{(\widehat{u}(t), \widehat{v}(t), \widehat{\theta}(t))\}_{t\in[0,T]}$ almost everywhere $(\omega, t) \in \Omega \times [0,T]$; then from (34) we obtain

$$\begin{aligned}
(u(t), \varphi_1) &= (u_0, \varphi_1) + \int_0^t (v(s), \varphi_1)\, ds, \\
(v(t), \varphi_2) &= (v_0, \varphi_2) + \int_0^t (\alpha_1 v_{xx} + u_{xx} + \beta f^* + \alpha_2\theta, \varphi_2)\, ds \\
&\quad + \int_0^t \int_Z (\sigma_1^*(s,z), \varphi_2)\, \widetilde{\eta_1}(dz, ds), \\
(\theta(t), \varphi_3) &= (\theta_0, \varphi_3) + \int_0^t (\theta_{xx} - \alpha_2 v + g^*, \varphi_3)\, ds \\
&\quad + \int_0^t \int_Z (\sigma_2^*(s,z), \varphi_3)\, \widetilde{\eta_2}(dz, ds).
\end{aligned} \tag{37}$$

Now, we consider the stopping time; for each $N \in N^+$,

$$\tau_N = \begin{cases} \inf\left\{t \in [0,T] : \|u(t)\|_1^2 \vee \|v(t)\|^2 \vee \|\theta(t)\|^2 > N\right\}, \\ T, \quad \|u(t)\|_1^2 \vee \|v(t)\|^2 \vee \|\theta(t)\|^2 \le N. \end{cases} \tag{38}$$

We claim that $\{(u(t), v(t), \theta(t))\}_{t\in[0,T]}$ holds:

$$\begin{aligned}
&\lim_{n\to\infty} E\left(\|(u-u_n)(\tau_N)\|_1^2 + \|(v-v_n)(\tau_N)\|^2 + \|(\theta_n-\theta_n)(\tau_N)\|^2\right) = 0, \\
&\lim_{n\to\infty} E\int_0^{\tau_N} \left[\|(v-v_n)(s)\|_1^2 + \|(\theta-\theta_n)(s)\|_1^2\right] ds = 0, \\
&\lim_{n\to\infty} E\int_0^{\tau_N} \|\sigma_1^* - P_n\sigma_1(u)\|_{L^2(Z,\nu;L^2)}^2 ds = 0, \\
&\lim_{n\to\infty} E\int_0^{\tau_N} \|\sigma_2^* - P_n\sigma_2(\theta)\|_{L^2(Z,\nu;L^2)}^2 ds = 0.
\end{aligned} \tag{39}$$

From (17) and (37), for any $(\phi_i, \phi_j, \phi_k) \in H_0^1 \times (H^2 \cap H_0^1) \times H_0^1$, $i, j, k \in [1, n]$, we have

$$\begin{aligned}
((u-u_n)(t), \phi_i) &= \int_0^t ((v-v_n)(s), \phi_i)\, ds, \\
((v-v_n)(t), \phi_j) &= \int_0^t \left(\alpha_1 (v-v_n)_{xx} + (u-u_n)_{xx} + \beta(f^* - P_n \sin u_n), \phi_j\right) ds \\
&\quad + \int_0^t \left(\alpha_2(\theta-\theta_n), \phi_j\right) ds \\
&\quad + \int_0^t \int_Z \left(\sigma_1^*(s,z) - P_n\sigma_1(u_n), \phi_j\right) \widetilde{\eta_1}(dz, ds), \\
((\theta-\theta_n)(t), \phi_k) &= \int_0^t ((\theta-\theta_n)_{xx} - \alpha_2(v-v_n) + (g^* - P_n g(\theta_n)), \phi_k)\, ds \\
&\quad + \int_0^t \int_Z (\sigma_2^*(s,z) - P_n\sigma_2(\theta_n), \phi_k)\, \widetilde{\eta_2}(dz, ds),
\end{aligned} \tag{40}$$

a.s., for all $t \in [0,T]$.

For each $n \in N^+$, let

$$\begin{aligned}
P_n(u(t)) &= \widetilde{u}_n(t) = \sum_{i=1}^n (u(t), \phi_i)\, \phi_i, \\
P_n(v(t)) &= \widetilde{v}_n(t) = \sum_{j=1}^n (v(t), \phi_j)\, \phi_j, \\
P_n(\theta(t)) &= \widetilde{\theta}_n(t) = \sum_{k=1}^n (\theta(t), \phi_k)\, \phi_k.
\end{aligned} \tag{41}$$

