Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 392018, 7 pages
http://dx.doi.org/10.1155/2014/392018

Research Article

Some New Coincidence Theorems in Product GFC-Spaces with Applications

Jianrong Zhao

School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China

Correspondence should be addressed to Jianrong Zhao; mathzjr@swufe.edu.cn

Received 28 February 2014; Accepted 12 April 2014; Published 6 May 2014

Academic Editor: Xie-ping Ding

We first propose a new concept of GFC-subspace. Using this notion, we obtain a new continuous selection theorem. As a consequence, we establish some new collective fixed point theorems and coincidence theorems in product GFC-spaces. Finally, we give some applications of our theorems.

1. Introduction and Preliminaries

Recently, Ding [1, 2], X. P. Ding and T. M. Ding [3], Tang et al. [4], Fang and Huang [5] established some collective fixed point theorems, coincidence theorems, and KKM-type theorems for the families of set-valued mappings, which are defined on FC-spaces or product FC-spaces without any convexity structure.

Very recently, Khanh and Quan [6], Khanh et al. [7] defined a new notion by GFC-space which is a generalization of FC-space and proved some continuous selections theorems, collectively fixed point theorems, coincidence theorems, and KKM-type theorems in this new GFC-space.

Motivated and inspired by the work mentioned above, we will give some new collective fixed point and coincidence theorems in product GFC-spaces in the present paper. For this purpose, we first propose a new concept of GFC-subspace. Using this notion, we obtain a new continuous selection theorem. As a consequence, we establish some new collective fixed point theorems and coincidence theorems in product GFC-spaces. Finally, we give some applications of our theorems.

For our purpose, first, we present some known definitions and preliminary results. For a nonempty set A, $\langle A\rangle$ denotes the family of all nonempty finite subsets and 2^A denotes the family of all subsets of A. We denote the standard n-simplex with vertices $\{e_i\}_{i=0}^n$ by Δ_n. The following notion was introduced by Khanh and Quan [6].

Definition 1 (see [6]). (a) A generalized finitely continuous topological space or a GFC-space $(X, Y, \{\varphi_N\})$ consists of a topological space X and a nonempty set Y such that for each finite subset $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y\rangle$, one has a continuous mapping $\varphi_N : \Delta_n \to X$.

(b) Let $A, B \subset Y$ and $S : Y \to 2^X$ be given. Then B is called an S-subset of Y with respect to A if for any $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i0}, \ldots, y_{ik}\} \subset A \cap N$, one has $\varphi_N(\Delta_k) \subset S(B)$, where $\Delta_k = \mathrm{co}(\{e_{i0}, \ldots, e_{ik}\})$.

Now we give a new concept as follows.

Definition 2. Assume that $(X, Y, \{\varphi_N\})$ is a GFC-space, $C \subset Y$, and $D \subset X$. Then D is said to be a GFC-subspace of X with respect to C if for each $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i0}, \ldots, y_{ik}\} \subset C \cap N$, one has $\varphi_N(\Delta_k) \subset D$, where $\Delta_k = \mathrm{co}(\{e_{i0}, \ldots, e_{ik}\})$.

Remark 3. By the Definition 2, we know that $(D, C, \{\varphi_N\})$ is also a GFC-space. Clearly, if B is an S-subset of Y with respect to A, then $S(B)$ is a GFC-subspace of X with respect to A. In addition, if $X = Y$, then a GFC-subspace D of X with respect to C coincides with an FC-subspace D of X with respect to C (see [1]).

Assume that $\{D_i\}_{i\in I}$ is a family of GFC-subspace of X with respect to C and $\bigcap_{i\in I} D_i \neq \emptyset$, where I is an index set. It then follows from Definition 2 that $\bigcap_{i\in I} D_i$ is also a GFC-subspace of X with respect to C. For any given subset A of X, we define GFC-hull of A with respect to C by

$$\begin{aligned} &\mathrm{GFC}(A, C) \\ &\quad = \cap \{B \subset X : A \subset B, \\ &\qquad B \text{ is a GFC-subspace of } X \text{ with respect to } C\}. \end{aligned} \tag{1}$$

We simply write $\mathrm{GFC}(S(C))$ instead of $\mathrm{GFC}(S(C), C)$ when $A = S(C)$. We can easily show that $\mathrm{GFC}(A, C)$ is also a GFC-subspace of X with respect to C.

Let X be a topological space. A set $A \subset X$ is called compactly closed (resp., compactly open) if for each nonempty compact set $K \subset X$ such that $A \cap K$ is closed (resp., open) in K. And the compact interior and the compact closure of A (see [8]) are defined by

$$\begin{gathered} \operatorname{cint}(A) = \cup \{B \subset X : B \text{ is compactly open in } X, B \subset A\}, \\ \operatorname{ccl}(A) = \cap \{B \subset X : B \text{ is compactly closed in } X, B \subset A\}. \end{gathered} \tag{2}$$

Clearly if K is a nonempty compact subset of X, then we deriver that $\operatorname{cint}(A) \cap K = \operatorname{int}_K(A \cap K)$, $\operatorname{ccl}(A) \cap K = \operatorname{cl}_K(A \cap K)$, and $\operatorname{ccl}(A \setminus K) = X \setminus \operatorname{cint}(A)$.

Assume that Y and Z are nonempty sets and X is a topological space. Define two set-valued mappings $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$. $F(x, y)$ is called transfer compactly upper semicontinuous in x with respect to C (see [9]) if for any nonempty compact subset K of X and any $x \in K$, the set $\{y \in Y : F(x, y) \in C(x)\} \neq \emptyset$ means that there is a relatively open neighborhood $N(x)$ of x in K and a point $y' \in Y$ such that $F(z, y') \subset C(z)$ for all $z \in N(x)$.

