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Research Article

On the Ideal Convergence of Double Sequences in Locally Solid Riesz Spaces

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The aim of this paper is to define the notions of ideal convergence, I-bounded for double sequences in setting of locally solid Riesz spaces and study some results related to these notions. We also define the notion of I^*-convergence for double sequences in locally solid Riesz spaces and establish its relationship with ideal convergence.

1. Introduction and Preliminaries

In 1951, Fast [1] and Steinhaus [2] introduced the concept of statistical convergence for single sequences, independently. Some basic and important properties of this concept were studied by Buck [3], Šalát [4], Schoenberg [5], and Fridy [6]. Later, the notion of statistical convergence for single sequences was further defined in various spaces; see Çakalli and Khan [7–9], Di Maio et al. [10, 11], Hazarika [12–14], Maddox [15], Mohiuddine et al. [16–19], and so forth. Some application of statistical summability methods is presented in [20, 21]. In 2003, the notion of statistical convergence for single sequences has been extended to double sequences by Mursaleen and Edely [22]. Recently, the statistical convergence and statistical Cauchy for double sequences have been defined in the framework fuzzy and intuitionistic normed spaces by Mohiuddine et al. [23] and Mursaleen and Mohiuddine [24], respectively, and established some interesting results related to the concept of statistical convergence and statistical Cauchy double sequences. Recently, it was defined and studied by Mohiuddine et al. [25] in the setting of locally solid Riesz spaces while for single sequences this concept was first studied by Albayrak and Pehlivan [26] (also see [27–29]). An application of locally solid Riesz spaces in economics can be found in [30].

The notion of ideal convergence for single sequences, which is a generalization of the concept of statistical convergence, was first defined and studied by Kostyrko et al. [31]. Let us recall the notion of ideal convergence and related concepts by Kostyrko et al. [31] as follows. Let $\mathbb{N}$ be a nonempty set. Then a family of sets $I \subseteq P(\mathbb{N})$ (power set of $\mathbb{N}$) is said to be an ideal if I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if, for each $A, B \in I$, we have $A \cup B \in I$ and, for each $A \in I$ and each $B \subset A$, we have $B \in I$. A nonempty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\Phi \notin \mathcal{F}$; for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $A \subset B$, we have $B \in \mathcal{F}$. An ideal I is called nontrivial ideal if $I \neq \Phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a nontrivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A nontrivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A nontrivial ideal I is *maximal* if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset.

We remark that if $I = I_f = \{A \subseteq \mathbb{N} : A$ is a finite subset$\}$, then the corresponding convergence coincides with the usual convergence. Also, if $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, then the corresponding convergence coincides with the statistical convergence (where $\delta(A)$ denotes the natural density of the set A). In the above cases, both I_f and I_δ are nontrivial admissible ideals of $\mathbb{N}$.

Kumar [32] defined the notions of I and I^*-convergence of double sequence and studied some properties of these notions. Recently, Das et al. [33] introduced the concepts of I and I^*-convergence of double sequences in the setting of metric space and established some relationship between these types of convergence. Quite recently, Mursaleen and Mohiuddine defined and studied the notion of I-convergence, I^*-convergence, I-limit points, and I-cluster points for single and double sequences, in [34, 35], respectively, in probabilistic normed spaces. Şahiner et al. [36] and Gürdal and Açik [37] introduced the notion of ideal convergence and I-Cauchy sequence in 2-normed spaces, respectively. Mursaleen and Alotaibi [38] introduced the notion of ideal convergence in random 2-normed spaces and later on it was extended by Mohiuddine et al. [39] from single to double sequences. For more details on these concepts, one can be referred to [40–52].

Now we recall the definition of locally solid Riesz spaces and some related concepts as follows. Let X be a real vector space and let $\leq$ be a partial order on this space. X is said to be an *ordered vector space* if it satisfies the following properties:

(1) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$;

(2) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partial ordering, then X is said to be a *Riesz space* (or a *vector lattice*)(see [53]).