Set $\rho(t) = e^{-\delta t}$, where δ is a positive constant to be defined later. By applying Itô's formula to the processes $\rho(t)\|\widetilde{v}_n(t) - v_n(t)\|^2$, $\rho(t)\|\widetilde{u}_n(t) - u_n(t)\|_1^2$, and $\rho(t)\|\widetilde{\theta}_n(t) - \theta_n(t)\|^2$, respectively, we obtain

$$\begin{aligned}
&\rho(t)\|\widetilde{v}_n(t) - v_n(t)\|^2 + \rho(t)\|\widetilde{u}_n(t) - u_n(t)\|_1^2 \\
&= -2\alpha_1 \int_0^t \rho(s)\|\widetilde{v}_n(s) - v_n(s)\|_1^2 ds + \int_0^t \rho'(s)\|\widetilde{u}_n(s) - u_n(s)\|_1^2 ds \\
&\quad + 2\beta \int_0^t \rho(s)\left(\widetilde{v}_n(s) - v_n(s), P_n(f^*(s) - \sin u_n(s))\right) ds \\
&\quad + 2\alpha_2 \int_0^t \rho(s)\left(\widetilde{v}_n(s) - v_n(s), \widetilde{\theta}_n(s) - \theta_n(s)\right) ds + \int_0^t \rho'(s)\|\widetilde{v}_n(s) - v_n(s)\|^2 ds \\
&\quad + 2\int_0^t \int_Z \rho(s)\left(\widetilde{v}_n(s-) - v_n(s-), P_n\sigma_1^*(s,z) - P_n\sigma_1(u_n)\right) \widetilde{\eta_1}(dz, ds) \\
&\quad + \int_0^t \int_Z \rho(s)\|P_n\sigma_1^*(s,z) - P_n\sigma_1(u_n)\|^2 \eta_1(dz, ds), \\
&\rho(t)\|\widetilde{\theta}_n(t) - \theta_n(t)\|^2 \\
&= -2\int_0^t \rho(s)\|\widetilde{\theta}_n(s) - \theta_n(s)\|_1^2 ds + \int_0^t \rho'(s)\|\widetilde{\theta}_n(s) - \theta_n(s)\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&- 2\alpha_2 \int_0^t \rho(s) \left(\widetilde{v}_n(s) - v_n(s), \widetilde{\theta}_n(s) - \theta_n(s)\right) ds \\
&+ \int_0^t \int_Z \rho(s) \left\|P_n\sigma_2^*(s,z) - P_n\sigma_2(\theta_n)\right\|^2 \eta_2(dz,ds) \\
&+ 2\int_0^t \rho(s) \left(\widetilde{\theta}_n(s) - \theta_n(s), P_n g^*(s) - P_n g(\theta_n)\right) ds \\
&+ 2\int_0^t \int_Z \rho(s) \left(\widetilde{\theta}_n(s-) - \theta_n(s-),\right. \\
&\qquad\qquad \left. P_n\sigma_2^*(s,z) - P_n\sigma_2(\theta_n)\right)(dz,ds).
\end{aligned} \tag{42}
$$

Taking the mathematical expectations in (42) yields

$$
\begin{aligned}
&E\left\{\rho(t)\left(\left\|\widetilde{v}_n(t) - v_n(t)\right\|^2 + \left\|\widetilde{u}_n(t) - u_n(t)\right\|_1^2\right.\right. \\
&\qquad\qquad \left.\left. + \left\|\widetilde{\theta}_n(t) - \theta_n(t)\right\|^2\right)\right\} \\
&\quad + E\int_0^t \rho(s)\left[2\alpha_1\left\|\widetilde{v}_n(s) - v_n(s)\right\|_1^2 + \left\|\widetilde{\theta}_n(s) - \theta_n(s)\right\|_1^2\right] ds \\
&= E\int_0^t \rho'(s)\left[\left\|\widetilde{u}_n(s) - u_n(s)\right\|_1^2 + \left\|\widetilde{v}_n(s) - v_n(s)\right\|^2\right. \\
&\qquad\qquad \left. + \left\|\widetilde{\theta}_n(s) - \theta_n(s)\right\|^2\right] ds \\
&\quad + 2\beta E\int_0^t \rho(s)\left(\widetilde{v}_n(s) - v_n(s), P_n\left(f^*(s) - \sin u_n(s)\right)\right) ds \\
&\quad + E\int_0^t \int_Z \rho(s)\left\|P_n\sigma_1^*(s,z) - P_n\sigma_1(u_n)\right\|^2 \eta_1(dz,ds) \\
&\quad + E\int_0^t \int_Z \rho(s)\left\|P_n\sigma_2^*(s,z) - P_n\sigma_2(\theta_n)\right\|^2 \eta_2(dz,ds) \\
&\quad + 2E\int_0^t \rho(s)\left(\widetilde{\theta}_n(s) - \theta_n(s), P_n g^*(s) - P_n g(\theta_n)\right) ds.
\end{aligned} \tag{43}
$$

