

Research Article

Green's Function and Positive Solutions for a Second-Order Singular Boundary Value Problem with Integral Boundary Conditions and a Delayed Argument

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This paper investigates the expression and properties of Green's function for a second-order singular boundary value problem with integral boundary conditions and delayed argument $-x''(t) - a(t)x'(t) + b(t)x(t) = \omega(t)f(t, x(\alpha(t)))$, $t \in (0, 1)$; $x'(0) = 0$, $x(1) - \int_0^1 h(t)x(t)dt = 0$, where $a \in ([0, 1], [0, +\infty))$, $b \in C([0, 1], (0, +\infty))$ and, ω may be singular at $t = 0$ or/and at $t = 1$. Furthermore, several new and more general results are obtained for the existence of positive solutions for the above problem by using Krasnosel'skii's fixed point theorem. We discuss our problems with a delayed argument, which may concern optimization issues of some technical problems. Moreover, the approach to express the integral equation of the above problem is different from earlier approaches. Our results cover a second-order boundary value problem without deviating arguments and are compared with some recent results.

1. Introduction

Boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [1], semiconductor problems [2], hydrodynamic problems [3], and so on. It is interesting to point out that such problems include two-, three-, and multipoint and nonlocal boundary value problems as special cases and have been extensively studied in the last ten years; see, for example, [4–18]. Recently, Feng et al. [19] applied the fixed point theory in a cone for strict set contraction operators to study the existence and multiplicity of positive solutions for the problem given by

$$\begin{aligned} x'' + f(t, x) &= \theta, \quad t \in (0, 1), \\ x(0) &= \theta, \quad x(1) = \int_0^1 g(t)x(t)dt, \end{aligned} \quad (1)$$

where θ is the zero element of a real Banach space E .

At the same time, a class of boundary value problems with deviating arguments are receiving much attention. For

example, in [20], Yang et al. studied the existence and multiplicity of positive solutions to a three-point boundary value problem with an advanced argument:

$$\begin{aligned} x''(t) + a(t)f(x(\alpha(t))) &= 0, \quad t \in (0, 1), \\ x(0) &= 0, \quad bx(\eta) = x(1), \end{aligned} \quad (2)$$

where $0 < \eta < 1$, $b > 0$, and $1 - b\eta > 0$. The main tool is the fixed point index theory. It is clear that the solution of [20] is concave when $a(t) \geq 0$ on $[0, 1]$ and $f(x) \geq 0$ on $[0, \infty)$. However, few papers have reported the same problems where the solution is without concavity; for example, see some recent excellent results and applications of the case of ordinary differential equations with deviating arguments to a variety of problems from Jankowski [21–23], Jiang and Wei [24], Wang [25], Wang et al. [26], and Hu et al. [27]. This paper will resolve this problem.

Consider the second-order singular boundary value problem with integral boundary conditions and a delayed argument:

$$\begin{aligned} Lx &= \omega(t) f(t, x(\alpha(t))), \quad t \in (0, 1), \\ x'(0) &= 0, \quad x(1) - \int_0^1 h(t) x(t) dt = 0, \end{aligned} \tag{3}$$

where L denotes the linear operator and

$$Lx := -x'' - ax' + bx, \tag{4}$$

where $a \in C([0, 1], [0, +\infty))$, $b \in C([0, 1], (0, +\infty))$, $f \in C([0, 1] \times [0, +\infty) \rightarrow [0, +\infty))$, and ω may be singular at $t = 0$ or/and at $t = 1$.

Throughout this paper, we assume that $\alpha(t) \neq t$ on $J = [0, 1]$. In addition, ω, f, α , and h satisfy the following:

- (H₁) $\omega \in C((0, 1), [0, +\infty))$ with $0 < \int_0^1 \omega(s) ds < \infty$ and ω does not vanish on any subinterval of $(0, 1)$;
- (H₂) $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, $\alpha \in C(J, J)$ with $\alpha(t) \leq t$ on J ;
- (H₃) $h \in C[0, 1]$ is nonnegative with $v \in [0, 1)$, where

$$v = \int_0^1 h(t) \phi(t) dt, \tag{5}$$

where ϕ satisfies

$$-\phi''(t) - a\phi'(t) + b\phi(t) = 0, \quad \phi'(0) = 0, \quad \phi(1) = 1. \tag{6}$$

Remark 1. Generally, when $y(t) \geq 0$ on J , the solution x is not concave for the linear equation

$$Lx - y(t) = 0. \tag{7}$$

This means that the method depending on concavity is no longer valid, and we need to introduce a new method to study this kind of problems.

