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Research Article

Solving Split Common Fixed-Point Problem of Firmly Quasi-Nonexpansive Mappings without Prior Knowledge of Operators Norms

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Very recently, Moudafi introduced alternating CQ-algorithms and simultaneous iterative algorithms for the split common fixed-point problem concerned two bounded linear operators. However, to employ Moudafi's algorithms, one needs to know a prior norm (or at least an estimate of the norm) of the bounded linear operators. To estimate the norm of an operator is very difficult, if it is not an impossible task. It is the purpose of this paper to introduce a viscosity iterative algorithm with a way of selecting the stepsizes such that the implementation of the algorithm does not need any prior information about the operator norms. We prove the strong convergence of the proposed algorithms for split common fixed-point problem governed by the firmly quasi-nonexpansive operators. As a consequence, we obtain strong convergence theorems for split feasibility problem and split common null point problems of maximal monotone operators. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let I denote the identity operator on H. Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T.

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point as follows:

$$x \in C \quad \text{such that } Ax \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2].

Note that if the split feasibility problem (1) is consistent (i.e., (1) has a solution), then (1) can be formulated as a fixed point equation by using the following fact:

$$P_C\left(I-\gamma A^*\left(I-P_Q\right)A\right)x^* = x^*, \tag{2}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A. That is, x^* solves SFP (1) if and only if x^* solves fixed point equation (2) (see [3] for the details). This implies that we can use fixed point algorithms (see [3–6]) to solve SFP. To solve (2), Byrne [2] proposed his CQ algorithm which generates a sequence $\{x_k\}$ by

$$x_{k+1} = P_C\left(I-\gamma A^*\left(I-P_Q\right)A\right)x_k, \quad k \in N, \tag{3}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

Censor and Segal [7] introduced the following split common fixed-point problem (SCFP):

$$\text{find } x^* \in F(U) \quad \text{such that } Ax^* \in F(T), \tag{4}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonexpansive operators with nonempty fixed-point sets $F(U) = C$ and $F(T) = Q$. SCFP is in itself at the core of the modeling of many inverse problems in various areas of mathematics and physical sciences and has been used to model significant real-world inverse problems in many areas (see [8]).

To solve (4), Censor and Segal [7] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U\left(x_k + \gamma A^t (T - I) A x_k\right), \quad k \in N, \tag{5}$$

where $\gamma \in (0, 2/\lambda)$, with λ being the largest eigenvalue of the matrix $A^t A$ (A^t stands for matrix transposition).

Let H_1, H_2, and H_3 be real Hilbert spaces; let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators; let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators. In [9], Moudafi introduced the following split common fixed-point problem (SCFP):

$$\text{find } x \in F(U), \quad y \in F(T), \quad \text{such that } Ax = By, \tag{6}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance in decomposition methods for PDE's, in a applications in game theory, and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [10]). In IMRT, these amounts envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [11]).

If $H_2 = H_3$ and $B = I$, then SCFP (6) reduces to SCFP (4). For solving SCFP (6), Moudafi [9] introduced the following alternating algorithm:

$$\begin{aligned} x_{k+1} &= U\left(x_k - \gamma_k A^* \left(A x_k - B y_k\right)\right), \\ y_{k+1} &= T\left(y_k + \gamma_k B^* \left(A x_{k+1} - B y_k\right)\right), \end{aligned} \tag{7}$$

for firmly quasi-nonexpansive operators U and T, where nondecreasing sequence $\gamma_k \in (\varepsilon, \min(1/\lambda_A, 1/\lambda_B) - \varepsilon)$ and λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively.

Very recently, Moudafi and Al-Shemas [12] introduced the following simultaneous iterative method to solve SCFP (6):

$$\begin{aligned} x_{k+1} &= U\left(x_k - \gamma_k A^* \left(A x_k - B y_k\right)\right), \\ y_{k+1} &= T\left(y_k + \gamma_k B^* \left(A x_k - B y_k\right)\right), \end{aligned} \tag{8}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_k \in (\varepsilon, (2/(\lambda_A + \lambda_B)) - \varepsilon)$ and λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively.