A mapping $T : X \to 2^Y$ is called transfer compactly open-valued (resp., transfer compactly closed-valued) on X (see [8]) if for each $x \in X$ and each nonempty compact subset K of Y, $y \in T(x) \cap K$ (resp., $y \notin T(x) \cap K$) means that there is $x' \in X$ such that $y \in \operatorname{int}_K(T(x') \cap K)$ (resp., $y \notin \operatorname{cl}_K(T(x') \cap K)$).

We need the following two results. The first statement was proved by Ding and Park [9].

Lemma 4 (see [9]). *Let Y and Z be two nonempty sets and X a topological space. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be two set-valued mappings. Then $F(x, y)$ is transfer compactly upper semicontinuous in x with respect to C if and only if the mapping $T : Y \to 2^X$ defined by $T(y) = \{x \in X : F(x, y) \not\subset C(x)\}$ is transfer compactly closed-valued on Y.*

The second statement is Lemma 1.1 of Ding [8].

Lemma 5. *Assume that X and Y are two topological spaces and $F : X \to 2^Y$ is a set-valued mapping with nonempty values. Then the following statements are equivalent:*

(i) *$F^{-1} : Y \to 2^X$ is transfer compactly open-valued;*

(ii) *for any compact subset K of X and any $x \in K$, one has $y \in Y$ satisfying that $x \in \operatorname{cint} F^{-1}(y) \cap K$ and $K = \bigcup_{y \in Y} \operatorname{int}_K(F^{-1} \cap K)$.*

Throughout this paper, we always let I and J be any given index set. Now we give the following statement, which generalizes Lemma 1.1 of Ding [1].

Lemma 6. *Suppose that $(X_i, Y_i, \{\varphi_{N_i}\})$ is a GFC-space for each $i \in I$. If $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$, and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$, then $(X, Y, \{\varphi_N\})$ is also a GFC-space.*

Proof. Let $\pi_i : Y \to Y_i$ be the projective mapping from Y to Y_i for each $i \in I$. For any given $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$, we denote $N_i = \pi_i(N) = \{\pi_i(y_0), \ldots, \pi_i(y_n)\} \in \langle Y_i \rangle$. Note that $(X_i, Y_i, \{\varphi_{N_i}\})$ is a GFC-space. Then we have a continuous mapping $\varphi_{N_i} : \Delta_n \to X_i$ for each $i \in I$. So we may let a mapping $\varphi_N : \Delta_n \to X$ by $\varphi_N(\alpha) = \prod_{i \in I} \varphi_{N_i}(\alpha)$, for any $\alpha \in \Delta_n$. It follows that φ_N is a continuous mapping, which means that $(X, Y, \{\varphi_N\})$ is a GFC-space. So Lemma 6 is proved. □

2. Continuous Selection and Collective Fixed Points

Theorem 7. *Assume that $(X, Y, \{\varphi_N\})$ is a GFC-space and Z is a compact topological space. Suppose that $F : Z \to 2^Y$ and $G : Z \to 2^X$ are such that*

(i) *$Z = \bigcup_{y \in Y} \operatorname{cint} F^{-1}(y)$;*

(ii) *for any given $z \in Z$, $G(z)$ is a GFC-subspace of X with respect to $F(z)$.*

Then one has a continuous selection $g : Z \to X$ of G satisfying $g = \varphi \circ \psi$, where $\varphi : \Delta_n \to X$ and $\psi : Z \to \Delta_n$ are continuous for some $n \in \mathbb{Z}^+$.

Proof. By condition (i), we know that there exists $N = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$ such that $Z = \bigcup_{i=0}^n \operatorname{cint} F^{-1}(y_i)$ since Z is compact. Assume that $\{\psi_i\}_{i=0}^n$ is the continuous partition of unity subordinated to the open covering $\{\operatorname{cint} F^{-1}(y_i)\}_{i=0}^n$; then for any $i \in \{0, 1, \ldots, n\}$ and $z \in Z$, one has

$$\psi_i(z) \neq 0 \Longleftrightarrow z \in \operatorname{cint} F^{-1}(y_i) \subset F^{-1}(y_i) \Longrightarrow y_i \in F(z). \tag{3}$$

Let $\psi : Z \to \Delta_n$ be a mapping with $\psi(z) = \sum_{i=0}^n \psi_i(z) e_i$. Clearly ψ is continuous and for any $z \in Z$, one has $\psi(z) = \sum_{i \in J(x)} \psi_j(z) e_j$ where $J(x) = \{j \in \{0, 1, \ldots, n\} : \psi_i(z) \neq 0\}$. So by (3), we get $\{y_j : j \in J(x)\} \subset F(z) \cap N$. It then follows from condition (ii) that for $z \in Z$, $\varphi_N(\Delta_{J(x)}) \subset G(z)$. It is easy to see that $g(z) = \varphi_N \circ \psi(z) \in \varphi_N(\Delta_{J(x)}) \subset G(z)$, which implies that $g = \varphi_N \circ \psi$ is a continuous selection of G. The proof of Theorem 7 is completed. □

Remark 8. Applying the definition of GFC-subspace, we extend Theorem 2.2 of Tarafdar [10], Proposition 1 of Browder [11], and Theorem 2.1 of Ding [2] to GFC-spaces without any convexity.

Theorem 9. *Assume that $X = \prod_{i \in I} X_i$ and $(X_i, Y_i, \{\varphi_{N_i}\})$ is a compact GFC-space for each $i \in I$. Suppose that $F_i : X \to 2^{Y_i}$*

and $G_i : X \to 2^{X_i}$ are two set-valued mappings satisfying the following conditions:

(i) $X = \bigcup_{y_i \in Y_i} \operatorname{cint} F_i^{-1}(y_i)$;

(ii) *for any given $x \in X$, $G_i(x)$ is a GFC-subspace of X_i with respect to $F_i(x)$.*

Then one has a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $\widehat{x}_i \in G_i(\widehat{x})$ for $i \in I$.