Remark 2. For simplicity we only consider Neumann boundary conditions since all the results obtained in this paper can also be adapted with minor changes to the other boundary conditions.

Some special cases of (3), such as boundary value problems with delay, have been investigated [28–32]. It is not difficult to see that the corresponding function f appearing on the right-hand side depends on $x(t-\tau)$, $\tau > 0$, where initial function x is given on the initial set, for example, $[-\tau, 0]$. T. Jankowski and R. Jankowski [33, 34] pointed out that, in such cases $\alpha(t) = t - \tau$, there are some problems with a constant delay τ . If we consider the differential problem on intervals $[0, k]$, where $k \leq \tau$, then it means that we have no delays; we have such a situation in [32]. If $k > \tau$, then it is easy to solve the differential equation on $[0, \tau]$, since we have the solution on the initial set $[-\tau, 0]$. Continuing this process, we can find a solution on the whole interval $[0, k]$, by using the method of steps. In our paper, for example, the deviating argument α can have a form $\alpha(t) = \rho t = t - (1 - \rho)t$ with a fixed number

$\rho \in (0, 1)$, so the delay $(1 - \rho)t$ is a function of t . In this case, the initial set reduces to one point $t = 0$, and we cannot apply the step method. To our knowledge, it is the first paper in which positive solution has been investigated for a second-order singular differential equation with a delayed argument under the case that $Lx := -x'' - ax' + bx$.

Being directly inspired by [5, 12, 20, 21], the authors will prove several new and more general results for the existence of positive solutions for problem (3) by using fixed point theories in a cone. Another contribution of this paper is to study the expression and properties of Green's function associated with problem (3). The expression of the integral equation is simpler than that of [5, 12].

The organization of this paper is as follows. In Section 2, we present the expression and properties of Green's function associated with the problem (3). In Section 3, we present some definitions and lemmas which are needed throughout this paper. In Section 4, we use fixed point theorem to obtain the existence of positive solutions for problem (3) with a delayed argument α . In particular, our results in these sections are new when $\alpha(t) \equiv t$ on $t \in J$. Finally, in Section 5, three examples are also included to illustrate the main results.

2. Expression and Properties of Green's Function

Theorem 3. Assume that $v \neq 1$. Then for any $y \in C[0, 1]$, the boundary value problem

$$\begin{aligned} -x''(t) - a(t)x'(t) + b(t)x(t) - y(t) &= 0, \quad t \in (0, 1), \\ x'(0) &= 0, \quad x(1) - \int_0^1 h(t)x(t) dt = 0 \end{aligned} \tag{8}$$

has a unique solution

$$x(t) = \int_0^1 H(t, s) q(s) y(s) ds, \tag{9}$$

where

$$\begin{aligned} q(t) &= \exp\left(\int_0^t a(s) ds\right), \quad H(t, s) = G(t, s) + G_1(t, s), \\ G(t, s) &= \frac{1}{\Delta} \begin{cases} \phi(s)\psi(t), & \text{if } 0 \leq s \leq t \leq 1, \\ \phi(t)\psi(s), & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \\ G_1(t, s) &= \frac{\phi(t)}{1 - v} \int_0^1 G(\tau, s) h(\tau) d\tau. \end{aligned} \tag{10}$$

Here ϕ and ψ satisfy (6) and

$$L\psi = 0, \quad \psi(0) = 1, \quad \psi(1) = 0, \tag{11}$$

respectively.

Proof. The proof is similar to that of Lemma 2.3 in [5] and Lemma 2.1 in [35]. □

Remark 4. It is not difficult from [5, 12] to show that $\Delta := -\phi(0)\psi'(0) > 0$ and (i) ϕ is nondecreasing on J and $\phi > 0$ on J and (ii) ψ is strictly decreasing on J .

Remark 5. The expression of the integral equation (9) is different from that of (2.10) in [5] and that of (2.9) in [12].

Remark 6. Noticing $a \in C([0, 1], [0, +\infty))$, it follows from the definition of q that

$$1 \leq q(t) \leq e^M \quad \text{for } t \in J, \tag{12}$$

where $M = \max_{t \in J} a(t)$.