For an element x of a Riesz space X, the positive part of x is defined by $x^+ = x \vee \overline{\theta} = \sup\{x, \overline{\theta}\}$, the negative part of x by $x^- = (-x) \vee \overline{\theta}$, and the absolute value of x by $|x| = x \vee (-x)$, where $\overline{\theta}$ is the zero element of X.

A subset S of X is said to be *solid* if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A topology τ on a real vector space X that makes the addition and scalar multiplication continuous is said to be a linear topology, that is, when the mappings

$$\begin{gathered}(x, y) \longrightarrow (x + y) \quad (\text{from } (X \times X, \tau \times \tau) \longrightarrow (X, \tau)), \\ (\lambda, x) \longrightarrow (\lambda x) \quad \left(\text{from } \left(\mathbb{R} \times X, \tau' \times \tau\right) \longrightarrow (X, \tau)\right)\end{gathered} \tag{1}$$

are continuous, where τ' is the usual topology on $\mathbb{R}$. In this case the pair (X, τ) is called a *topological vector space*.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties.

(1) Each $Y \in N$ is a *balanced set*; that is, $ax \in Y$ holds for all $x \in Y$ and every $a \in \mathbb{R}$ with $|a| \leq 1$.

(2) Each $Y \in N$ is an *absorbing set*; that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.

(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally solid* (see [54]) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space X equipped with a locally solid topology τ. For more details on these concepts, one can be referred to [55–57].

Throughout the paper, the symbol N_{sol} will stand for a base at zero consisting of solid sets and satisfying conditions (1), (2), and (3) in a locally solid topology. Also we assume that I_2 is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$.

2. Ideal Convergence of Double Sequences in LSR-Spaces

Throughout the paper X will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability. For our convenience, here and in what follows, we will write an LSR-space instead of a locally solid Riesz space.

The notion of convergence for double sequence was first introduced by Pringsheim [58] as follows. We say that a double sequence $x = (x_{j,k})_{j,k\in\mathbb{N}}$ of reals is convergent to L in Pringsheim's sense (briefly, P-convergent) provided that given $\epsilon > 0$ there exists a positive integer N such that $|x_{j,k} - L| < \epsilon$ whenever $j, k \geq N$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m, n)$ denotes the number of (i, j) in K such that $i \leq m$ and $j \leq n$ (see [22]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty}(|K(m, n)|/mn)$. In this case, the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense; then we say that K has a *double natural density* and is defined by $P - \lim_{m,n\to\infty}(|K(m, n)|/mn) = \delta_2(K)$.

In the recent past, Mohiuddine et al. [25] introduced the notion of statistical convergence of double sequences in LSR-space as follows. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $S_2(\tau)$-*convergent* to an element x_0 if for each τ-neighborhood V of zero

$$\delta_2\left(\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\}\right) = 0. \tag{2}$$

Now we introduce the notions of $I_2(\tau)$-convergence and $I_2(\tau)$-bounded double sequences in LSR-spaces.

Definition 1. Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$-*convergent* to an element x_0 of X if for each τ-neighborhood V of zero

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\right\} \in I_2. \tag{3}$$

That is,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\right\} \in \mathcal{F}. \tag{4}$$

In this case, one writes $I_2(\tau)\text{-}\lim_{k,l\to\infty} x_{k,l} = x_0$ or $(x_{k,l}) \xrightarrow{I_2(\tau)} x_0$.

Definition 2. Let (X, τ) be a LSR-space. Then, a double sequence $(x_{k,l})$ of points in X is said to be $I_2(\tau)$-*bounded* in X if, for each τ-neighborhood V of zero, there is some $a > 0$,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : a x_{k,l} \notin V\right\} \in I_2. \tag{5}$$

Definition 3. Let (X, τ) be a LSR-space. One says that a double sequence $x = (x_{k,l})$ is $I_2(\tau)$-*Cauchy* in X if, for each τ-neighborhood V of zero, there exist $p, q \in \mathbb{N}$ such that, for all $k, m \geq p$ and $l, n \geq q$,

$$\left\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{m,n} \notin V\right\} \in I_2. \tag{6}$$

Definition 4. Let (X, τ) be a LSR-space. Then, a double sequence $x = (x_{k,l})$ in X is said to be $I_2^*(\tau)$-convergent to x_0 if there is a set $K = \{(k, l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $k, l = 1, 2, \ldots$, with $K \in F$ such that $\lim_{k,l} x_{k,l} = x_0$. In this case, one writes $I_2^*(\tau)$-$\lim_{k,l} x_{k,l} = x_0$.