In [13], Zhao and He introduced the following alternating mann iterative algorithms for SCFP (6) governed by quasi-nonexpansive mappings and obtained weak convergence results:

$$\begin{aligned} u_k &= x_k - \gamma_k A^* \left(A x_k - B y_k\right), \\ x_{k+1} &= \alpha_k u_k + \left(1 - \alpha_k\right) U\left(u_k\right), \\ v_{k+1} &= y_k + \gamma_k B^* \left(A x_{k+1} - B y_k\right), \\ y_{k+1} &= \beta_k v_{k+1} + \left(1 - \beta_k\right) T\left(v_{k+1}\right). \end{aligned} \tag{9}$$

Note that, in (7), (8), and (9) mentioned above, the determination of the stepsize $\{\gamma_k\}$ depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the above algorithms for solving SCFP (6), one has first to compute (or, at least, estimate) operator norms of A and B, which is in general not an easy work in practice. To overcome this difficulty, López et al [14] and Zhao and Yang [15] presented a helpful method for estimating the stepsizes which do not need prior knowledge of the operator norms for solving the split feasibility problems and multiple-set split feasibility problems, respectively. Inspired by them, in this paper, we introduce a new choice of the stepsize sequence $\{\gamma_k\}$ for the viscosity iterative algorithm to solve SCFP (6) governed by firmly quasi-nonexpansive operators as follows:

$$\gamma_k \in \left(\epsilon, \frac{2\left\|A x_k - B y_k\right\|^2}{\left\|A^* \left(A x_k - B y_k\right)\right\|^2 + \left\|B^* \left(A x_k - B y_k\right)\right\|^2} - \epsilon\right). \tag{10}$$

The advantage of our choice (9) of the stepsizes lies in the fact that no prior information about the operator norms of A and B is required, and still convergence is guaranteed.

Some algorithms have been invented to solve SCFP (6) (see [16, 17] and references therein). In this paper, inspired and motivated by the works mentioned above, to get the strong convergence of the algorithm, we introduce the viscosity iterative algorithm without prior knowledge of operator norms for solving SCFP (6) governed by firmly quasi-nonexpansine operators. The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the iterative algorithm in Section 2. In Section 3, the strong convergence theorem of the proposed viscosity iterative algorithm is obtained. At last, we provide some applications.

2. Preliminaries

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_k\}$ and use Γ to stand for the solution set of SCFP (6).

Definition 1. An operator $T : H \to H$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$,

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and if $\|Tx - q\| \leq \|x - q\|$ for all $x \in H$ and $q \in F(T)$,

(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$ for all $x, y \in H$,

(iv) firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and if $\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2$ for all $x \in H$ and $q \in F(T)$.

Remark 2. A firmly quasi-nonexpansive operator is also called a separating operator [18], cutter operator [19], directed operators [7, 20], or class-$\mathfrak{T}$ operator which was introduced by Bauschke and Combettes [21]. Firmly quasi-nonexpansive operators are important because they include many types of nonlinear operators arising in applied mathematics such as approximation and convex optimization. For instance, the subgradient projection T of a continuous convex function $f : H \to \mathbb{R}$ is a firmly quasi-nonexpansive operator. Recall that the subgradient projection T is defined by, assuming that the level set $\{x \in H : f(x) \leq 0\} \neq \emptyset$,

$$Tx := \begin{cases} x - \dfrac{f(x)}{\|g(x)\|^2} g(x), & f(x) > 0, \\ x, & f(x) \leq 0, \end{cases} \tag{11}$$

where g is a selection of the subdifferential ∂f (i.e., $g(x) \in \partial f(x)$ for all $x \in H$).

Particularly, projections are firmly quasi-nonexpansive operators. Recall that, given a closed convex subset C of a Hilbert space H, the projection $P_C : H \to C$ assigns each $x \in H$ to its closest point from C defined by

$$P_C x = \arg\min_{z \in C} \|x - z\|. \tag{12}$$

It is well known that $P_C x$ is characterized by the inequality

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x \rangle \leq 0, \quad z \in C. \tag{13}$$

Lemma 3 (see [19, 21]). *The fixed point set of a firmly quasi-nonexpansive operator is closed convex.*

We also need other classes of operators.

Definition 4. An operator $T : H \to H$ called demiclosed at the origin if whenever the sequence $\{x_n\}$ converges weakly to x and the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

We remark here that a firmly quasi-nonexpansive operator T may be not nonexpansive; even $T - I$ is demiclosed at origin. See the following example [22].