Lemma 7. Let $\xi \in (0, 1)$, G , G_1 , and H be given as in Theorem 3. Then one has the following results:

$$G(t, s) \leq G(s, s), \quad G_1(t, s) \leq G(1, s),$$

$$H(t, s) \leq H(s) \leq H^0, \tag{13}$$

$$\forall t, s \in J,$$

$$G(t, s) \geq \delta G(s, s), \quad G_1(t, s) \geq \phi(0) G_1(1, s),$$

$$H(t, s) \geq \delta H(s) \geq \delta H_0, \tag{14}$$

$$\forall t \in [0, \xi], \quad s \in J,$$

where

$$H(s) = G(s, s) + G_1(1, s), \quad H^0 = \max_{s \in J} H(s),$$

$$H_0 = \min_{s \in J} H(s), \quad \delta = \min \{ \psi(\xi), \phi(0) \}. \tag{15}$$

Proof. Noticing Remark 4, it follows from the definition of $G(t, s)$, $G_1(t, s)$, and $H(t, s)$ that (13) holds. Now, we show that (14) also holds.

In fact, for $t \in [0, \xi]$ and $s \in J$, we have

$$\frac{G(t, s)}{G(s, s)} \geq \min \left\{ \frac{\psi(t)}{\psi(s)}, \frac{\phi(t)}{\phi(s)} \right\} \geq \min \left\{ \frac{\psi(0)}{\psi(0)}, \frac{\phi(\xi)}{\phi(1)} \right\}$$

$$= \{ \psi(\xi), \phi(0) \} =: \delta. \tag{16}$$

Similarly, we can prove that $G_1(t, s) \geq G_1(0, s) \geq \phi(0)G_1(1, s)$ for $t \in [0, \xi]$ and $s \in J$. This and (15) imply that

$$H(t, s) \geq \delta G(s, s) + \phi(0) G_1(1, s) = \delta H(s),$$

$$\forall t \in [0, \xi], \quad s \in J. \tag{17}$$

This gives the proof of Lemma 7. □

Remark 8. It follows from (13) and (14) that

$$\delta H_0 \leq H(t, s) \leq H^0, \quad \forall t \in [0, \xi], \quad s \in J. \tag{18}$$

3. Preliminaries

In this section, we first present some definitions and lemmas which are needed throughout this paper.

Definition 9 (see [36]). Let E be a real Banach space over \mathbb{R} . A nonempty closed set $P \subset E$ is said to be a cone provided that

- (i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$;
- (ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Let $E = C[0, 1]$. Then E is a real Banach space with the norm $\| \cdot \|$ defined by

$$\|x\| = \max_{t \in J} |x(t)|, \quad x \in E. \tag{19}$$

Definition 10. A function $x \in E \cap C^2(0, 1)$ is called a solution of (3) if it satisfies (3). If $x(t) \geq 0$ and $x(t) \neq 0$ on J , then x is called a positive solution of (3).

Define a cone K in E by

$$K = \left\{ x \in E : x(t) \geq 0, \min_{t \in [0, \xi]} x(t) \geq \delta \|x\| \right\}. \tag{20}$$

Also, define, for a positive number r , Ω_r by

$$\Omega_r = \{x \in E : \|x\| < r\}. \tag{21}$$

Note that $\partial\Omega_r = \{x \in E : \|x\| = r\}$.

Define $T : K \rightarrow K$ by

$$(Tx)(t) = \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds. \tag{22}$$

Lemma 11. Assume that (H_1) – (H_3) hold. Then, $T(K) \subset K$ and $T : K \rightarrow K$ are completely continuous.

Proof. For $x \in K$, it follows from (13) and (22) that

$$\|Tx\| = \max_{t \in J} \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds$$

$$\leq \int_0^1 H(s) q(s) \omega(s) f(s, x(\alpha(s))) ds. \tag{23}$$

It follows from (14), (22), and (23) that

$$\min_{t \in [0, \xi]} (Tx)(t) = \min_{t \in [0, \xi]} \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds$$

$$\geq \delta \int_0^1 H(s) q(s) \omega(s) f(s, x(\alpha(s))) ds$$

$$\geq \delta \|Tx\|. \tag{24}$$

Thus, $T(K) \subset K$.

Next, by standard methods and Ascoli-Arzelà theorem, one can prove that $T : K \rightarrow K$ is completely continuous. So it is omitted, and the lemma is proved. □

Remark 12. From (22), we know that $x \in E$ is a solution of problem (3) if and only if x is a fixed point of operator T .