Theorem 5. *Let (X, τ) be a LSR-space. Every $I_2(\tau)$-convergent sequence in X has only one limit.*

Proof. Suppose that $x = (x_{k,l})$ is a double sequence in X such that $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = y_0$. Let V be any τ-neighborhood of zero. Also for each τ-neighborhood V of zero there is a set $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let W in N_{sol} be such that $W + W \subseteq Y$. We define the sets A_1 and A_2 as follows:

$$A_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}, \quad A_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - y_0 \in W\}. \tag{7}$$

Since $I_2(\tau)\text{-}\lim x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim x_{k,l} = y_0$, we get $A_1, A_2 \in \mathcal{F}$. Now, let $A = A_1 \cap A_2$. Then we have

$$x_0 - y_0 = x_0 - x_{k,l} + x_{k,l} - y_0 \in W + W \subseteq Y \subseteq V. \tag{8}$$

As we know, intersection of all τ-neighborhoods V of zero is the singleton set $\{\overline{\theta}\}$ because (X, τ) is Hausdorff. Hence $x_0 - y_0 = 0$; that is, $x_0 = y_0$. □

Theorem 6. *Let (X, τ) be a LSR-space and let $(x_{k,l})$ and $(y_{k,l})$ be two double sequences of points in X. Then,*

(i) *if $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = y_0$, then $I_2(\tau)\text{-}\lim_{k,l}(x_{k,l} + y_{k,l}) = x_0 + y_0$;*

(ii) *if $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$, then $I_2(\tau)\text{-}\lim_{k,l} a x_{k,l} = a x_0$ for $a \in \mathbb{R}$.*

Proof. Assume that $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = y_0$. Suppose that V is an arbitrary τ-neighborhood of zero. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Thus, we can write

$$B_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}, \quad B_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - y_0 \in W\}. \tag{9}$$

Then we have $B_1, B_2 \in \mathcal{F}$.

Let $B = B_1 \cap B_2$. Hence we have $B \in \mathcal{F}$ and

$$(x_{k,l} + y_{k,l}) - (x_0 + y_0) = (x_{k,l} - x_0) + (y_{k,l} - y_0) \in W + W \subseteq Y \subseteq V. \tag{10}$$

Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} + y_{k,l}) - (x_0 + y_0) \in V\} \in \mathcal{F}. \tag{11}$$

Since V is arbitrary, we have $I_2(\tau)\text{-}\lim(x_{k,l} + y_{k,l}) = x_0 + y_0$.

(ii) Suppose that $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$ and also suppose that V is an arbitrary τ-neighborhood of zero. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$, so we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in Y\} \in \mathcal{F}. \tag{12}$$

Since Y is balanced, $a(x_{k,l} - x_0) \in Y$ holds for all $x_{k,l} - x_0 \in Y$ and for every $a \in \mathbb{R}$ with $|a| \le 1$. Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in Y\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : a x_{k,l} - a x_0 \in Y\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\}. \tag{13}$$

Thus, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in V\} \in \mathcal{F} \tag{14}$$

for each τ-neighborhood V of zero. Now let $|a| > 1$ and $[|a|]$ be the smallest integer greater than or equal to $|a|$. Then there exists $W \in N_{\text{sol}}$ such that $[|a|]W \subseteq Y$. From our assumption that $I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0$, we obtain that

$$K = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\} \in \mathcal{F}. \tag{15}$$

Therefore

$$|a x_{k,l} - a x_0| = |a| \, |x_{k,l} - x_0| \le [|a|] \, |x_{k,l} - x_0| \in [|a|] \, W \subseteq Y \subseteq V. \tag{16}$$