Let us analyze each term of (43). By the conditions C_1 and C_2, we have

$$
\begin{aligned}
&\left(\widetilde{\theta}_n - \theta_n, P_n g^* - P_n g(\theta_n)\right) \\
&\quad = \left(\widetilde{\theta}_n - \theta_n, P_n g^* - P_n g(\widetilde{\theta}_n)\right) \\
&\qquad + \left(\widetilde{\theta}_n - \theta_n, P_n g(\widetilde{\theta}_n) - P_n g(\theta_n)\right) \\
&\quad = \left(\widetilde{\theta}_n - \theta_n, P_n g(\widetilde{\theta}_n) - P_n g(\theta_n)\right) \\
&\qquad + \left(\widetilde{\theta}_n - \theta_n, P_n g^* - P_n g(\theta)\right) \\
&\qquad + \left(\widetilde{\theta}_n - \theta_n, P_n g(\theta) - P_n g(\widetilde{\theta}_n)\right) \\
&\quad \le 2K_2\left\|\widetilde{\theta}_n - \theta_n\right\|^2 + 4K_2\left\|\widetilde{\theta}_n - \theta\right\|^2 \\
&\qquad + \left(\widetilde{\theta}_n - \theta_n, P_n g^* - P_n g(\theta)\right), \\
&\left(\widetilde{v}_n(s) - v_n(s), P_n\left(f^*(s) - \sin u_n(s)\right)\right) \\
&\quad \le \left\|\widetilde{v}_n - v_n\right\|^2 + \frac{1}{2}\left\|\widetilde{u}_n - u_n\right\|^2 + \frac{1}{2}\left\|\widetilde{u}_n - u\right\|^2 \\
&\qquad + \left(\widetilde{v}_n - v_n, P_n f^* - P_n \sin u\right), \\
&\left\|P_n\sigma_1^*(s,z) - P_n\sigma_1(u_n)\right\|^2 \\
&\quad = \left\|P_n\sigma_1(u) - P_n\sigma_1(u_n)\right\|^2 - \left\|P_n\sigma_1^* - P_n\sigma_1(u)\right\|^2 \\
&\qquad + 2\left(P_n\sigma_1^* - P_n\sigma_1(u),\right. \\
&\qquad\qquad \left. P_n\sigma_1(u) - P_n\sigma_1(u_n)\right).
\end{aligned} \tag{44}
$$

Hence, using the conditions C_1 and C_2 again, we get

$$
\begin{aligned}
&E\int_0^t \int_Z \rho(s)\left\|P_n\sigma_1^* - P_n\sigma_1(u_n)\right\|^2 \eta_1(dz,ds) \\
&\quad = E\int_0^t \rho(s)\int_Z \left\|P_n\sigma_1^* - P_n\sigma_1(u_n)\right\|^2 \nu(dz)\, ds \\
&\quad = E\int_0^t \rho(s)\int_Z \left[\left\|P_n\sigma_1(u) - P_n\sigma_1(u_n)\right\|^2\right. \\
&\qquad\qquad \left. - \left\|P_n\sigma_1^* - P_n\sigma_1(u)\right\|^2\right] \nu(dz)\, ds \\
&\qquad + 2E\int_0^t \rho(s)\int_Z \left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u)\right. \\
&\qquad\qquad \left. - P_n\sigma_1(u_n)\right) \nu(dz)\, ds \\
&\quad \le k_2 E\int_0^t \rho(s)\left\|u - u_n\right\|^2 ds \\
&\qquad - E\int_0^t \rho(s)\left\|P_n\sigma_1^* - P_n\sigma_1(u)\right\|_{L^2(Z,\nu;L^2)}^2 ds \\
&\qquad + 2E\int_0^t \rho(s)\int_Z \left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u)\right. \\
&\qquad\qquad \left. - P_n\sigma_1(u_n)\right) \nu(dz)\, ds,
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
&E\int_0^t \int_Z \rho(s)\left\|P_n\sigma_2^* - P_n\sigma_2(\theta_n)\right\|^2 \eta_2(dz,ds) \\
&\quad \le k_2 E\int_0^t \rho(s)\left\|\theta - \theta_n\right\|^2 ds \\
&\qquad - E\int_0^t \rho(s)\left\|P_n\sigma_2^* - P_n\sigma_2(\theta)\right\|_{L^2(Z,\nu;L^2)}^2 ds \\
&\qquad + 2E\int_0^t \rho(s)\int_Z \left(P_n\sigma_2^* - P_n\sigma_2(\theta), P_n\sigma_2(\theta)\right. \\
&\qquad\qquad \left. - P_n\sigma_2(\theta_n)\right) \nu(dz)\, ds.
\end{aligned} \tag{46}
$$