In the rest of this section, we state a well known fixed point theorem which we need later.

Lemma 13 (see [36]). *Let P be a cone in a real Banach space E . Assume Ω_1 and Ω_2 are bounded open sets in E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow P \tag{25}$$

is completely continuous such that either

- (i) $\|Ax\| \leq \|x\|$, for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \geq \|x\|$, for all $x \in P \cap \partial\Omega_2$, or
- (ii) $\|Ax\| \geq \|x\|$, for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \leq \|x\|$, for all $x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4. Existence of Single or Twin Positive Solutions

For convenience, we introduce the following notations:

$$\begin{aligned} f^0 &= \lim_{y \rightarrow 0} \sup \max_{t \in J} \frac{f(t, y)}{y}, & f_0 &= \lim_{y \rightarrow 0} \inf \min_{t \in J} \frac{f(t, y)}{y}, \\ f^\infty &= \lim_{y \rightarrow \infty} \sup \max_{t \in J} \frac{f(t, y)}{y}, & f_\infty &= \lim_{y \rightarrow \infty} \inf \min_{t \in J} \frac{f(t, y)}{y}. \end{aligned} \tag{26}$$

We also define as [37] i_0 = number of zeros in the set $\{f^0, f^\infty\}$ and i_∞ = number of infinities in the set $\{f_0, f_\infty\}$. Sun and Li [38] pointed out that $i_0, i_\infty = 0, 1$, or 2 , and there are six possible cases: (i) $i_0 = 0$ and $i_\infty = 0$; (ii) $i_0 = 0$ and $i_\infty = 1$; (iii) $i_0 = 0$ and $i_\infty = 2$; (iv) $i_0 = 1$ and $i_\infty = 0$; (v) $i_0 = 1$ and $i_\infty = 1$; and (vi) $i_0 = 2$ and $i_\infty = 0$. By using Krasnoselii's fixed point theorem in a cone, some results are obtained for the existence of at least one or two positive solutions of problem (3) for $\alpha(t) \leq t$ on J under the above six possible cases.

4.1. For the Case $i_0 = 1$ and $i_\infty = 1$. In this subsection, we discuss the existence of single positive solution for problem (3) under $i_0 = 1$ and $i_\infty = 1$.

For convenience, we introduce the following notations:

$$\gamma = \int_0^1 \omega(s) ds, \quad \gamma_1 = \int_\xi^1 \omega(s) ds. \tag{27}$$

Theorem 14. *Assume that (H_1) – (H_3) hold. If $i_0 = 1$ and $i_\infty = 1$, then problem (3) has at least one positive solution.*

Proof. First, we consider the case $f^0 = 0$ and $f_\infty = \infty$. Since $f^0 = 0$, then there exists $r > 0$ such that

$$f(t, x) \leq \frac{1}{H^0 \gamma e^M} x \quad \forall t \in J, \quad 0 \leq x \leq r. \tag{28}$$

Since $0 \leq \alpha(t) \leq t \leq 1$ on $[0, \xi]$, it follows from $0 \leq x(t) \leq r$ on J that

$$0 \leq x(\alpha(t)) \leq r \quad \text{for } t \in J. \tag{29}$$

Consequently, for any $t \in J$ and $x \in K \cap \partial\Omega_r$, (13) and (22) imply

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\leq H^0 e^M \int_0^1 \omega(s) \frac{1}{e^M \gamma H^0} x(\alpha(s)) ds \\ &\leq \frac{1}{\gamma} \int_0^1 \omega(s) ds \|x\| \\ &= \|x\|, \end{aligned} \tag{30}$$

which implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_r. \tag{31}$$

Next, turning to $f_\infty = \infty$, there exists R satisfying $0 < r < R$ such that

$$f(t, x) \geq \frac{1}{\delta H_0 \gamma_1} x, \quad \forall t \in J, \quad x \geq R. \tag{32}$$