Since Y is solid, $a x_k - a x_0 \in Y$. It follows that $a x_{k,l} - a x_0 \in V$. Thus,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : a x_{k,l} - a x_0 \in V\} \in \mathcal{F}, \tag{17}$$

for each τ-neighborhood V of zero. We conclude that $I_2(\tau)\text{-}\lim_{k,l} a x_{k,l} = a x_0$. □

Theorem 7. *Let (X, τ) be a LSR-space. If a double sequence $(x_{k,l})$ in X is $I_2(\tau)$-convergent, then it is $I_2(\tau)$-bounded.*

Proof. Assume that $I_2(\tau)\text{-}\lim_{k,l \to \infty} x_{k,l} = x_0$. Suppose V is an arbitrary τ-neighborhood of zero. Then, there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Using our assumption, we obtain that

$$A = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \in I_2. \tag{18}$$

Since W is absorbing, there exists $a > 0$ such that $a x_0 \in W$. Let b be such that $|b| \le 1$ and $b \le a$. Since W is solid and $|b x_0| \le |a x_0|$, we have $b x_0 \in W$. Also, since W is balanced, $x_{k,l} - x_0 \in W$ implies $b(x_{k,l} - x_0) \in W$. Then we have

$$b x_{k,l} = b(x_{k,l} - x_0) + b x_0 \in W + W \subseteq V, \quad \text{for each } k, l \in \mathbb{N} - A. \tag{19}$$

Thus

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : b x_{k,l} \notin W\} \in I_2. \tag{20}$$

Hence $(x_{k,l})$ is $I_2(\tau)$-bounded. □

Theorem 8. *Let (X, τ) be a LSR-space and let $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$ be three double sequences of points in X such that*

(i) *$x_{k,l} \le y_{k,l} \le z_{k,l}$, for all $k, l \in \mathbb{N}$,*

(ii) *$I_2(\tau)\text{-}\lim_{k,l} x_{k,l} = x_0 = I_2(\tau)\text{-}\lim_{k,l} z_{k,l}$.*

Then $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = x_0$.

Proof. Suppose that the given conditions (i) and (ii) hold for the double sequences $(x_{k,l})$, $(y_{k,l})$, and $(z_{k,l})$. Suppose V is an arbitrary τ-neighborhood of zero. Then, there exists $Y \in N_{\text{sol}}$

such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. It follows from (ii) that $P, Q \in F$, where

$$P = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}, \quad Q = \{(k,l) \in \mathbb{N} \times \mathbb{N} : z_{k,l} - x_0 \in W\}. \tag{21}$$

Also from the given condition (i), we have

$$x_{k,l} - x_0 \leq y_{k,l} - x_0 \leq z_{k,l} - x_0 \Longrightarrow |y_{k,l} - x_0| \leq |x_{k,l} - x_0| + |z_{k,l} - x_0| \in W + W \subseteq Y. \tag{22}$$

Since Y is solid, we have $y_{k,l} - x_0 \in Y \subseteq V$. Thus,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \in V\} \in \mathcal{F}, \tag{23}$$

for each τ-neighborhood V of zero. Thus $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = x_0$. □

Theorem 9. *Let (X, τ) be a LSR-space. A double sequence $(x_{k,l})$ is $I_2(\tau)$-convergent to x_0 in X if and only if for each τ-neighborhood V of zero there exists a subsequence $(x_{k'(r),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty} x_{k'(r),l'(s)} = x_0$ and*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{24}$$

Proof. Suppose that $I_2(\tau)\text{-}\lim_{k,l\to\infty} x_{k,l} = x_0$. Also, suppose that V is an arbitrary τ-neighborhood of zero. Let $\{V_i\}$ be a sequence of nested base of τ-neighborhoods of zero. For each $i \in \mathbb{N}$, put

$$E^{(i)} = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V_i\}. \tag{25}$$