Substituting (44)–(46) into (43), we have

$$
\begin{aligned}
&E\left\{\rho(t)\left(\left\|\widetilde{v}_n(t) - v_n(t)\right\|^2 + \left\|\widetilde{u}_n(t) - u_n(t)\right\|_1^2\right.\right. \\
&\qquad\qquad \left.\left. + \left\|\widetilde{\theta}_n(t) - \theta_n(t)\right\|^2\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ E\int_0^t \rho(s)\left[2\alpha_1\|\widetilde{v}_n(s)-v_n(s)\|_1^2 + \left\|\widetilde{\theta}_n(s)-\theta_n(s)\right\|_1^2\right]ds \\
&+ E\int_0^t \rho(s)\|P_n\sigma_1^* - P_n\sigma_1(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&+ E\int_0^t \rho(s)\|P_n\sigma_2^* - P_n\sigma_2(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&\le E\int_0^t \rho'(s)\left[\|\widetilde{u}_n(s)-u_n(s)\|_1^2 + \|\widetilde{v}_n(s)-v_n(s)\|^2 + \left\|\widetilde{\theta}_n(s)-\theta_n(s)\right\|^2\right]ds \\
&+ E\int_0^t \rho(s)\left[\beta\lambda\|\widetilde{u}_n(s)-u_n(s)\|_1^2 + 2\beta\|\widetilde{v}_n(s)-v_n(s)\|^2 + 4K_2\left\|\widetilde{\theta}_n(s)-\theta_n(s)\right\|^2\right]ds \\
&+ E\int_0^t \rho(s)\left[\|\widetilde{u}_n-u\|^2 + 2\left(\widetilde{v}_n-v_n, P_nf^* - P_n\sin u\right)\right]ds \\
&+ 2E\int_0^t \rho(s)\left[4K_2\left\|\widetilde{\theta}_n-\theta\right\|^2 + \left(\widetilde{\theta}_n-\theta_n, P_ng^* - P_ng(\theta)\right)\right]ds \\
&+ 2E\int_0^t \rho(s)\int_Z\left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u) - P_n\sigma_1(u_n)\right)\nu(dz)\,ds \\
&+ 2E\int_0^t \rho(s)\int_Z\left(P_n\sigma_2^* - P_n\sigma_2(\theta), P_n\sigma_2(\theta) - P_n\sigma_2(\theta_n)\right)\nu(dz)\,ds \\
&+ k_2E\int_0^t \rho(s)\left[\|u-u_n\|^2 + \|\theta-\theta_n\|^2\right]ds,
\end{aligned}
\tag{47}
$$

where λ is the Sobolev embedding constant such that $\|u(\cdot)\|^2 \le \lambda\|u(\cdot)\|_1^2$. Choosing $\delta = \max\{2\beta, \beta\lambda, 4K_2\}$, we get $\rho'(s) + \delta\rho(s) = 0$; hence

$$
\begin{aligned}
&E\left\{\rho(t)\left(\|\widetilde{v}_n(t)-v_n(t)\|^2 + \|\widetilde{u}_n(t)-u_n(t)\|_1^2 + \left\|\widetilde{\theta}_n(t)-\theta_n(t)\right\|^2\right)\right\} \\
&+ E\int_0^t \rho(s)\left[2\alpha_1\|\widetilde{v}_n(s)-v_n(s)\|_1^2 + \left\|\widetilde{\theta}_n(s)-\theta_n(s)\right\|_1^2\right]ds \\
&+ E\int_0^t \rho(s)\|P_n\sigma_1^* - P_n\sigma_1(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&+ E\int_0^t \rho(s)\|P_n\sigma_2^* - P_n\sigma_2(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&\le E\int_0^t \rho(s)\left[\|\widetilde{u}_n-u\|^2 + 2\left(\widetilde{v}_n-v_n, P_nf^* - P_n\sin u\right)\right]ds \\
&+ 2E\int_0^t \rho(s)\left[4K_2\left\|\widetilde{\theta}_n-\theta\right\|^2 + \left(\widetilde{\theta}_n-\theta_n, P_ng^* - P_ng(\theta)\right)\right]ds \\
&+ 2E\int_0^t \rho(s)\int_Z\left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u) - P_n\sigma_1(u_n)\right)\nu(dz)\,ds \\
&+ 2E\int_0^t \rho(s)\int_Z\left(P_n\sigma_2^* - P_n\sigma_2(\theta), P_n\sigma_2(\theta) - P_n\sigma_2(\theta_n)\right)\nu(dz)\,ds \\
&+ k_2E\int_0^t \rho(s)\left[\|u-u_n\|^2 + \|\theta-\theta_n\|^2\right]ds.
\end{aligned}
\tag{48}
$$