Since $0 \leq \alpha(t) \leq t \leq \xi$ on $[0, \xi]$, it follows from $x(t) \geq R$ on J that

$$x(\alpha(t)) \geq R \quad \text{for } t \in [0, \xi]. \tag{33}$$

Hence, for $x \in K \cap \partial\Omega_R$, it follows from Remark 8 and (22) that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^1 q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^\xi \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^\xi \omega(s) \frac{1}{\delta H_0 \gamma_1} x(\alpha(s)) ds \\ &\geq \frac{1}{\delta \gamma_1} \int_0^\xi \omega(s) ds \delta \|x\| \\ &= \|x\|, \end{aligned} \tag{34}$$

which implies

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_R. \tag{35}$$

Thus, by (i) of Lemma 13, it follows that T has a fixed point x in $K \cap (\overline{\Omega}_R \setminus \Omega_r)$ with

$$r \leq \|x\| \leq R. \tag{36}$$

Remark 12 shows that problem (3) has at least one positive solution x with

$$r \leq \|x\| \leq R. \tag{37}$$

Next, we consider the case $f_0 = \infty$ and $f^\infty = 0$. Since $f_0 = \infty$, we can choose $r > 0$ such that

$$f(t, x) \geq \frac{1}{\delta H_0 \gamma_1} x, \quad \forall t \in J, \quad 0 \leq x \leq r. \quad (38)$$

Since $0 \leq \alpha(t) \leq t \leq \xi$ on $[0, \xi]$, it follows from $0 \leq x(t) \leq r$ on J that

$$0 \leq x(\alpha(t)) \leq r \quad \text{for } t \in [0, \xi]. \quad (39)$$

Consequently, for $x \in K \cap \partial\Omega_r$, it follows from Remark 8 and (22) that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^1 q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^\xi \omega(s) f(s, x(\alpha(s))) ds \\ &\geq \delta H_0 \int_0^\xi \omega(s) \frac{1}{\delta H_0 \gamma_1} x(\alpha(s)) ds \\ &\geq \frac{1}{\delta \gamma_1} \int_0^\xi \omega(s) ds \delta \|x\| \\ &= \|x\|, \end{aligned} \quad (40)$$

which implies

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_r. \quad (41)$$

If $f^\infty = 0$, we can choose $0 < \varepsilon < 1/H^0 \gamma e^M$ and $l > 0$ such that

$$f(t, x) \leq \varepsilon x \quad \text{for } t \in J, \quad x \geq l. \quad (42)$$

Letting $\zeta = \max_{t \in J, x \in [0, l]} f(t, x)$, then

$$0 \leq f(t, x) \leq \varepsilon x + \zeta \quad \text{for } t \in J, \quad x \in [0, \infty). \quad (43)$$

Since $0 \leq \alpha(t) \leq t \leq 1$ on J , it follows from $x(t) \geq l$ or $0 \leq x(t) \leq l$ on J that

$$x(\alpha(t)) \geq l \quad \text{or } 0 \leq x(\alpha(t)) \leq l \quad \text{for } t \in J. \quad (44)$$

Let $R \geq \max\{2r, e^M H^0 \gamma \zeta / (1 - e^M H^0 \gamma \varepsilon)\}$. Then, for $t \in J$ and $x \in K \cap \partial\Omega_R$, (13) and (22) imply

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\leq H^0 e^M \int_0^1 \omega(s) f(s, x(\alpha(s))) ds \\ &\leq H^0 e^M \int_0^1 \omega(s) (\varepsilon x(\alpha(s)) + \zeta) ds \\ &\leq H^0 e^M \int_0^1 \omega(s) (\varepsilon \|x\| + \zeta) ds \\ &\leq H^0 e^M \gamma (\varepsilon R + \zeta) \\ &\leq R, \end{aligned} \quad (45)$$

which implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_R. \quad (46)$$

Thus, by (ii) of Lemma 13, it follows from (41) and (46) that T has a fixed point x in $K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \leq \|x\| < R$. Remark 12 shows that problem (3) has at least one positive solution x with $r \leq \|x\| < R$. This gives the proof of Theorem 14. \square

4.2. For the Case $i_0 = 0$ and $i_\infty = 0$. In this subsection, we discuss the existence for the positive solutions of problem (3) under $i_0 = 0$ and $i_\infty = 0$. For convenience, we introduce the following notation:

$$f_0^\rho = \max \left\{ \max_{t \in J} \frac{f(t, x)}{\rho} : x \in [0, \rho] \right\}. \quad (47)$$

Now, we will state and prove the following main result.

Theorem 15. Suppose (H_1) – (H_3) hold. In addition, let the following two conditions hold:

(H_4) there exists $\rho_1 > 0$ such that $f_0^{\rho_1} \leq 1/e^M H^0 \gamma$;

(H_5) there exist $\eta > 0$ and $\rho_2 > 0$ such that $f(t, x) \geq \eta$ for $t \in J, x \geq \rho_2$; furthermore, $\rho_1 \neq \rho_2$.