Example 5. Let $H = R_1$ and define a mapping by $T : H \to H$ by

$$Tx := \begin{cases} \dfrac{x}{2} \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{14}$$

Then, $F(T) = \{0\}$ and

$$\begin{aligned} |Tx - 0|^2 &= \frac{x^2}{4} \left(\sin \frac{1}{x} \right)^2 \leq x^2 - \left(x - \frac{x}{2} \sin \frac{1}{x} \right)^2 \\ &= |x - 0|^2 - |x - Tx|^2. \end{aligned} \tag{15}$$

So, T is firmly quasi-nonexpansive but not nonexpansive. It is easy to see that $T - I$ is demiclosed at origin.

Definition 6. An operator $T : H \to H$ is called contraction with constant $\rho \in [0, 1)$ if, for any $x, y \in H$,

$$\|Tx - Ty\| \leq \rho \|x - y\|. \tag{16}$$

In real Hilbert space, we easily get the following equality:

$$\begin{aligned} 2 \langle x, y \rangle &= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \\ &= \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in H. \end{aligned} \tag{17}$$

We end this section by the following lemmas, which are important in convergence analysis for our iterative algorithm.

Lemma 7 (see [23]). *Assume $\{s_k\}$ is a sequence of nonnegative real numbers such that*

$$\begin{aligned} s_{k+1} &\leq (1 - \lambda_k) s_k + \lambda_k \delta_k, \quad k \geq 0, \\ s_{k+1} &\leq s_k - \eta_k + \mu_k, \quad k \geq 0, \end{aligned} \tag{18}$$

where $\{\lambda_k\}$ is a sequence in $(0, 1)$, $\{\eta_k\}$ is a sequence of nonnegative real numbers, and $\{\delta_k\}$ and $\{\mu_k\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{k=1}^{\infty} \lambda_k = \infty$;

(ii) $\lim_{k \to \infty} \mu_k = 0$;

(iii) $\lim_{l \to \infty} \eta_{k_l} = 0$ *implies* $\limsup_{l \to \infty} \delta_{k_l} \leq 0$ *for any subsequence* $\{k_l\} \subset \{k\}$.

Then, $\lim_{k \to \infty} s_k = 0$.

Lemma 8 (see [24, Lemma 1.3]). *Let $\{\delta_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\delta_{n_j}\}_{j \geq 0}$ of $\{\delta_n\}$ which satisfies $\delta_{n_j} < \delta_{n_j + 1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined by*

$$\tau(n) = \max \{ k \leq n \mid \delta_k < \delta_{k+1} \}. \tag{19}$$

Then, $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \tau(n) = \infty$ and, for all $n \geq n_0$, it holds that $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$ and we have

$$\delta_n \leq \delta_{\tau(n)+1}. \tag{20}$$

3. Viscosity Iterative Algorithm without Prior Knowledge of Operator Norms

In this section, we introduce a viscosity iterative algorithm where the stepsizes γ_k do not depend on the operator norms $\|A\|$ and $\|B\|$ and prove the strong convergence of algorithm without prior knowledge of operator norms.

Algorithm 9. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants $\rho_1, \rho_2 \in [0, 1)$, and $\alpha_k \in [0, 1]$. Choose an initial guess $x_0 \in H_1$, $y_0 \in H_2$ arbitrarily. Assume that the kth iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed; then, we calculate the $(k+1)$th iterate (x_{k+1}, y_{k+1}) via the formula

$$\begin{aligned} u_k &= x_k - \gamma_k A^* (Ax_k - By_k), \\ x_{k+1} &= \alpha_k f_1 (x_k) + (1 - \alpha_k) U (u_k), \\ v_k &= y_k + \gamma_k B^* (Ax_k - By_k), \\ y_{k+1} &= \alpha_k f_2 (x_k) + (1 - \alpha_k) T (v_k). \end{aligned} \tag{21}$$

The stepsize γ_k is chosen in such a way that

$$\gamma_k \in \left(\epsilon, \frac{2\|Ax_k - By_k\|^2}{\|A^* (Ax_k - By_k)\|^2 + \|B^* (Ax_k - By_k)\|^2} - \epsilon \right), \quad k \in \Omega, \tag{22}$$

otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{k : Ax_k - By_k \neq 0\}$.

Remark 10. Note that, in (22), the choice of the stepsize γ_k is independent of the norms $\|A\|$ and $\|B\|$. The value of γ does not influence the considered algorithm, but it was introduced just for the sake of clarity. Furthermore, we will see from Lemma 3 that γ_k is well defined.