Then, $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F$. Let $m(1)$ and $n(1)$ be such that $r > m(1)$ and $s > n(1)$, respectively. Then $E^{(1)} \neq \phi$. For $r, s \in \mathbb{N}$ such that $m(1) \leq r < m(2)$ and $n(1) \leq s < n(2)$, choose $k'(r), l'(s) \in E^{(i)}$; that is, $x_{k'(r),l'(s)} - x_0 \in V_1$. In general, choose $m(p+1) > m(p)$ and $n(p+1) > n(p)$ such that $r > m(p+1)$ and $s > n(p+1)$ hold. Then $E^{(p+1)} \neq \phi$. Therefore for all r, s which satisfy $m(p) \leq r < m(p+1)$ and $n(p) \leq s < n(p+1)$, choose $k'(r), l'(s) \in E^{(p)}$; that is, $x_{k'(r),l'(s)} - x_0 \in V_p$. Hence, it follows that $\lim_{r,s} x_{k'(r),l'(s)} = x_0$.

Since V is an arbitrary τ-neighborhood of zero, there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Now

$$x_{k,l} - x_{k'(r),l'(s)} = x_{k,l} - x_0 + x_{k'(r),l'(s)} - x_0 \in W + W \subseteq Y \subseteq V. \tag{26}$$

Also $I_2(\tau)\text{-}\lim_{k,l\to\infty} x_{k,l} = x_0$ and $\lim_{r\to\infty} x_{k'(r),l'(s)} = x_0$ imply that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{27}$$

Next suppose for an arbitrary τ-neighborhood V of zero that there exists a subsequence $(x_{k'(r),l'(s)})$ of $(x_{k,l})$ such that $\lim_{r,s\to\infty} x_{k'(r),l'(s)} = x_0$ and

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r),l'(s)} \notin V\} \in I_2. \tag{28}$$

Since V is any τ-neighborhood of zero, we choose $W \in N_{\text{sol}}$ such that $W + W \subseteq V$. Then we have

$$x_{k,l} - x_0 = x_{k,l} - x_{k'(r),l'(s)} + x_{k'(r),l'(s)} - x_0 \in W + W \subseteq V. \tag{29}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{k'(r)} \notin W\} \cup \{(r,s) \in \mathbb{N} \times \mathbb{N} : x_{k'(r),l'(s)} - x_0 \notin W\}. \tag{30}$$

Therefore

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \tag{31}$$

□

Theorem 10. *If $\lim_{k,l\to\infty} x_{k,l} = x_0$ and $I_2(\tau)\text{-}\lim_{k,l\to\infty} y_{k,l} = 0$, then $I_2(\tau)\text{-}\lim_{k,l\to\infty}(x_{k,l} + y_{k,l}) = \lim_{k,l\to\infty} x_{k,l}$.*

Proof. Let V be any τ-neighborhood of 0. Then there exists $Y \in N_{\text{sol}}$ such that $Y \subseteq V$. Let $W \in N_{\text{sol}}$ such that $W + W \subseteq Y$. Since $\lim_{k,l\to\infty} x_{k,l} = x_0$, then there exist integers n_0, m_0 such that $k \geq n_0, l \geq m_0$ implies that $x_{k,l} - x_0 \in W$. Hence

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \subseteq \mathbb{N} \times \mathbb{N} - \{(n_0, m_0)\}. \tag{32}$$

By the assumption $I_2(\tau)\text{-}\lim_{k,l\to\infty} y_{k,l} = 0$, $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \notin W\} \in I_2$. Thus

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} - x_0) + y_{k,l} \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin W\} \cup \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \notin W\}. \tag{33}$$

That is,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : (x_{k,l} - x_0) + y_{k,l} \notin V\} \in I_2. \tag{34}$$

This implies that $I_2(\tau)\text{-}\lim_{k,l\to\infty}(x_{k,l} + y_{k,l}) = \lim_{k,l\to\infty} x_{k,l}$. □

Theorem 11. *Let (X, τ) be a LSR-space and let $x = (x_{k,l})$ be a double sequence in X. If there is a $I_2(\tau)$-convergent sequence $y = (y_{k,l})$ in X such that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ then x is also $I_2(\tau)$-convergent.*