Replacing t by τ_N in (48),

$$
\begin{aligned}
&E\left\{\rho(\tau_N)\left(\|(\widetilde{v}_n-v_n)(\tau_N)\|^2 + \|(\widetilde{u}_n-u_n)(\tau_N)\|_1^2 + \left\|(\widetilde{\theta}_n-\theta_n)(\tau_N)\right\|^2\right)\right\} \\
&+ E\int_0^{\tau_N} \rho(s)\left[2\alpha_1\|\widetilde{v}_n(s)-v_n(s)\|_1^2 + \left\|\widetilde{\theta}_n(s)-\theta_n(s)\right\|_1^2\right]ds \\
&+ E\int_0^{\tau_N} \rho(s)\|P_n\sigma_1^* - P_n\sigma_1(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&+ E\int_0^{\tau_N} \rho(s)\|P_n\sigma_2^* - P_n\sigma_2(u)\|^2_{L^2(Z,\nu;L^2)}ds \\
&\le E\int_0^{\tau_N} \rho(s)\left[\|\widetilde{u}_n-u\|^2 + 2\left(\widetilde{v}_n-v_n, P_nf^* - P_n\sin u\right)\right]ds \\
&+ 2E\int_0^{\tau_N} \rho(s)\left[4K_2\left\|\widetilde{\theta}_n-\theta\right\|^2 + \left(\widetilde{\theta}_n-\theta_n, P_ng^* - P_ng(\theta)\right)\right]ds \\
&+ 2E\int_0^{\tau_N} \rho(s)\int_Z\left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u) - P_n\sigma_1(u_n)\right)\nu(dz)\,ds \\
&+ 2E\int_0^{\tau_N} \rho(s)\int_Z\left(P_n\sigma_2^* - P_n\sigma_2(\theta), P_n\sigma_2(\theta) - P_n\sigma_2(\theta_n)\right)\nu(dz)\,ds \\
&+ k_2E\int_0^{\tau_N} \rho(s)\left[\|u-u_n\|^2 + \|\theta-\theta_n\|^2\right]ds.
\end{aligned}
\tag{49}
$$

Since $1_{[0,\tau_N]}\rho(s)(\sigma_1^* - \sigma_1(u)) \in L^2(Z,\nu;L^2)$ implies that $1_{[0,\tau_N]}\rho(s)(P_n\sigma_1^* - P_n\sigma_1(u)) \in L^2(Z,\nu;L^2)$, due to (31) and Sobolev embedding theorem, we have $\|u_n - \widehat{u}\|_{L^2(0,L)} \to 0$ as $n \to \infty$; hence

$$\begin{aligned} E\int_0^{\tau_N} \rho(s)\|P_n\sigma_1(u) - P_n\sigma_1(u_n)\|^2_{L^2(Z,\nu;L^2)}ds &\le E\int_0^{\tau_N} \rho(s)\|\sigma_1(u) - \sigma_1(u_n)\|^2_{L^2(Z,\nu;L^2)}ds \\ &\le k_2 E\int_0^{\tau_N} \rho(s)\|u - u_n\|^2_{L^2(0,L)}ds \\ &= k_2 E\int_0^{\tau_N} \rho(s)\|\widehat{u} - u_n\|^2_{L^2(0,L)}ds \longrightarrow 0 \quad (n \longrightarrow \infty); \end{aligned} \tag{50}$$

then we have

$$\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\int_Z \left(P_n\sigma_1^* - P_n\sigma_1(u), P_n\sigma_1(u) - P_n\sigma_1(u_n)\right)\nu(dz)\,ds = 0. \tag{51}$$

Similarly, we can prove that

$$\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\int_Z \left(P_n\sigma_2^* - P_n\sigma_2(\theta), P_n\sigma_2(\theta) - P_n\sigma_2(\theta_n)\right)\nu(dz)\,ds = 0. \tag{52}$$

Owing to (31)–(34) the sequence $P_ng^* - P_ng(\theta)$ is bounded in $L^2(\Omega\times[0,T];L^2)$, and by the Sobolev embedding theorem,

$$\begin{aligned} \lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\left(\widetilde{\theta}_n - \theta_n, P_ng^* - P_ng(\theta)\right)ds &\le C\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\left\|\widetilde{\theta}_n - \theta_n\right\|^2 ds \\ &= C\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\|P_n\theta - P_n\theta_n\|^2 ds \\ &\le C\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\|\theta - \theta_n\|^2 ds \\ &= C\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\left\|\widehat{\theta} - \theta_n\right\|^2 ds = 0. \end{aligned} \tag{53}$$

Similarly, we can prove that

$$\begin{aligned} &\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\left(\widetilde{v}_n - v_n, P_nf^* - P_n\sin u\right)ds \le C\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\|\widetilde{v}_n - v_n\|^2 ds = 0, \\ &\lim_{n\to\infty} E\int_0^{\tau_N} \rho(s)\left[\|u - u_n\|^2 + \|\widetilde{u}_n - u\|^2 + \left\|\widetilde{\theta}_n - \theta\right\|^2 + \|\theta - \theta_n\|^2\right]ds = 0. \end{aligned} \tag{54}$$

Therefore, the limits (51)–(54) imply that (39) holds true.