Lemma 11. *Assume the solution set* Γ *of* (6) *is nonempty. Then,* γ_k *defined by* (22) *is well defined.*

Proof. Taking $(x, y) \in \Gamma$, that is, $x \in F(U)$, $y \in F(T)$, and $Ax = By$, we have

$$\begin{aligned} \langle A^* (Ax_k - By_k), x_k - x \rangle &= \langle Ax_k - By_k, Ax_k - Ax \rangle, \\ \langle B^* (Ax_k - By_k), y - y_k \rangle &= \langle Ax_k - By_k, By - By_k \rangle. \end{aligned} \tag{23}$$

By adding the two above equalities and by taking into account the fact that $Ax = By$, we obtain

$$\begin{aligned} \|Ax_k - By_k\|^2 &= \langle A^* (Ax_k - By_k), x_k - x \rangle \\ &\quad + \langle B^* (Ax_k - By_k), y - y_k \rangle \\ &\le \|A^* (Ax_k - By_k)\| \cdot \|x_k - x\| \\ &\quad + \|B^* (Ax_k - By_k)\| \cdot \|y - y_k\|. \end{aligned} \tag{24}$$

Consequently, for $k \in \Omega$, that is, $\|Ax_k - By_k\| > 0$, we have $\|A^*(Ax_k - By_k)\| \neq 0$ or $\|B^*(Ax_k - By_k)\| \neq 0$. This leads that γ_k is well defined. □

Theorem 12. *Let H_1, H_2, and H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be firmly quasi-nonexpansive operators with the solution set Γ of* (6) *being nonempty. Let the sequence $\{(x_k, y_k)\}$ be generated by Algorithm 9. Assume that the following conditions are satisfied:*

(1) $\rho_1, \rho_2 \in [0, 1/\sqrt{2})$;

(2) $\lim_{k \to \infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$.

(3) $U - I$ *and* $T - I$ *are demiclosed at origin;*

Then, sequence $\{(x_k, y_k)\}$ strongly converges to a solution (x^*, y^*) of (6) *which solves the variational inequality problem*

$$\begin{aligned} \langle (I - f_1) x^*, x - x^* \rangle &\ge 0, \\ \langle (I - f_2) y^*, y - y^* \rangle &\ge 0, \end{aligned} \tag{25}$$

Proof. Let $(x^*, y^*) \in \Gamma$ be the solution of the variational inequality problem (25). Then, $x^* \in F(U)$, $y^* \in F(T)$, and $Ax^* = By^*$. We have

$$\begin{aligned} &\|u_k - x^*\|^2 \\ &\quad = \|x_k - \gamma_k A^* (Ax_k - By_k) - x^*\|^2 \\ &\quad = \|x_k - x^*\|^2 - 2\gamma_k \langle x_k - x^*, A^* (Ax_k - By_k) \rangle \\ &\qquad + \gamma_k^2 \|A^* (Ax_k - By_k)\|^2. \end{aligned} \tag{26}$$

Using (17), we have

$$\begin{aligned} &- 2 \langle x_k - x^*, A^* (Ax_k - By_k) \rangle \\ &\quad = -2 \langle Ax_k - Ax^*, Ax_k - By_k \rangle \\ &\quad = -\|Ax_k - Ax^*\|^2 - \|Ax_k - By_k\|^2 \\ &\qquad + \|By_k - Ax^*\|^2. \end{aligned} \tag{27}$$

By (26) and (27), we obtain

$$\begin{aligned} &\|u_k - x^*\|^2 \\ &\quad \le \|x_k - x^*\|^2 - \gamma_k \|Ax_k - Ax^*\|^2 \\ &\qquad - \gamma_k \|Ax_k - By_k\|^2 + \gamma_k \|By_k - Ax^*\|^2 \\ &\qquad + \gamma_k^2 \|A^* (Ax_k - By_k)\|^2. \end{aligned} \tag{28}$$

Similarly, we have

$$\begin{aligned} &\|v_k - y^*\|^2 \\ &\quad \le \|y_k - y^*\|^2 - \gamma_k \|By_k - By^*\|^2 - \gamma_k \|Ax_k - By_k\|^2 \\ &\qquad + \gamma_k \|Ax_k - By^*\|^2 + \gamma_k^2 \|B^*(Ax_k - By_k)\|^2. \end{aligned} \tag{29}$$