Proof. For any given $i \in I$, using Theorem 7, we obtain that there are continuous mappings $\varphi_{N_i} : \Delta_{n_i} \to X_i$ and $\psi_i : X \to \Delta_{n_i}$ such that $g_i = \varphi_{N_i} \circ \psi_i$ is a continuous selection of G_i for some positive integer n_i. Assume that E_i is the linear hull of the set $\{e_0, e_1, \ldots, e_{n_i}\}$ for each $i \in I$. Clearly E_i is a locally convex Hausdorff topological vector space as it is finite dimensional and Δ_{n_i} is a compact convex subset of E_i. Moreover, $E = \prod_{i\in I} E_i$ is a locally convex Hausdorff topological vector space and $\Delta = \prod_{i\in I} \Delta_{n_i}$ is a compact convex subset of E. Now let $\Phi : \Delta \to X$ and $\Psi : X \to \Delta$ be two continuous mappings with

$$\Phi(\delta) = \prod_{i\in I} \varphi_{N_i}(P_i(\delta)), \qquad \Psi(x) = \prod_{i\in I} \psi_i(x), \tag{4}$$

where $P_i : \Delta \to \Delta_{n_i}$ is the projection of Δ onto Δ_{n_i} for any given $i \in I$. It then follows from the Tychonoff fixed point theorem that the continuous mapping $\Psi \circ \Phi : \Delta \to \Delta$ has a fixed point $\delta \in \Delta$; that is, $\delta = \Psi \circ \Phi(\delta)$. Let $\widehat{x} = \Phi(\delta)$. It follows that

$$\begin{aligned} \widehat{x} &= \Psi \circ \Phi(\widehat{x}) = \Phi\left(\prod_{i\in I} \psi_i(\widehat{x})\right) \\ &= \prod_{i\in I} \varphi_{N_i}\left(P_i\left(\prod_{i\in I} \psi_i(\widehat{x})\right)\right) = \prod_{i\in I} \varphi_{N_i} \circ \psi_i(\widehat{x}). \end{aligned} \tag{5}$$

This means that $\widehat{x} = \varphi_{N_i} \circ \psi_i(\widehat{x}) \in G_i(\widehat{x})$ for each $i \in I$. So Theorem 9 is proved. □

Theorem 10. *Assume that Z_i is a topological space and $(X_i, Y_i, \{\varphi_{N_i}\})$ is a GFC-space for each $i \in I$. Suppose that $X = \prod_{i\in I} X_i, S_i : Z_i \to 2^{Y_i}$ and $T_i : Z_i \to 2^{X_i}$ and $g_i : X \to Z_i$ is a continuous mapping such that*

(i) *for any compact subset D_i of Z_i, $D_i = \bigcup_{y_i \in Y_i} (\operatorname{cint} S_i^{-1}(y_i) \cap D_i)$;*

(ii) *for any $z_i \in Z_i, T_i(z_i)$ is a GFC-subspace of X_i with respect to $S_i(z_i)$;*

(iii) *there is a nonempty subset of $Y_i^0 \subset Y_i$ such that the set $B_i = \bigcap_{y_i \in Y_i^0} (\operatorname{cint} S_i^{-1}(y_i))^c$ is empty or compact in Z_i, and for any $N_i \in \langle Y_i \rangle$, there exists a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$.*

Then one has a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $\widehat{x}_i \in T_i(g_i(\widehat{x}))$ for each $i \in I$.

Proof. For any given $i \in I$, if B_i is a nonempty compact subset of Z_i, then by (i) one has

$$B_i = \bigcup_{y_i \in Y_i} \left(\operatorname{cint} S_i^{-1}(y_i) \cap B_i\right) \subset \bigcup_{y_i \in Y_i} \operatorname{cint} S_i^{-1}(y_i). \tag{6}$$

Moreover, noticing that B_i is compact, we can find that there is a finite set $N_i \in \langle Y_i \rangle$ such that

$$B_i = \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c \subset \bigcup_{y_i \in N_i} \operatorname{cint} S_i^{-1}(y_i). \tag{7}$$

It then follows from (7) that

$$Z_i = \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i) \cup \left(\bigcup_{y_i \in N_i} \operatorname{cint} S_i^{-1}(y_i)\right). \tag{8}$$

If B_i is empty in (iii), then we derive that

$$Z_i = Z_i \setminus B_i = Z_i \setminus \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c = \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i). \tag{9}$$

From condition (iii), we know that there exists a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$. So by (8) and (9), we get

$$Z_i = \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} S_i^{-1}(y_i)\right). \tag{10}$$

Now let $K_N = \prod_{i\in I} K_{N_i}$, $L_N = \prod_{i\in I} L_{N_i}$, and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. By condition (iii), we deduce that $(K_{N_i}, L_{N_i}, \{\varphi_{N_i}\})$ is a compact GFC-space. It then follows from Lemma 6 that $(K_N, L_N, \{\varphi_N\})$ is also a compact GFC-space. Let $P_i : K_N \to 2^{L_{N_i}}$ and $Q_i : K_N \to 2^{K_{N_i}}$ be two set-valued mappings with

$$\begin{aligned} P_i(x) &= S_i(g_i(x)) \cap L_{N_i}, \\ Q_i(x) &= T_i(g_i(x)) \cap K_{N_i} \quad \text{for } x \in K_N. \end{aligned} \tag{11}$$