Next, due to the property of g, we see that, for all $\psi \in L^2(\Omega\times[0,T];L^2(0,L))$,

$$\begin{aligned} &\lim_{n\to\infty} E\int_0^{\tau_N} \left(g(\theta_n(s)) - g(\theta(s)), \psi(s)\right)ds \le C\lim_{n\to\infty} E\int_0^{\tau_N} \|\theta_n(s) - \theta(s)\|\,\|\psi(s)\|\,ds = 0, \\ &\lim_{n\to\infty} E\int_0^{\tau_N} \left(\sin u_n(s) - \sin u(s), \psi(s)\right)ds = 0. \end{aligned} \tag{55}$$

From this, since $P_ng(\theta_n) \rightharpoonup g^*$, $P_n\sin u_n \rightharpoonup f^*$ in $L^2(\Omega\times[0,T];L^2(0,L))$, and (39), we have

$$\begin{aligned} I_{[0,\tau_N]}(s)\,\sigma_1^*(s) &= I_{[0,\tau_N]}(s)\,\sigma_1(u(s)), \\ I_{[0,\tau_N]}(s)\,\sigma_2^*(s) &= I_{[0,\tau_N]}(s)\,\sigma_2(\theta(s)), \\ I_{[0,\tau_N]}(s)\,g^*(s) &= I_{[0,\tau_N]}(s)\,g(\theta(s)), \\ I_{[0,\tau_N]}(s)\,f^*(s) &= I_{[0,\tau_N]}(s)\sin u(s), \end{aligned} \tag{56}$$

for almost everywhere $(\omega,t)\in\Omega\times[0,T]$.

Putting (56) into (37), we get

$$\begin{aligned} (u(t),\varphi_1) &= (u_0,\varphi_1) + \int_0^t (v(s),\varphi_1)\,ds, \\ (v(t),\varphi_2) &= (v_0,\varphi_2) + \int_0^t (\alpha_1 v_{xx} + u_{xx} + \beta\sin u + \alpha_2\theta, \varphi_2)\,ds + \int_0^t\int_Z (\sigma_1(u),\varphi_2)\,\widetilde{\eta_1}(dz,ds), \\ (\theta(t),\varphi_3) &= (\theta_0,\varphi_3) + \int_0^t (\theta_{xx} - \alpha_2 v + g(\theta),\varphi_3)\,ds + \int_0^t\int_Z (\sigma_2(\theta),\varphi_3)\,\widetilde{\eta_2}(dz,ds), \end{aligned} \tag{57}$$

a.s., $\forall(\varphi_1,\varphi_2,\varphi_3)\in L^2\times H_0^1\times H_0^1$, $t\in[0,T]$.

From the property of τ_N, we obtain $P(\bigcup_{N=1}^{\infty}\{\tau_N = T\}) = 1$; let

$$\Omega' := \left\{\omega\in\Omega;\ \omega\in\bigcup_{N=1}^{\infty}\{\tau_N = T\},\ (u,v,\theta)(\omega,t)\ \text{satisfies (57)}\right\}. \tag{58}$$

and $P(\Omega') = 1$. For $\omega\in\Omega'$ there exists $N' > 0$ such that $\tau_N = T$ for all $N \ge N'$.

Then, $(u(t), v(t), \theta(t))$ is a solution to (10).

Uniqueness. Set (u_1, v_1, θ_1) and (u_2, v_2, θ_2) as two solutions of (10); thus

$$\begin{aligned} ((u_1 - u_2)(t), \varphi_1) &= \int_0^t (v(s), \varphi_1)\, ds, \\ ((v_1 - v_2)(t), \varphi_2) &= \int_0^t (\alpha_1 (v_1 - v_2)_{xx} + (u_1 - u_2)_{xx} \\ &\quad + \beta(\sin u_2 - \sin u_2), \varphi_2)\, ds \\ &\quad + \int_0^t \beta\alpha_2 ((\theta_1 - \theta_2), \varphi_2)\, ds \\ &\quad + \int_0^t \int_Z (\sigma_1(u_1) - \sigma_1(u_2), \varphi_2) \\ &\quad \times \widetilde{\eta_1}(dz, ds), \\ ((\theta_1 - \theta_2)(t), \varphi_3) &= \int_0^t ((\theta_1 - \theta_2)_{xx} - \alpha_2 (v_1 - v_2) \\ &\quad + g(\theta_1) - g(\theta_2), \varphi_3)\, ds \\ &\quad + \int_0^t \int_Z (\sigma_2(\theta_1) - \sigma_2(\theta_2), \varphi_3) \\ &\quad \times \widetilde{\eta_2}(dz, ds), \end{aligned} \tag{59}$$

a.s., $\forall(\varphi_1, \varphi_2, \varphi_3) \in L^2 \times H_0^1 \times H_0^1$, $t \in [0, T]$.

From a similar argument as in the proof of existence, by the B-D-G inequality, conditions C_1, C_2, and the Gronwall lemma, one can easily show that

$$E\left(\|u_1(t) - u_2(t)\|_1^2 + \|\theta_1(t) - \theta_2(t)\|_1^2\right) = 0 \tag{60}$$

for $t \in [0, T]$; thus $P(u_1(t) = u_2(t),\ \theta_1(t) = \theta_2(t)) = 1$ for all $t \in [0, T]$.

We complete the proof of the theorem. □

4. Asymptotic Behavior

In this section, we briefly discuss the long time behavior of the strong solutions of the stochastic nonlinear thermoelastic system coupled sine-Gordon equation driven by jump noise.