By adding the two last inequalities and by taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned} &\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \\ &\quad \le \|x_k - x^*\|^2 + \|y_k - y^*\|^2 \\ &\qquad - \gamma_k \big[2\|Ax_k - By_k\|^2 \\ &\qquad - \gamma_k \big(\|A^*(Ax_k - By_k)\|^2 \\ &\qquad + \|B^*(Ax_k - By_k)\|^2 \big) \big]. \end{aligned} \tag{30}$$

With assumption on γ_k, we obtain

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2. \quad (31)$$

Setting $\rho = \max\{\rho_1, \rho_1\}$, we have $\rho \in [0, 1/\sqrt{2})$. By U and T being firmly quasi-nonexpansive operators, it follows that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\quad \leq \alpha_k \|f_1(x_k) - x^*\|^2 + (1-\alpha_k)\|U(u_k) - x^*\|^2 \\
&\quad \leq \alpha_k (\|f_1(x_k) - f_1(x^*)\| + \|f_1(x^*) - x^*\|)^2 \\
&\qquad + (1-\alpha_k)\|u_k - x^*\|^2 - (1-\alpha_k)\|u_k - U(u_k)\|^2 \\
&\quad \leq 2\alpha_k \left(\|f_1(x_k) - f_1(x^*)\|^2 + \|f_1(x^*) - x^*\|^2\right) \\
&\qquad + (1-\alpha_k)\|u_k - x^*\|^2 - (1-\alpha_k)\|u_k - U(u_k)\|^2 \\
&\quad \leq 2\alpha_k \rho_1^2 \|x_k - x^*\|^2 + 2\alpha_k \|f_1(x^*) - x^*\|^2 \\
&\qquad + (1-\alpha_k)\|u_k - x^*\|^2 - (1-\alpha_k)\|u_k - U(u_k)\|^2, \\
&\|y_{k+1} - y^*\|^2 \\
&\quad \leq 2\alpha_k \rho_2^2 \|y_k - y^*\|^2 + 2\alpha_k \|f_2(y^*) - y^*\|^2 \\
&\qquad + (1-\alpha_k)\|v_k - y^*\|^2 - (1-\alpha_k)\|v_k - T(v_k)\|^2.
\end{aligned} \quad (32)$$

Adding up the last two inequalities and using (31), setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$\begin{aligned}
s_{k+1} \leq & \left(1 - \alpha_k \left(1 - 2\rho^2\right)\right) s_k \\
& + 2\alpha_k \left(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2\right) \\
& - (1-\alpha_k)\left(\|u_k - U(u_k)\|^2 + \|v_k - T(v_k)\|^2\right),
\end{aligned} \quad (33)$$

which implies

$$\begin{aligned}
s_{k+1} \leq & \left(1 - \alpha_k \left(1 - 2\rho^2\right)\right) s_k + \alpha_k \left(1 - 2\rho^2\right) \frac{2}{1 - 2\rho^2} \\
& \times \left(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2\right).
\end{aligned} \quad (34)$$

It follows from induction that

$$s_k \leq \max\left\{s_0, \frac{2}{1-2\rho^2}\left(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2\right)\right\} \quad k \geq 0, \quad (35)$$

which implies that $\{x_k\}$ and $\{y_k\}$ are bounded. It follows that $\{u_k\}, \{v_k\}, \{f_1(x_k)\}$ and $\{f_2(y_k)\}$ are bounded.