In order to show that the conditions (i) and (ii) of Theorem 9 hold, we only need to show that $Q_i(x)$ is a compact GFC-subspace of K_{N_i} with respect to $P_i(x)$, and $K_N = \bigcup_{y_i \in L_{N_i}} \operatorname{cint} P_i^{-1}(y_i)$. By conditions (ii) and (iii), it is easy to see that $Q_i(x)$ is a compact GFC-subspace of K_{N_i} with respect to $P_i(x)$. It remains to show that $K_N = \bigcup_{y_i \in L_{N_i}} \operatorname{cint} P_i^{-1}(y_i)$. On one hand, for each $y_i \in L_{N_i}$, we deduce that

$$\begin{aligned} &P_i^{-1}(y_i) \\ &\quad = \{x \in K_N : y_i \in P_i(x)\} \\ &\quad = \left\{x \in K_N : y_i \in S_i(g_i(x)) \cap L_{N_i}\right\} \\ &\quad = \{x \in K_N : y_i \in S_i(g_i(x))\} \\ &\quad = \left\{x \in K_N : x \in g_i^{-1}\left(S_i^{-1}(y_i)\right)\right\} \\ &\quad = g_i^{-1}\left(S_i^{-1}(y_i)\right). \end{aligned} \tag{12}$$

On the other hand, by (10), we obtain

$$g_i(K_N) \subset Z_i = \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} S_i^{-1}(y_i)\right). \tag{13}$$

It then follows form (12), (13), and the continuity of g_i that

$$\begin{aligned} K_N &\subset g_i^{-1}\left(\bigcup_{y_i \in L_{N_i}} \operatorname{cint} S_i^{-1}(y_i)\right) \\ &= \bigcup_{y_i \in L_{N_i}} g_i^{-1}\left(\operatorname{cint} S_i^{-1}(y_i)\right) \\ &= \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} P_i^{-1}(y_i)\right) \subset K_N. \end{aligned} \tag{14}$$

Hence $K_N = \bigcup_{y_i \in L_{N_i}} \operatorname{cint} P_i^{-1}(y_i)$. Then Theorem 9 tells us that there exists a point $\hat{x} \in K_N \subset X$ such that $\hat{x}_i \in Q_i(\hat{x}) = T_i(g_i(\hat{x})) \cap K_{N_i} \subset T_i(g_i(\hat{x}))$ for each $i \in I$. This completes the proof of Theorem 10. □

Theorem 11. *Let $X = \prod_{i\in I} X_i$, Z_i be a topological space and $(X_i, Y_i, \{\varphi_{N_i}\})$ a GFC-space for each $i \in I$. Assume that $S_i : Z_i \to 2^{Y_i}$, $T_i : Z_i \to 2^{X_i}$, and $R_i : Y_i \to 2^{X_i}$ are set-valued mappings and $g_i : X \to Z_i$ is a continuous mapping such that*

(i) *for any compact subset D_i of Z_i, $D_i = \bigcup_{y_i \in Y_i}(\operatorname{cint} S_i^{-1}(y_i) \cap D_i)$;*

(ii) *for any $z_i \in Z_i$, $GFC(R_i(S_i(z_i))) \subset T_i(z_i)$;*

(iii) *if Z_i is not compact, then there is a nonempty subset of $Y_i^0 \subset Y_i$ such that a compact subset D_i of Z_i satisfying $Z_i \setminus D_i \subset \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i)$, and for any $N_i \in \langle Y_i \rangle$, there exists a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$.*

Then we have a point $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ such that $\hat{x}_i \in T_i(g_i(\hat{x}))$ for each $i \in I$.

Proof. Using the similar argument of Theorem 10, we only need to show that the conditions (ii) and (iii) of Theorem 10 are satisfied. By the definition of $\mathrm{GFC}(R_i(S_i(z_i)))$, we know that $\mathrm{GFC}(R_i(S_i(z_i)))$ is a GFC-subspace of X_i with respect to $S_i(z_i)$. It then follows from condition (ii) that $T_i(z_i)$ is a GFC-subspace of X_i with respect to $S_i(z_i)$, which means that the condition (i) of Theorem 10 holds. It remains to deal with condition (iii). If Z_i is noncompact, by (iii), we get

$$B_i = \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c = Y_i \setminus \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i) \subset D_i. \tag{15}$$

Clearly B_i is a closed subset of compact set D_i. If B_i is nonempty, then B_i is compact in Z_i. This means that the condition (iii) of Theorem 10 is satisfied. So the statement of Theorem 11 follows immediately from Theorem 10. □

Remark 12. (a) Let V_j be a topological spaces for each $j \in J$. By Lemma 5, if using assumption (i) in Lemma 5 for assumption (i) in Theorems 9 and 10, then the statements of Theorems 9 and 10 are still true.

(b) The results of Theorems 10 and 11 generalize Theorem 2.1 of Lan and Webb [12], Theorem 1 of Ansari and Yao [13], Theorem 2.2 of Ding and Park [14], Theorem 3.1 of Ding and Park [15], and Theorem 3.1 of Lin and Ansari [16] and Ding [2] to GFC-spaces.

3. Coincidence Theorems for Two Families of Set-Valued Mappings

Theorem 13. *Let $(X_i, Y_i, \{\varphi_{N_i}\})$ and $(U_j, V_j, \{\varphi_{N_j}\})$ be GFC-spaces for any $i \in I$ and $j \in J$. Assume that $X = \prod_{i\in I} X_i$ and $U = \prod_{i\in J} U_j$. Let $S_i : U \to 2^{Y_i}$, $T_i : U \to 2^{X_i}$, $F_j : X \to 2^{V_j}$, and $G_j : X \to 2^{U_j}$ be set-valued mappings satisfying the following conditions:*

(i) *for any compact subset A of X, $A = \bigcup_{v_j \in V_j}(\operatorname{cint} F_j^{-1}(v_j) \cap A)$;*

(ii) *for any $x \in X$, $G_j(x)$ is a GFC-subspace of U_j with respect to $F_j(x)$;*

(iii) *there exists a nonempty subset of Y_i^0 of Y_i such that the set $D_i = \bigcap_{y_i \in Y_i^0}(\operatorname{cint} S_i^{-1}(y_i))^c$ is empty or compact in U and for each $N_i \in \langle Y_i \rangle$, there is a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$;*

(iv) *for any compact subset B of U, $B = \bigcup_{y_i \in Y_i}(\operatorname{cint} S_i^{-1}(y_i) \cap B)$;*

(v) *for any $u \in U$, $T_i(u)$ is a GFC-subspace of X_i with respect to $S_i(u)$.*

Then one has $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ and $\hat{u} = (\hat{u}_j)_{j\in J} \in U$ such that $\hat{x}_i \in T_i(\hat{u})$ and $\hat{u}_j \in G_j(\hat{x})$ for any $i \in I$ and $j \in J$.