Following the idea in [6], we assume that there exists a constant $a > 0$ such that

$$\begin{aligned} \gamma &= \max\{a + \beta, a + 1 + K_1 + k_1\} > 0, \\ \xi &= \min\left\{\frac{2\alpha_1}{\lambda}, \frac{2}{\lambda}\right\} > 0, \end{aligned} \tag{61}$$

where λ is defined in Section 3.

Theorem 4. *Suppose that the conditions for Theorem 3 hold true. If $\xi - \gamma - a < 0$, then the solution $\{(u(t), v(t), \theta(t))\}_{t\in[0,T]}$ of the problem* (10) *satisfies*

$$E\left(\|u_t(t)\|^2 + \|\theta(t)\|^2\right) \le e^{(\xi-\gamma-a)t}\left(C_0 + C_1 t + C_2 t^2\right), \tag{62}$$

where C_i $(i = 0, 1, 2)$ are the positive constants.

Proof. Set $\rho(t) = e^{at}$, using Itô's formula to the processes $\rho(t)\|v(t)\|^2$, $\rho(t)\|u(t)\|_1^2$, and $\rho(t)\|\theta(t)\|^2$, respectively; we obtain

$$\begin{aligned} &\rho(t)\|v(t)\|^2 + \rho(t)\|u(t)\|_1^2 \\ &\quad = \|v_0\|^2 + \|u_0\|_1^2 + \int_0^t \rho'(s)\|v(s)\|^2 ds \\ &\quad - 2\alpha_1 \int_0^t \rho(s)\|v(s)\|_1^2 ds \\ &\quad + \int_0^t \rho'(s)\|u(s)\|_1^2 ds \\ &\quad + 2\beta \int_0^t \rho(s)(v(s), \sin u(s))\, ds \\ &\quad + 2\alpha_2 \int_0^t \rho(s)(v(s), \theta(s))\, ds \\ &\quad + 2\int_0^t \int_Z \rho(s)(v(s-), \sigma_1(u))\,\widetilde{\eta_1}(dz, ds) \\ &\quad + \int_0^t \int_Z \rho(s)\|\sigma_1(u)\|^2 \eta_1(dz, ds), \\ &\rho(t)\|\theta(t)\|^2 = \|\theta_0\|^2 - 2\int_0^t \rho(s)\|\theta(s)\|_1^2 ds \\ &\quad + \int_0^t \rho'(s)\|\theta(s)\|^2 ds \\ &\quad - 2\alpha_2 \int_0^t \rho(s)(v(s), \theta(s))\, ds \\ &\quad + 2\int_0^t \rho(s)(\theta(s), g(\theta(s)))\, ds \\ &\quad + 2\int_0^t \int_Z \rho(s)(\theta(s-), \sigma_2(\theta))\,\widetilde{\eta_2}(dz, ds) \\ &\quad + \int_0^t \int_Z \rho(s)\|\sigma_2(\theta)\|^2 \eta_2(dz, ds). \end{aligned} \tag{63}$$

Taking the mathematical expectations in (63) yields

$$\begin{aligned} &E\rho(t)\left[\|v(t)\|^2 + \|u(t)\|_1^2 + \|\theta(t)\|^2\right] \\ &\quad + 2\alpha_1 E\int_0^t \rho(s)\|v(s)\|_1^2 ds + 2E\int_0^t \rho(s)\|\theta(s)\|_1^2 ds \\ &= \left[\|v_0\|^2 + \|u_0\|_1^2 + \|\theta_0\|^2\right] \\ &\quad + E\int_0^t \rho'(s)\left[\|u(s)\|_1^2 + \|v(s)\|^2 + \|\theta(s)\|^2\right] ds \\ &\quad + 2\beta E\int_0^t \rho(s)(v(s), \sin u(s))\, ds \end{aligned}$$

$$+ 2E\int_0^t \rho(s)(\theta(s), g(\theta(s)))\,ds$$
$$+ E\int_0^t\int_Z \rho(s)\|\sigma_1(u)\|^2\eta_1(ds,dz)$$
$$+ E\int_0^t\int_Z \rho(s)\|\sigma_2(\theta)\|^2\eta_2(ds,dz). \tag{64}$$

From Hölder's inequality, Young's inequality, and condition C_1, we have

$$\begin{aligned}
&2\beta E\int_0^t \rho(s)(v(s),\sin u(s))\,ds \\
&\quad \le \beta E\int_0^t \rho(s)\|v(s)\|^2 ds + \beta E\int_0^t \rho(s)\|u(s)\|^2 ds, \\
&2E\int_0^t \rho(s)(\theta(s), g(\theta(s)))\,ds \\
&\quad \le E\int_0^t \rho(s)\|\theta(s)\|^2 ds + K_1 E\int_0^t \rho(s)\|\theta(s)\|^2 ds, \\
&E\int_0^t\int_Z \rho(s)\|\sigma_1(u)\|^2\eta_1(ds,dz) \\
&\quad = E\int_0^t\int_Z \rho(s)\|\sigma_1(u)\|^2 \nu(dz)\,ds \\
&\quad \le k_1 E\int_0^t \rho(s)\|u\|^2 ds, \\
&E\int_0^t\int_Z \rho(s)\|\sigma_2(\theta)\|^2\eta_2(ds,dz) \\
&\quad = E\int_0^t\int_Z \rho(s)\|\sigma_2(\theta)\|^2 \nu(dz)\,ds \\
&\quad \le k_1 E\int_0^t \rho(s)\|\theta\|^2 ds.
\end{aligned} \tag{65}$$