Proof. Suppose that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \in I_2$ and $I_2(\tau)\text{-}\lim_{k,l} y_{k,l} = x_0$. Then for an arbitrary τ-neighborhood V of zero, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\} \in I_2. \tag{35}$$

Now,

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} \neq x_{k,l} \notin V\} \cup \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - x_0 \notin V\}. \tag{36}$$

Therefore, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \in I_2. \quad (37)$$

□

Theorem 12. *Let (X, τ) be a LSR-space. If a double sequence $x = (x_{k,l})$ is $I_2^*(\tau)$-convergent to x_0, then it is $I_2(\tau)$-convergent to x_0.*

Proof. Suppose that $I_2^*(\tau)$-$\lim_k x_{k,l} = x_0$. Let V be an arbitrary τ-neighborhood V of zero. Since $I_2^*(\tau)$-$\lim_{k,l} x_{k,l} = x_0$, there is a set $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $(k, l \in \mathbb{N})$ with $K \in F$ such that $k \geq n$, $l \geq m$ and $(k,l) \in K$ implies $x_{k,l} - x_0 \in V$. Then

$$\begin{aligned} K_1 &= \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\} \\ &\subseteq \mathbb{N} \times \mathbb{N} - \{(k_{n+1}, l_{m+1}), (k_{n+2}, l_{m+2}), \ldots\}. \end{aligned} \quad (38)$$

Therefore

$$K_1 \in I_2. \quad (39)$$

Hence x is $I_2(\tau)$-convergent to x_0. □

Theorem 13. *The sequential method $I_2(\tau)$ is regular.*

Proof of the theorem is straightforward, so it is omitted.

From Theorem 12, we can easily obtain the following useful result.

Theorem 14. *The sequential method $I_2(\tau)$ is subsequential.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. Fast, "Sur la convergence statistique," *Colloquium Mathematicum*, vol. 2, pp. 241–244, 1951.

[2] H. Steinhaus, "Sur la convergence ordinate et la convergence asymptotique," *Colloquium Mathematicum*, vol. 2, pp. 73–84, 1951.

[3] R. C. Buck, "Generalized asymptotic density," *American Journal of Mathematics*, vol. 75, pp. 335–346, 1953.

[4] T. Šalát, "On statistical convergence of real numbers," *Mathematica Slovaca*, vol. 30, pp. 139–150, 1980.

[5] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *The American Mathematical Monthly*, vol. 66, pp. 361–375, 1959.

[6] J. A. Fridy, "On statistical convergence," *Analysis*, vol. 5, no. 4, pp. 301–313, 1985.

[7] H. Çakalli, "Lacunary statistical convergence in topological groups," *Indian Journal of Pure and Applied Mathematics*, vol. 26, no. 2, pp. 113–119, 1995.

[8] H. Çakalli, "On statistical convergence in topological groups," *Pure and Applied Mathematika Sciences*, vol. 43, no. 1-2, pp. 27–31, 1996.

[9] H. Çakalli and M. K. Khan, "Summability in topological spaces," *Applied Mathematics Letters*, vol. 24, no. 3, pp. 348–352, 2011.

[10] G. Di Maio and L. D. R. Kočinac, "Statistical convergence in topology," *Topology and Its Applications*, vol. 156, no. 1, pp. 28–45, 2008.

[11] A. Caserta, G. Di Maio, and L. D. R. Kočinac, "Statistical convergence in function spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 420419, 11 pages, 2011.

[12] B. Hazarika, "On ideal convergence in topological groups," *Scientia Magna*, vol. 7, no. 4, pp. 80–86, 2011.

[13] B. Hazarika, "On generalized difference ideal convergence in random 2-normed spaces," *Filomat*, vol. 26, no. 6, pp. 1273–1282, 2012.

[14] B. Hazarika, "Lacunary generalized difference statistical convergence in random 2-normed spaces," *Proyecciones. Journal of Mathematics*, vol. 31, no. 4, pp. 373–390, 2012.

[15] I. J. Maddox, "Statistical convergence in a locally convex space," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 104, no. 1, pp. 141–145, 1988.