Proof. For any given $i \in I$, if D_i is empty in (iii), it is easy to see that

$$U = U \setminus D_i = U \setminus \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c = \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i). \tag{16}$$

If D_i is nonempty compact set in U, by (iv), we know that

$$D_i = \bigcup_{y_i \in Y_i} \left(\operatorname{cint} S_i^{-1}(y_i) \cap D_i\right) \subset \bigcup_{y_i \in Y_i} \operatorname{cint} S_i^{-1}(y_i). \tag{17}$$

Note that D_i is compact. Then we can find a finite set $N_i \in \langle Y_i \rangle$ such that

$$D_i = \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c \subset \bigcup_{y_i \in N_i} \operatorname{cint} S_i^{-1}(y_i). \tag{18}$$

It then follows from (16) and (18) that if either D_i is empty or compact in U, we get

$$U = \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i) \cup \left(\bigcup_{y_i \in N_i} \operatorname{cint} S_i^{-1}(y_i)\right). \tag{19}$$

Furthermore, by (iii), there exists a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$. So by (19), we get

$$U = \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} S_i^{-1}(y_i)\right). \tag{20}$$

Assume that $K_N = \prod_{i\in I} K_{N_i}, L_N = \prod_{i\in I} L_{N_i}$, and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. It then follows from Lemma 6 that K_N is a compact GFC-subspace of X with respect to L_N. So $(K_N, L_N, \{\varphi_N\})$ is a compact GFC-space. We consider the restrictions $F_j|_{K_N}$ and $G_j|_{K_N}$ of F_j and G_j on K_N, respectively, for each $j \in J$. Then by condition (ii), we have $G_j|_{K_N}(x)$ as a GFC-subspace of U_j with respect to $F_j|_{K_N}(x)$ for any $x \in X$. From condition (i), we obtain

$$K_N = \bigcup_{v_j \in V_j} \left(\operatorname{cint} F_j^{-1}(v_j) \cap K_N\right) = \bigcup_{v_j \in V_j} \operatorname{cint} F_j^{-1}|_{K_N}(v_j). \tag{21}$$

Moreover, by Theorem 7, there exists a continuous selection $g_j : K_N \to U_j$ of $G_j|_{K_N}$ for any $j \in J$. Let $g : K_N \to U$ be a mapping with $g(x) = \prod_{j\in J} g_j(x)$ for any $x \in L_N$. Clearly g is a continuous mapping. Define two set-valued mappings $P_i : K_N \to 2^{L_{N_i}}$ and $Q_i : K_N \to 2^{K_{N_i}}$ by

$$P_i(x) = S_i(g(x)) \cap L_{N_i}, \qquad Q_i(x) = T_i(g(x)) \cap K_{N_i}. \tag{22}$$

Since K_{N_i} is a compact GFC-subspace of X_i relative to L_{N_i}, by (v), we know that $Q_i(x)$ is also a compact GFC-subspace of X_i with respect to $P_i(x)$.

Now we claim that $K_N = \bigcup_{y_i \in L_{N_i}} \operatorname{cint} P_i^{-1}(y_i)$. For any given $y_i \in L_{N_i}$, we deduce that

$$\begin{aligned} P_i^{-1}(y_i) &= \{x \in K_N : y_i \in P_i(x)\} \\ &= \{x \in K_N : y_i \in S_i(g(x)) \cap L_{N_i}\} \\ &= \{x \in K_N : y_i \in S_i(g(x))\} \\ &= \{x \in K_N : x \in g^{-1}(S_i^{-1}(y_i))\} \\ &= g^{-1}(S_i^{-1}(y_i)). \end{aligned} \tag{23}$$

Then by (20) we obtain

$$g(K_N) \subset U = \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} S_i^{-1}(y_i)\right). \tag{24}$$

It then follows form (23), (24), and the continuity of g that

$$\begin{aligned} K_N &\subset g^{-1}\left(\bigcup_{y_i \in L_{N_i}} \operatorname{cint} S_i^{-1}(y_i)\right) \\ &= \bigcup_{y_i \in L_{N_i}} g^{-1}\left(\operatorname{cint} S_i^{-1}(y_i)\right) \\ &= \bigcup_{y_i \in L_{N_i}} \left(\operatorname{cint} P_i^{-1}(y_i)\right) \subset K_N. \end{aligned} \tag{25}$$

Hence $K_N = \bigcup_{y_i \in L_{N_i}} \operatorname{cint} P_i^{-1}(y_i)$. So the claim is proved.