Then problem (3) has at least one positive solution.

Proof. Without loss of generality, we may assume that $\rho_1 < \rho_2$. Considering $f_0^{\rho_1} \leq 1/e^M H^0 \gamma$, we have $f(t, x) \leq (1/e^M H^0 \gamma) \rho_1$ for $0 \leq x \leq \rho_1, t \in J$.

Since $0 \leq \alpha(t) \leq t \leq 1$ on J , it follows from $0 \leq x(t) \leq \rho_1$ on J that

$$0 \leq x(\alpha(t)) \leq \rho_1. \quad (48)$$

Consequently, for any $t \in J$ and $y \in K \cap \partial\Omega_{\rho_1}$, (13) and (22) imply

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) q(s) \omega(s) f(s, x(\alpha(s))) ds \\ &\leq H^0 e^M \int_0^1 \frac{1}{H^0 \gamma e^M} \rho_1 \omega(s) ds \\ &= \rho_1, \end{aligned} \quad (49)$$

which implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_{\rho_1}. \quad (50)$$

On the other hand, from (H_5) , when a ρ_2 is fixed, then there exists a $\eta > 0$ such that

$$f(t, x) \geq \eta \geq \frac{\rho_2}{\delta H_0 \gamma_1} \quad (51)$$

for $t \in J$ and $x \geq \rho_2$. Since $0 \leq \alpha(t) \leq t \leq \xi$ on $[0, \xi]$, it follows from $x(t) \geq \rho_2$ on J that

$$x(\alpha(t)) \geq \rho_2. \quad (52)$$

Hence, for $x \in K \cap \partial\Omega_{\rho_2}$, it follows from Remark 8 and (22) that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t,s)q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \min_{t \in [0,\xi]} \int_0^1 H(t,s)q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta H_0 \int_0^1 q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta H_0 \int_0^\xi \omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta \eta H_0 \int_0^\xi \omega(s)ds \\ &= \rho_2, \end{aligned} \tag{53}$$

which implies

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_{\rho_2}. \tag{54}$$

Thus, by (i) of Lemma 13, it follows that T has a fixed point x in $K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$ with

$$\rho_1 \leq \|x\| \leq \rho_2. \tag{55}$$

Thus, it follows from Remark 12 that problem (3) has at least one positive solution x with $\rho_1 \leq \|x\| \leq \rho_2$. This finishes the proof of Theorem 15. \square

We remark that condition (H_4) in Theorem 15 can be replaced by the following condition:

$$(H_4)' \quad f^0 \leq 1/H^0\gamma e^M,$$

which is a special case of (H_5) .

Corollary 16. *Suppose (H_1) – (H_3) , $(H_4)'$, and (H_5) hold. Then problem (3) has at least one positive solution.*

Proof. We show that $(H_4)'$ implies (H_4) . Suppose that $(H_4)'$ holds. Then, there exists a positive number $\rho_1 \neq \rho_2$ such that

$$\frac{f(t,x)}{x} \leq \frac{1}{H^0\gamma e^M}, \quad t \in J, \quad 0 < x \leq \rho_1. \tag{56}$$

Hence, we obtain

$$f(t,x) \leq \frac{1}{H^0\gamma e^M}x \leq \frac{1}{H^0\gamma e^M}\rho_1, \quad t \in J, \quad 0 < x \leq \rho_1. \tag{57}$$

Therefore, (H_4) holds. Hence, by Theorem 15, problem (3) has at least one positive solution. \square

Theorem 17. *Suppose (H_1) – (H_4) hold. In addition, let the following condition hold:*

$$(H_6) \quad f_\infty \geq 1/\delta H_0\gamma_1.$$

Then problem (3) has at least one positive solution.