Note that U is a firmly quasi-nonexpansive operator; we have

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&= \alpha_k^2 \|f_1(x_k) - x^*\|^2 + 2\alpha_k (1-\alpha_k) \\
&\quad \times \langle f_1(x_k) - x^*, U(u_k) - x^* \rangle \\
&\quad + (1-\alpha_k)^2 \|U(u_k) - x^*\|^2 \\
&= \alpha_k^2 \|f_1(x_k) - x^*\|^2 + 2\alpha_k (1-\alpha_k) \\
&\quad \times \langle f_1(x_k) - f_1(x^*), U(u_k) - x^* \rangle \\
&\quad + 2\alpha_k (1-\alpha_k) \langle f_1(x^*) - x^*, U(u_k) - x^* \rangle \\
&\quad + (1-\alpha_k)^2 \|U(u_k) - x^*\|^2 \\
&\leq \alpha_k^2 \|f_1(x_k) - x^*\|^2 + \alpha_k (1-\alpha_k) \\
&\quad \times \left(\|f_1(x_k) - f_1(x^*)\|^2 + \|U(u_k) - x^*\|^2\right) \\
&\quad + 2\alpha_k (1-\alpha_k) \langle f_1(x^*) - x^*, U(u_k) - x^* \rangle \\
&\quad + (1-\alpha_k)^2 \|U(u_k) - x^*\|^2 \\
&\leq \alpha_k^2 \|f_1(x_k) - x^*\|^2 + \alpha_k (1-\alpha_k) \rho_1^2 \|x_k - x^*\|^2 \\
&\quad + \alpha_k (1-\alpha_k) \|U(u_k) - x^*\|^2 \\
&\quad + 2\alpha_k (1-\alpha_k) \langle f_1(x^*) - x^*, U(u_k) - x^* \rangle \\
&\quad + (1-\alpha_k)^2 \|U(u_k) - x^*\|^2 \\
&\leq (1-\alpha_k) \|u_k - x^*\|^2 + \alpha_k (1-\alpha_k) \rho_1^2 \|x_k - x^*\|^2 \\
&\quad + \alpha_k \left[\alpha_k \|f_1(x_k) - x^*\|^2 + 2(1-\alpha_k)\right. \\
&\qquad \left. \times \langle f_1(x^*) - x^*, U(u_k) - x^* \rangle\right].
\end{aligned} \quad (36)$$

Similarly, we have

$$\begin{aligned}
&\|y_{k+1} - y^*\|^2 \\
&\quad \leq (1-\alpha_k) \|v_k - y^*\|^2 + \alpha_k (1-\alpha_k) \rho_2^2 \|y_k - y^*\|^2 \\
&\qquad + \alpha_k \left[\alpha_k \|f_2(y_k) - y^*\|^2 + 2(1-\alpha_k)\right. \\
&\qquad\quad \left. \times \langle f_2(y^*) - y^*, T(v_k) - y^* \rangle\right].
\end{aligned} \quad (37)$$

So, by (31), (36), and (37), we obtain

$$\begin{aligned} s_{k+1} &\leq (1-\alpha_k) s_k + \alpha_k (1-\alpha_k) \rho^2 s_k \\ &\quad + \alpha_k \left[\alpha_k \left(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2 \right) \right. \\ &\quad + 2(1-\alpha_k) \left(\langle f_1(x^*) - x^*, U(u_k) - x^* \rangle \right. \\ &\quad \left. \left. + \langle f_2(y^*) - y^*, T(v_k) - y^* \rangle \right) \right] \\ &= (1-\lambda_k) s_k + \lambda_k \delta_k, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \lambda_k &= \alpha_k \left(1 - (1-\alpha_k) \rho^2 \right), \\ \delta_k &= \left(2(1-\alpha_k) \left(\langle f_1(x^*) - x^*, U(u_k) - x^* \rangle \right. \right. \\ &\quad \left. \left. + \langle f_2(y^*) - y^*, T(v_k) - y^* \rangle \right) \right) \\ &\quad \times \left(1 - (1-\alpha_k) \rho^2 \right)^{-1} \\ &\quad + \frac{\alpha_k \left(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2 \right)}{1 - (1-\alpha_k) \rho^2}. \end{aligned} \tag{39}$$

On the other hand, from (21), we have

$$\begin{aligned} &\| x_{k+1} - x^* \|^2 \\ &\quad \leq \alpha_k \| f_1(x_k) - x^* \|^2 + (1-\alpha_k) \| U(u_k) - x^* \|^2 \\ &\quad \leq \alpha_k \| f_1(x_k) - x^* \|^2 + (1-\alpha_k) \| u_k - x^* \|^2 \\ &\quad\quad - (1-\alpha_k) \| U(u_k) - u_k \|^2, \\ &\| y_{k+1} - y^* \|^2 \\ &\quad \leq \alpha_k \| f_2(y_k) - y^* \|^2 + (1-\alpha_k) \| v_k - y^* \|^2 \\ &\quad\quad - (1-\alpha_k) \| T(v_k) - v_k \|^2. \end{aligned} \tag{40}$$