Note that $Q_i(x)$ is a compact GFC-subspace of X_i relative to $P_i(x)$. So by the above claim and Theorem 9, we know that there exists a point $\hat{x} \in K_N \subset X$ such that $\hat{x}_i \in Q_i(\hat{x}) = T_i(g(\hat{x})) \cap K_{N_i} \subset T_i(g(\hat{x}))$ for any given $i \in I$. Assume that $\hat{u} = g(\hat{x}) = \prod_{j\in J} g_j(\hat{x})$. Thus we derive that there exist $\hat{x} \in X$ and $\hat{u} \in U$ such that $\hat{x}_i \in T_i(\hat{u})$ and $\hat{u}_j \in G_j(\hat{x})$ for any $i \in I$ and $j \in J$, which implies that Theorem 13 is true. □

Theorem 14. *Let $(X_i, Y_i, \{\varphi_{N_i}\})$ and $(U_j, V_j, \{\varphi_{N_j}\})$ be GFC-spaces for any $i \in I$ and $j \in J$. Assume that $X = \prod_{i\in I} X_i$ and $U = \prod_{i\in J} U_j$. Let $S_i : U \to 2^{Y_i}$, $T_i : U \to 2^{X_i}$, $R_i : Y_i \to 2^{X_i}$, $F_j : X \to 2^{V_j}$, $G_j : X \to 2^{U_j}$, and $H_j : V_j \to 2^{U_j}$ be set-valued mappings for each $i \in I$ and $j \in J$. If the following conditions hold:*

(i) *for any compact subset A of X, $A = \bigcup_{v_j \in V_j} (\operatorname{cint} F_j^{-1}(v_j) \cap A)$,*

(ii) *for any $x \in X$, $GFC(H_j(F_j(x))) \subset G_j(x)$,*

(iii) *if U is not compact, then there exists a nonempty subset of Y_i^0 of Y_i as well as a compact subset K of U such that $U \setminus K \subset \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i)$ and for each $N_i \in \langle Y_i \rangle$, there is a compact GFC-subspace K_{N_i} of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$,*

(iv) *for any compact subset B of U, $B = \bigcup_{v_j \in V_j} (\operatorname{cint} F_j^{-1}(v_j) \cap B)$,*

(v) *for any $u \in U$, $GFC(R_i(S_i(u))) \subset T_i(u)$,*

then there exist $\hat{x} = (\hat{x}_i)_{i\in I} \in X$ and $\hat{u} = (\hat{u}_j)_{j\in J} \in U$ such that $\hat{x}_i \in T_i(\hat{u})$ and $\hat{u}_j \in G_j(\hat{x})$ for any $i \in I$ and $j \in J$.

Proof. By the definition of the GFC-hull with respect to $F_j(x)$ and condition (ii), it is easy to see that $G_j(x)$ is a GFC-subspace of U_j with respect to $F_j(x)$ for any $x \in X$ and $j \in J$. Similarly, by (v), we obtain that $T_i(u)$ is a GFC-subspace of X_i with respect to $S_i(u)$ for any $u \in U$ and $i \in I$. Then conditions (ii) and (v) of Theorem 13 hold. On the other hand, by (iii), if U is not compact, we know that

$$D = \bigcap_{y_i \in Y_i^0} \left(\operatorname{cint} S_i^{-1}(y_i)\right)^c = U \setminus \bigcup_{y_i \in Y_i^0} \operatorname{cint} S_i^{-1}(y_i) \subset K. \tag{26}$$

If D is nonempty, then D is a closed subset of compact set K. This implies that D is compact in U. So condition (iii) tells us condition (iii) of Theorem 13 holds. Then the statement of Theorem 14 follows immediately from Theorem 13. □

Remark 15. Theorems 13 and 14 improve Theorem 9 of Yu and Lin [17], Theorem 3.3 of Lin and Ansari [16], and Theorems 3.1 and 3.2 of Ding [2] to GFC-spaces without any convexity structure.

4. Applications

In the current section, we will give some applications of our theorems.

Theorem 16. *Let $(X_i, Y_i, \{\varphi_{N_i}\})$ be a GFC-space and Z_i a topological space for each $i \in I$. Assume that A_i and B_i are a subset of $Y_i \times Z_i$ and $X_i \times Z_i$, respectively. Let $X = \prod_{i\in I} X_i$ and for each $i \in I$, let $g_i : X \to Z_i$ be a continuous mapping and $R_i : Y_i \to 2^{X_i}$ a set-valued mapping satisfying the following conditions:*

(i) *for any compact subset D_i of Z_i, $D_i = \bigcup_{y_i\in Y_i}(\mathrm{cint}\{z_i \in Z_i : (y_i, z_i) \in A_i\} \cap D_i)$;*

(ii) *for any $z_i \in Z_i$, $GFC(R_i(\{y_i \in Y_i : (y_i, z_i) \in A_i\})) \subset \{x_i \in X_i : (x_i, z_i) \in B_i\}$;*

(iii) *if Z_i is not compact, then there exists a nonempty subset of Y_i^0 of Y_i such that a compact subset D_i of Z_i such that $Z_i \setminus D_i \subset \bigcup_{y_i\in Y_i^0} \mathrm{cint}\{z_i \in Z_i : (y_i, z_i) \in A_i\}$ and for each $N_i \in \langle Y_i\rangle$, there is a compact GFC-subspace of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$.*

Then we have a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $(\widehat{x}_i, g_i(\widehat{x})) \in B_i$ for each $i \in I$.

Proof. For any given $i \in I$, we define two set-valued mappings $S_i : Z_i \to 2^{Y_i}$ and $T_i : Z_i \to 2^{X_i}$ by

$$S_i(z_i) = \{y_i \in Y_i : (y_i, z_i) \in A_i\},$$
$$T_i(z_i) = \{x_i \in X_i : (x_i, z_i) \in B_i\} \tag{27}$$

for all $z_i \in Z_i$. It then follows from assumptions (i)–(iii) and Theorem 11 that for any $i \in I$, there exists a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $\widehat{x}_i \in T_i(g_i(\widehat{x}))$, which implies that $(\widehat{x}_i, g_i(\widehat{x})) \in B_i$. So Theorem 16 is true. □

Corollary 17. *For each $i \in I$, let $(X_i, Y_i, \{\varphi_{N_i}\})$ be a GFC-space, $R_i : Y_i \to 2^{X_i}$ a set-valued mapping, $X = \prod_{i\in I} X_i$, and $X^i = \prod_{j\in I, j\neq i} X_j$. Assume that $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are the families of subsets of $Y_i \times X^i$ and $X_i \times X^i$, respectively. If the following conditions hold:*