Proof. The proof is similar to that of (50) and (35), respectively. \square

Corollary 18. *Suppose (H_1) – (H_3) , $(H_4)'$, and (H_6) hold. Then problem (3) has at least one positive solution.*

4.3. *For the Case $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$.* In this subsection, we discuss the existence for the positive solutions of problem (3) for the case $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$. For convenience, we introduce the following notation:

$$l = \frac{1}{H^0\gamma e^M}, \quad L = \frac{1}{\delta^2 H_0\gamma_1}. \tag{58}$$

Theorem 19. *Suppose (H_1) – (H_3) hold, $f^0 \in [0, l)$, and $f_\infty \in (L, \infty)$. Then problem (3) has at least one positive solution.*

Proof. The proof is similar to that of Theorem 15. \square

Theorem 20. *Suppose (H_1) – (H_3) hold, $f_0 \in (L, \infty)$, and $f^\infty \in [0, l)$. Then problem (3) has at least one positive solution.*

Proof. Considering $f_0 \in (L, \infty)$, then there exists $\rho_1 > 0$ such that $f(t,x) > Lx$ for $0 \leq x \leq \rho_1, t \in J$.

Since $0 \leq \alpha(t) \leq t \leq 1$ on J , it follows from $0 \leq x(t) \leq \rho_1$ on J that

$$0 \leq x(\alpha(t)) \leq \rho_1. \tag{59}$$

Consequently, for $x \in K \cap \partial\Omega_{\rho_1}$, it follows from Remark 8 and (22) that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t,s)q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \min_{t \in [0,\xi]} \int_0^1 H(t,s)q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta H_0 \int_0^1 q(s)\omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta H_0 \int_0^\xi \omega(s)f(s,x(\alpha(s)))ds \\ &\geq \delta H_0 \int_0^\xi \omega(s)Lx(\alpha(s))ds \\ &\geq \delta H_0 \int_0^\xi \omega(s)L\delta \|x\| ds \\ &\geq \|x\|, \end{aligned} \tag{60}$$

which shows

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_{\rho_1}. \tag{61}$$

Next, we turn to $f^\infty \in [0, l)$. In fact, we can show that $f^\infty \in [0, l)$ implies (H_4) .

Let $\tau \in (f^\infty, l)$. Then, there exists $r > \tau$ such that $\max_{t \in J} f(t, x) \leq \tau x$ for $y \in [r, \infty)$. Let

$$\beta = \max \left\{ \max_{t \in J} f(t, x) : 0 \leq x \leq r \right\},$$

$$\rho_1^* > \max \left\{ \frac{\beta}{l - \tau}, \rho \right\}. \tag{62}$$

Then, we have

$$\max_{0 \leq t \leq 1} f(t, x) \leq \tau x + \beta \leq \tau \rho_1^* + \beta < l \rho_1^*, \quad \forall x \in [0, \rho_1^*]. \tag{63}$$

This implies that $f_0^{\rho_1^*} \leq l$. Hence, $f^\infty \in [0, l)$ implies that (H_4) .

Similar to the proof of (46), we have

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_{\rho^*}. \tag{64}$$

Thus, by (ii) of Lemma 13, it follows that T has a fixed point x in $K \cap (\bar{\Omega}_{\rho_1^*} \setminus \Omega_\rho)$ with

$$\rho \leq \|x\| \leq \rho_1^*. \tag{65}$$

This finishes the proof of Theorem 20. □

From Theorems 19 and 20, we have the following result.

Corollary 21. *Assume that (H_1) – (H_3) hold. Furthermore, suppose that $f^0 = 0$ and condition (H_5) in Theorem 15 hold. Then problem (3) has at least one positive solution.*

Theorem 22. *Suppose (H_1) – (H_3) hold, $f^0 \in (0, l)$, and $f_\infty = \infty$. Then problem (3) has at least one positive solution.*

Proof. The proof is similar to that of Theorem 15. □

Theorem 23. *Suppose (H_1) – (H_3) hold, $f_0 = \infty$, and $f^\infty \in (0, l)$. Then problem (3) has at least one positive solution.*

Proof. The proof is similar to that of Theorem 15. □

From Theorems 22 and 23, the following corollaries are easily obtained.

Corollary 24. *Assume that (H_1) – (H_3) hold. Furthermore, suppose that $f_0 = \infty$ and condition (H_4) in Theorem 15 hold. Then problem (3) has at least one positive solution.*

Corollary 25. *Assume that (H_1) – (H_3) hold. Furthermore, suppose that $f_\infty = \infty$ and condition (H_4) in Theorem 15 hold. Then problem (3) has at least one positive solution.*

4.4. For the Case $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$. In this subsection, we study the existence of multiple positive solutions for the problem (3) for the case $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$.

Combining the proof of Theorems 14 and 15, the following theorem is easily proven.