Therefore, from (64), we get

$$\begin{aligned}
&E\rho(t)\left[\|v(t)\|^2 + \|u(t)\|_1^2 + \|\theta(t)\|^2\right] \\
&\quad + 2\alpha_1 E\int_0^t \rho(s)\|v(s)\|_1^2 + 2E\int_0^t \rho(s)\|\theta(s)\|_1^2 ds \\
&\le \left[\|v_0\|^2 + \|u_0\|_1^2 + \|\theta_0\|^2\right] \\
&\quad + aE\int_0^t \rho(s)\left[\|u(s)\|_1^2 + \|v(s)\|^2 + \|\theta(s)\|^2\right] ds \\
&\quad + E\int_0^t \rho(s)\left[\beta\|v(s)\|^2 + (k_1+\beta)\|u(s)\|^2\right. \\
&\qquad \left. + (1 + K_1 + k_1)\|\theta(s)\|^2\right] ds.
\end{aligned} \tag{66}$$

Due to the embedding theorem and (61),

$$\begin{aligned}
&E\rho(t)\left[\|v(t)\|^2 + \|u(t)\|_1^2 + \|\theta(t)\|^2\right] \\
&\quad + \xi E\int_0^t \rho(s)\left[\|v(s)\|^2 + \|\theta(s)\|^2\right] ds \\
&\le E\rho(t)\left[\|v(t)\|^2 + \|u(t)\|_1^2 + \|\theta(t)\|^2\right] \\
&\quad + \frac{2\alpha_1}{\lambda}E\int_0^t \rho(s)\|v(s)\|^2 + \frac{2}{\lambda}E\int_0^t \rho(s)\|\theta(s)\|^2 ds \\
&\le \left[\|v_0\|^2 + \|u_0\|_1^2 + \|\theta_0\|^2\right] \\
&\quad + E\int_0^t \rho(s)(a + (k_1+\beta)\lambda)\|u(s)\|_1^2 ds \\
&\quad + E\int_0^t \rho(s)(a+\beta)\|v(s)\|^2 ds \\
&\quad + E\int_0^t \rho(s)(a + 1 + K_1 + k_1)\|\theta(s)\|^2 ds \\
&\le \left[\|v_0\|^2 + \|u_0\|_1^2 + \|\theta_0\|^2\right] \\
&\quad + E\int_0^t \rho(s)(a + (k_1+\beta)\lambda)\|u(s)\|_1^2 ds \\
&\quad + \gamma E\int_0^t \rho(s)\left[\|v(s)\|^2 + \|\theta(s)\|^2\right] ds.
\end{aligned} \tag{67}$$

Hence, by Theorem 3, we have

$$\begin{aligned}
&E\rho(t)\left[\|v(t)\|^2 + \|\theta(t)\|^2\right] + (\xi-\gamma)E \\
&\quad \times \int_0^t \rho(s)\left[\|v(s)\|^2 + \|\theta(s)\|^2\right] ds \\
&\le \left[\|v_0\|^2 + \|u_0\|_1^2 + \|\theta_0\|^2\right] + Ct = C_1 + C_2 t;
\end{aligned} \tag{68}$$

Gronwall's inequality leads to

$$\begin{aligned}
&E\rho(t)\left[\|v(t)\|^2 + \|\theta(t)\|^2\right] \\
&\quad \le e^{(\xi-\gamma)t}\left(\|v_0\|^2 + \|\theta_0\|^2 + C_1 t + C_2 t^2\right).
\end{aligned} \tag{69}$$

With the choice of a, ξ, γ, the assumption of Theorem 3 holds true; we obtain that

$$E\left[\|v(t)\|^2 + \|\theta(t)\|^2\right] \le e^{(\xi-\gamma-a)t}\left(C_0 + C_1 t + C_2 t^2\right). \tag{70}$$

This completes the proof of the theorem. □

Remark 5. Since $u(x,t,\omega)$ denotes the displacement at point (x,t) on an orbit $\omega \in \Omega$, a.s., $\partial u/\partial t = v$ means the velocity; the result in the Theorem 4 exhibits that the velocity is exponentially decay in time t in the sense of mean square; in the view of physics, one can obtain that the displacement u will tend to a constant in the large time in the sense of mean square.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the anonymous referees for helpful comments and suggestions that greatly improved the presentation of this paper. This work is supported by NSF of China (11272277 and 11226188) and FRF for the Central Universities of China (2013ZZGH027).

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