(i) *for any compact subset D^i of X^i, $D^i = \bigcup_{y_i\in Y_i}(\mathrm{cint}\{z_i \in Z_i : (y_i, x^i) \in A_i\} \cap D^i)$,*

(ii) *for all $x^i \in X^i$, $GFC(R_i(\{y_i \in Y_i : (y_i, x^i) \in A_i\})) \subset \{x_i \in X_i : (x_i, x^i) \in B_i\}$,*

(iii) *if X^i is not compact, then there exists a nonempty subset of Y_i^0 of Y_i such that a compact subset D^i of X^i such that $X^i \setminus D^i \subset \bigcup_{y_i\in Y_i^0} \mathrm{cint}\{x^i \in X^i : (y_i, z_i) \in A_i\}$ and for each $N_i \in \langle Y_i\rangle$, there is a compact GFC-subspace of X_i relative to L_{N_i} containing $Y_i^0 \cup N_i$,*

then $\bigcap_{i\in I} B_i \neq \emptyset$.

Proof. Define a mapping $g_i : X \to X^i$ as the projection of X onto $X^i = \prod_{j\in I, j\neq i} X_j$. Clearly g_i is a continuous mapping. Using Theorem 16 with $Z_i = X^i$ and $D_i = D^i$, we find that there exists a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in X$ such that $(\widehat{x}_i, g_i(\widehat{x})) \in B_i$ for any $i \in I$, which implies that $\widehat{x} = (\widehat{x}_i, g_i(\widehat{x})) \in \bigcap_{i\in I} B_i$. Hence $\bigcap_{i\in I} B_i \neq \emptyset$. □

In the rest of this section, for each $i \in I$ and $j \in J$, we always let Z_i and W_i be two nonempty set and $(X_i, Y_i, \{\varphi_{N_i}\})$ and $(U_j, V_j, \{\varphi_{N_j}\})$ be two GFC-spaces. Assume that $X = \prod_{i\in I} X_i$ and $U = \prod_{i\in J} U_j$. For any $i \in I$ and $j \in J$, let $R_i : Y_i \to 2^{X_i}$, $H_j : V_j \to 2^{U_j}$, $C_j : X \to 2^{Z_j}$, $D_i : U \to 2^{W_i}$, $B_j : X \times V_j \to 2^{Z_j}$, $A_j : X \times U_j \to 2^{Z_j}$, $Q_i : Y_i \times U \to 2^{W_i}$, and $P_i : X_i \times U \to 2^{W_i}$ be set-valued mappings. Then we have the follows results.

Theorem 18. *Suppose that the following conditions hold:*

(i) *for any compact subset A of X, $A = \bigcup_{v_j\in V_j}(\mathrm{cint}\{x \in X : B_j(x, v_j) \subset C_j(x)\} \cap A)$;*

(ii) *for any $x \in X$, $GFC(H_j(\{v_j \in V_j : B_j(x, v_j) \subset C_j(x)\})) \subset \{u_j \in U_j : A_j(x, u_j) \subset C_j(x)\}$;*

(iii) *if U is not compact, then there exists a nonempty subset of Y_i^0 of Y_i such that a compact subset K of U such that $U \setminus K \subset \bigcup_{y_i\in Y_i^0} \mathrm{cint}\{u \in U : Q_i(y_i, u) \subset D_i(u)\}$ and for each $N_i \in \langle Y_i\rangle$, there is a compact GFC-subspace of X_i with respect to L_{N_i} containing $Y_i^0 \cup N_i$;*

(iv) *for any compact subset B of U, $B = \bigcup_{y_i\in Y_i}(\mathrm{cint}\{y_i \in Y_i : Q_i(y_i, u) \subset D_i(u)\} \cap B)$;*

(v) *for any $u \in U$, $GFC(R_i(\{y_i \in Y_i : Q_i(y_i, u) \subset D_i(u)\})) \subset \{x_i \in X_i : P_i(x_i, u) \subset D_i(u)\}$.*

Then there exist $(\widehat{x}, \widehat{u}) \in X \times U$ and $\widehat{u} = (\widehat{u}_j)_{j\in J} \in U$ such that $A_j(\widehat{x}, \widehat{u}_j) \subset C_j(\widehat{x})$ and $P_i(\widehat{x}_i, \widehat{u}) \subset D_i(\widehat{u})$ for each $i \in I$ and $j \in J$.

Proof. Assume that $S_i : U \to 2^{Y_i}$, $T_i : U \to 2^{X_i}$, $F_j : X \to 2^{V_j}$, and $G_j : X \to 2^{U_j}$ are set-valued mappings as follows:

$$S_i(u) = \{y_i \in Y_i : Q_i(y_i, u) \subset D_i(u)\},$$
$$T_i(u) = \{x_i \in X_i : P_i(x_i, u) \subset D_i(u)\} \quad \forall u \in U, \tag{28}$$
$$F_j(x) = \{v_j \in V_j : B_j(x, v_j) \subset C_j(x)\},$$
$$G_j(x) = \{u_j \in U_j : A_j(x, u_j) \subset C_j(x)\} \quad \forall x \in X. \tag{29}$$

It then follows from Theorem 14 that there exist $\widehat{x} \in X$ and $\widehat{u} \in U$ such that $\widehat{x}_i \in T_i(\widehat{u})$ and $\widehat{u}_j \in G_j(\widehat{x})$ for any $i \in I$ and $j \in J$. By the above definition, we obtain that $A_j(\widehat{x}, \widehat{u}_j) \subset C_j(\widehat{x})$ and $P_i(\widehat{x}_i, \widehat{u}) \subset D_i(\widehat{u})$ for any $i \in I$ and $j \in J$. This completes the proof. □