[16] S. A. Mohiuddine and M. Aiyub, "Lacunary statistical convergence in random 2-normed spaces," *Applied Mathematics & Information Sciences*, vol. 6, no. 3, pp. 581–585, 2012.

[17] S. A. Mohiuddine, H. Şevli, and M. Cancan, "Statistical convergence in fuzzy 2-normed space," *Journal of Computational Analysis and Applications*, vol. 12, no. 4, pp. 787–798, 2010.

[18] S. A. Mohiuddine, A. Alotaibi, and M. Mursaleen, "A new variant of statistical convergence," *Journal of Inequalities and Applications*, vol. 2013, article 309, 8 pages, 2013.

[19] E. Savaş and S. A. Mohiuddine, "$\bar{\lambda}$-statistically convergent double sequences in probabilistic normed spaces," *Mathematica Slovaca*, vol. 62, no. 1, pp. 99–108, 2012.

[20] C. Belen and S. A. Mohiuddine, "Generalized weighted statistical convergence and application," *Applied Mathematics and Computation*, vol. 219, no. 18, pp. 9821–9826, 2013.

[21] N. L. Braha, H. M. Srivastava, and S. A. Mohiuddine, "A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean," *Applied Mathematics and Computation*, vol. 228, pp. 162–169, 2014.

[22] Mursaleen and O. H. H. Edely, "Statistical convergence of double sequences," *Journal of Mathematical Analysis and Applications*, vol. 288, no. 1, pp. 223–231, 2003.

[23] S. A. Mohiuddine, H. Şevli, and M. Cancan, "Statistical convergence of double sequences in fuzzy normed spaces," *Filomat*, vol. 26, no. 4, pp. 673–681, 2012.

[24] M. Mursaleen and S. A. Mohiuddine, "Statistical convergence of double sequences in intuitionistic fuzzy normed spaces," *Chaos, Solitons & Fractals*, vol. 41, no. 5, pp. 2414–2421, 2009.

[25] S. A. Mohiuddine, A. Alotaibi, and M. Mursaleen, "Statistical convergence of double sequences in locally solid Riesz spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 719729, 9 pages, 2012.

[26] H. Albayrak and S. Pehlivan, "Statistical convergence and statistical continuity on locally solid Riesz spaces," *Topology and Its Applications*, vol. 159, no. 7, pp. 1887–1893, 2012.

[27] S. A. Mohiuddine, B. Hazarika, and A. Alotaibi, "Double lacunary density and some inclusion results in locally solid

Riesz spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 507962, 8 pages, 2013.

[28] S. A. Mohiuddine and A. Alotaibi, "Statistical summability of double sequences through de la Vallée-Poussin mean in probabilistic normed spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 215612, 5 pages, 2013.

[29] S. A. Mohiuddine and M. A. Alghamdi, "Statistical summability through a lacunary sequence in locally solid Riesz spaces," *Journal of Inequalities and Applications*, vol. 2012, article 225, 9 pages, 2012.

[30] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with Applications to Economics*, vol. 105 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2nd edition, 2003.

[31] P. Kostyrko, T. Šalát, and W. Wilczyński, "$\mathcal{I}$-convergence," *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–685, 2000-2001.

[32] V. Kumar, "On I and I^*-convergence of double sequences," *Mathematical Communications*, vol. 12, no. 2, pp. 171–181, 2007.

[33] P. Das, P. Kostyrko, W. Wilczyński, and P. Malik, "I and I^*-convergence of double sequences," *Mathematica Slovaca*, vol. 58, no. 5, pp. 605–620, 2008.

[34] M. Mursaleen and S. A. Mohiuddine, "On ideal convergence in probabilistic normed spaces," *Mathematica Slovaca*, vol. 62, no. 1, pp. 49–62, 2012.

[35] M. Mursaleen and S. A. Mohiuddine, "On ideal convergence of double sequences in probabilistic normed spaces," *Mathematical Reports*, vol. 12(62), no. 4, pp. 359–371, 2010.