Theorem 26. *Suppose that (H_1) – (H_3) , $i_0 = 0$, and $i_\infty = 2$, and the condition (H_4) of Theorem 15 hold. Then problem (3) has at least two positive solutions.*

Corollary 27. *Suppose that (H_1) – (H_3) , $i_0 = 0$, and $i_\infty = 2$, and the condition (H_4) ' of Corollary 16 hold. Then problem (3) has at least two positive solutions.*

Theorem 28. *Suppose that (H_1) – (H_3) , $i_0 = 2$, and $i_\infty = 0$, and the condition (H_5) of Theorem 15 hold. Then problem (3) has at least two positive solutions.*

Corollary 29. *Suppose that (H_1) – (H_3) , $i_0 = 2$, and $i_\infty = 0$, and the condition (H_6) of Theorem 17 hold. Then problem (3) has at least two positive solutions.*

5. Three Examples

To illustrate how our main results can be used in practice, we present three examples.

Example 1. Consider the following boundary value problem:

$$-x''(t) + bx(t) = \omega(t) f(t, x(\alpha(t))), \quad t \in J, \tag{66}$$

$$x'(0) = 0, \quad x(1) = \int_0^1 x(t) dt,$$

where $\alpha \in C(J, J)$, $\alpha(t) \leq t$ on J , and

$$\omega(t) = \frac{1}{\sqrt{t}}, \quad f(t, x) = \sqrt[n]{1 + t^n x^n}; \tag{67}$$

here $n \geq 2$ is a positive integral number.

This means that problem (66) involves the advanced argument α . For example, we can take $\alpha(t) = t^2$. It is clear that ω may be singular at $t = 0$ and/or $t = 1$ and f is both nonnegative and continuous.

Problem (66) can be regarded as a problem of the form (3), where $a(t) \equiv 0$, $b(t) \equiv 1$, and $h(t) \equiv 1$.

Let ϕ and ψ satisfy

$$L\phi = 0, \quad \phi'(0) = 0, \quad \phi(1) = 1, \tag{68}$$

$$L\psi = 0, \quad \psi(0) = 1, \quad \psi(1) = 0,$$

where $Lx = -x'' + x(t)$ and

$$\phi(t) = \frac{e^{1-t} + e^{1+t}}{1 + e^2}, \quad \phi(0) = \frac{2e}{1 + e^2},$$

$$\psi(t) = \frac{-e^{2-t} + e^t}{1 - e^2}, \tag{69}$$

$$\psi'(0) = \frac{e^2 + 1}{1 - e^2}, \quad q(t) = 1,$$

$$\Delta := -\phi(0)\psi'(0) = \frac{2e}{e^2 - 1} > 0.$$

It follows from the definition of ω , f , and h that (H_1) – (H_3) hold, and

$$f^0 = 0, \quad f_\infty = \infty. \tag{70}$$

Hence, by Theorem 14, problem (66) has at least one positive solution.

Example 2. If we replace $f(t, x)$ in Example 1 by

$$f(t, x) = \left(\frac{1}{3} + \frac{1}{3}t\right) + 2 \sin x, \quad (71)$$

in this case, by Theorem 20, we obtain that if $L < 7/3$ and $l > 2/3$, then problem (66) admits at least one positive solution.

In fact, we can prove that

$$\begin{aligned} f_0 &= \liminf_{x \rightarrow 0} \min_{t \in J} \frac{f(t, x)}{x} = \frac{7}{3}, \\ f^\infty &= \limsup_{x \rightarrow \infty} \max_{t \in J} \frac{f(t, x)}{x} = \frac{2}{3}, \end{aligned} \quad (72)$$

which shows that $f_0 \in (L, \infty)$ and $f^\infty \in [0, l)$.

Example 3. If we replace $f(t, x)$ in Example 1 by

$$f(t, x) = (1+t)x^2 + x^{1/2}, \quad (73)$$

in this case, by Theorem 26, we obtain that problem (66) admits at least two positive solutions.

In fact, we can prove that

$$\begin{aligned} f_0 &= \liminf_{x \rightarrow 0} \min_{t \in J} \frac{f(t, x)}{x} = \infty, \\ f_\infty &= \limsup_{x \rightarrow \infty} \max_{t \in J} \frac{f(t, x)}{x} = \infty, \end{aligned} \quad (74)$$

which shows that $i_0 = 0$ and $i_\infty = 2$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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