Theorem 19. *For each $i \in I$ and $j \in J$, let Y_i and V_j be topological spaces. Suppose that the following conditions hold:*

(i) *for each $x \in X$, the set $\{v_j \in V_j : B_j(x, v_j) \subset C_j(x)\}$ is nonempty and $B_j(x, v_j)$ is transfer compactly upper semicontinuous in x with respect to C_j;*

(ii) *for any $x \in X$, $GFC(H_j(\{v_j \in V_j : B_j(x, v_j) \subset C_j(x)\})) \subset \{u_j \in U_j : A_j(x, u_j) \subset C_j(x)\}$;*

(iii) *if U is not compact, then there exists a nonempty subset of Y_i^0 of Y_i as well as a compact subset K of U such that*

$U \setminus K \subset \bigcup_{y_i \in Y_i^0} \operatorname{cint}\{u \in U : Q_i(y_i, u) \subset D_i(u)\}$ *and for each* $N_i \in \langle Y_i \rangle$, *there is a compact GFC-subspace of* X_i *with respect to* L_{N_i} *containing* $Y_i^0 \cup N_i$;

(iv) *for each* $u \in U$, *the set* $\{y_i \in Y_i : Q_i(y_i, u) \subset D_i(u)\}$ *is nonempty and* $Q_i(y_i, u)$ *is transfer compactly upper semicontinuous in* u *with respect to* D_i;

(v) *for any* $u \in U$, $GFC(R_i(\{y_i \in Y_i : Q_i(y_i, u) \subset D_i(u)\})) \subset \{x_i \in X_i : P_i(x_i, u) \subset D_i(u)\}$.

Then there exist $(\hat{x}, \hat{u}) \in X \times U$ *and* $\hat{u} = (\hat{u}_j)_{j \in J} \in U$ *such that* $A_j(\hat{x}, \hat{u}_j) \subset C_j(\hat{x})$ *and* $P_i(\hat{x}_i, \hat{u}) \subset D_i(\hat{u})$ *for all* $i \in I$ *and* $j \in J$.

Proof. Let S_i, T_i, F_j, and G_j be set-valued mappings as defined in the proof of Theorem 18. Using the same argument of Theorem 18, we only need to show that the assumptions (i) and (iv) of Theorem 18 hold. From assumption (i) and Lemma 4, we know that for any $x \in X, F_j^{-1}$ is transfer compactly open-valued. Thus Lemma 5 tells us that the assumption (i) of Theorem 18 holds. In a similar way, from (iv) and Lemma 4, we obtain that the assumption (iv) of Theorem 18 is satisfied. So the statement of Theorem 19 follows immediately from Theorem 18. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author would like to thank the anonymous referees for careful reading of the paper and for helpful comments and suggestions which improved its presentation.

References

[1] X. P. Ding, "Maximal element theorems in product FC-spaces and generalized games," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 29–42, 2005.

[2] X. P. Ding, "Collective fixed points and a system of coincidence theorems in product FC-spaces and applications," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 66, no. 11, pp. 2604–2617, 2007.

[3] X. P. Ding and T. M. Ding, "KKM type theorems and generalized vector equilibrium problems in noncompact FC-spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1230–1245, 2007.

[4] G. Tang, Q. Zhang, and C. Cheng, "W-G-F-KKM mapping, intersection theorems and minimax inequalities in FC-space," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 1481–1491, 2007.

[5] M. Fang and N. Huang, "KKM type theorems with applications to generalized vector equilibrium problems in FC-spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 3, pp. 809–817, 2007.

[6] P. Q. Khanh and N. H. Quan, "Intersection theorems coincidence theorems and maximal-element theorems in GFC-spaces," *Optimization*, vol. 59, pp. 29–42, 2010.

[7] P. Q. Khanh, N. H. Quan, and J. Yao, "Generalized KKM-type theorems in GFC-spaces and applications," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 3-4, pp. 1227–1234, 2009.

[8] X. P. Ding, "Coincidence theorems in topological spaces and their applications," *Applied Mathematics Letters*, vol. 12, no. 7, pp. 99–105, 1999.

[9] X. P. Ding and J. Y. Park, "Generalized vector equilibrium problems in generalized convex spaces," *Journal of Optimization Theory and Applications*, vol. 120, no. 2, pp. 327–353, 2004.

[10] E. Tarafdar, "Fixed point theorems in H-spaces and equilibrium points of abstract economies," *Journal of the Australian Mathematical Society. Series A*, vol. 53, no. 1, pp. 252–260, 1992.

[11] F. E. Browder, "Coincidence theorems, minimax theorems, and variational inequalities," *Contemporary Mathematics*, vol. 26, pp. 67–80, 1984.

[12] K. Lan and J. Webb, "New fixed point theorems for a family of mappings and applications to problems on sets with convex sections," *Proceedings of the American Mathematical Society*, vol. 126, no. 4, pp. 1127–1132, 1998.

[13] Q. H. Ansari and J. Yao, "A fixed point theorem and its applications to a system of variational inequalities," *Bulletin of the Australian Mathematical Society*, vol. 59, no. 3, pp. 433–442, 1999.

[14] X. Ding and J. Park, "New collectively fixed point theorems and applications in G-convex spaces," *Applied Mathematics and Mechanics. English Edition*, vol. 23, no. 11, pp. 1237–1249, 2002.

[15] X. P. Ding and J. Y. Park, "Collectively fixed-point theorems in noncompact G-convex spaces," *Applied Mathematics Letters*, vol. 16, no. 3, pp. 137–142, 2003.

[16] L. Lin and Q. H. Ansari, "Collective fixed points and maximal elements with applications to abstract economies," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 455–472, 2004.

[17] Z. Yu and L. Lin, "Continuous selection and fixed point theorems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 52, no. 2, pp. 445–455, 2003.