[36] A. Şahiner, M. Gürdal, S. Saltan, and H. Gunawan, "Ideal convergence in 2-normed spaces," *Taiwanese Journal of Mathematics*, vol. 11, no. 5, pp. 1477–1484, 2007.

[37] M. Gürdal and I. Açık, "On I-Cauchy sequences in 2-normed spaces," *Mathematical Inequalities & Applications*, vol. 11, no. 2, pp. 349–354, 2008.

[38] M. Mursaleen and A. Alotaibi, "On I-convergence in random 2-normed spaces," *Mathematica Slovaca*, vol. 61, no. 6, pp. 933–940, 2011.

[39] S. A. Mohiuddine, A. Alotaibi, and S. M. Alsulami, "Ideal convergence of double sequences in random 2-normed spaces," *Advances in Difference Equations*, vol. 2012, article 149, 8 pages, 2012.

[40] H. Çakalli and B. Hazarika, "Ideal quasi-Cauchy sequences," *Journal of Inequalities and Applications*, vol. 2012, article 234, 11 pages, 2012.

[41] E. Dündar, Ö. Talo, and F. Başar, "Regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequences of fuzzy numbers," *International Journal of Analysis*, vol. 2013, Article ID 749684, 7 pages, 2013.

[42] B. Hazarika and S. A. Mohiuddine, "Ideal convergence of random variables," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 148249, 7 pages, 2013.

[43] B. K. Lahiri and P. Das, "I and I^*-convergence in topological spaces," *Mathematica Bohemica*, vol. 130, no. 2, pp. 153–160, 2005.

[44] H. I. Miller, "A measure theoretical subsequence characterization of statistical convergence," *Transactions of the American Mathematical Society*, vol. 347, no. 5, pp. 1811–1819, 1995.

[45] M. Mursaleen, S. A. Mohiuddine, and O. H. H. Edely, "On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 603–611, 2010.

[46] F. Nuray and W. H. Ruckle, "Generalized statistical convergence and convergence free spaces," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 2, pp. 513–527, 2000.

[47] T. Šalát, B. C. Tripathy, and M. Ziman, "On some properties of I-convergence," *Tatra Mountains Mathematical Publications*, vol. 28, pp. 279–286, 2004.

[48] T. Šalát, B. C. Tripathy, and M. Ziman, "On I-convergence field," *Italian Journal of Pure and Applied Mathematics*, no. 17, pp. 45–54, 2005.

[49] B. C. Tripathy and B. Hazarika, "I-monotonic and I-convergent sequences," *Kyungpook Mathematical Journal*, vol. 51, no. 2, pp. 233–239, 2011.

[50] B. C. Tripathy and B. Hazarika, "I-convergent sequence spaces associated with multiplier sequences," *Mathematical Inequalities & Applications*, vol. 11, no. 3, pp. 543–548, 2008.

[51] B. Tripathy and B. C. Tripathy, "On I-convergent double sequences," *Soochow Journal of Mathematics*, vol. 31, no. 4, pp. 549–560, 2005.

[52] U. Yamancı and M. Gürdal, "$\mathcal{I}^{\mathcal{K}}$-convergence in the topology induced by random 2-normed spaces," *International Journal of Analysis*, vol. 2013, Article ID 451762, 7 pages, 2013.

[53] A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer, Berlin, Germany, 1997.

[54] G. T. Roberts, "Topologies in vector lattices," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 48, pp. 533–546, 1952.

[55] L. V. Kantorovich, "Lineare halbgeordnete Raume," *Recreational Mathematics*, vol. 2, pp. 121–168, 1937.

[56] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam, The Netherlands, 1971.

[57] F. Riesz, "Sur la décomposition des opérations fonctionelles linéaires," in *Atti Del Congr. Internaz. Dei Mat., 3, Bologna, 1928*, pp. 143–148, Zanichelli, 1930.

[58] A. Pringsheim, "Zur Theorie der zweifach unendlichen Zahlenfolgen," *Mathematische Annalen*, vol. 53, no. 3, pp. 289–321, 1900.