Adding up the last two inequalities and using (30), we obtain

$$\begin{aligned} s_{k+1} &\leq \| u_k - x^* \|^2 + \| v_k - y^* \|^2 \\ &\quad + \alpha_k \left(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2 \right) \\ &\quad - (1-\alpha_k) \left(\| U(u_k) - u_k \|^2 + \| T(v_k) - v_k \|^2 \right) \\ &\leq s_k - \gamma_k \left[2 \| Ax_k - By_k \|^2 \right. \\ &\quad - \gamma_k \left(\| A^* (Ax_k - By_k) \|^2 \right. \\ &\quad \left. \left. + \| B^* (Ax_k - By_k) \|^2 \right) \right] \\ &\quad + \alpha_k \left(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2 \right) \\ &\quad - (1-\alpha_k) \left(\| U(u_k) - u_k \|^2 + \| T(v_k) - v_k \|^2 \right). \end{aligned} \tag{41}$$

Now, by setting $\mu_k = \alpha_k(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2)$, $\eta_k = \gamma_k[2\| Ax_k - By_k \|^2 - \gamma_k(\| A^*(Ax_k - By_k) \|^2 + \| B^*(Ax_k - By_k) \|^2)]$, and $\theta_k = (1-\alpha_k)(\| U(u_k) - u_k \|^2 + \| T(v_k) - v_k \|^2)$, (41) can be rewritten as the following form:

$$s_{k+1} \leq s_k - \eta_k + \mu_k - \theta_k \leq s_k - \eta_k + \mu_k, \quad k \geq 0. \tag{42}$$

By the assumption on α_k, we get $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\lim_{k\to\infty} \mu_k = 0$ which thanks to the boundedness of $\{f_1(x_k)\}$ and $\{f_2(y_k)\}$.

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists k_0 such that $\{s_k\}_{k \geq k_0}$ is nonincreasing. In this situation, $\{s_k\}$ is convergent because it is nonnegative so that $\lim_{k\to\infty}(s_{k+1} - s_k) = 0$; hence, in light of (33) together with $\alpha_k \to 0$ and the boundedness of $\{s_k\}$, we obtain

$$\lim_{k\to\infty} \| u_k - U(u_k) \| = \lim_{k\to\infty} \| v_k - T(v_k) \| = 0. \tag{43}$$

To use Lemma 8, it suffices to verify that, for all subsequences $\{k_l\} \subset \{k\}$, $\lim_{l\to\infty} \eta_{k_l} = 0$ implies

$$\limsup_{l\to\infty} \delta_{k_l} \leq 0. \tag{44}$$

It follows from $\lim_{k\to\infty} \eta_{k_l} = 0$ that

$$\begin{aligned} \lim_{l\to\infty} \gamma_{k_l} &\left[2 \left\| Ax_{k_l} - By_{k_l} \right\|^2 \right. \\ &\quad - \gamma_{k_l} \left(\left\| A^* \left(Ax_{k_l} - By_{k_l} \right) \right\|^2 \right. \\ &\quad \left. \left. + \left\| B^* \left(Ax_{k_l} - By_{k_l} \right) \right\|^2 \right) \right] = 0, \end{aligned} \tag{45}$$

which yields $\lim_{l\to\infty} \| Ax_{k_l} - By_{k_l} \| = 0$ from the assumption on γ_k. So,

$$\begin{aligned} \lim_{l\to\infty} \left\| u_{k_l} - x_{k_l} \right\| &= \lim_{l\to\infty} \gamma_{k_l} \left\| A^* \left(Ax_{k_l} - By_{k_l} \right) \right\| = 0, \\ \lim_{l\to\infty} \left\| v_{k_l} - y_{k_l} \right\| &= \lim_{l\to\infty} \gamma_{k_l} \left\| B^* \left(Ax_{k_l} - By_{k_l} \right) \right\| = 0. \end{aligned} \tag{46}$$

Taking $(\tilde{x}, \tilde{y}) \in \omega_w(x_{k_l}, y_{k_l})$, from (46), we have $(\tilde{x}, \tilde{y}) \in \omega_w(u_{k_l}, v_{k_l})$. Combined with the demiclosednesses of $U - I$ and $T - I$ at 0, (43) yields $U\tilde{x} = \tilde{x}$ and $T\tilde{y} = \tilde{y}$. So, $\tilde{x} \in F(U)$ and $\tilde{y} \in F(T)$. On the other hand, $A\tilde{x} - B\tilde{y} \in \omega_w(Ax_{k_l} - By_{k_l})$ and weakly lower semicontinuity of the norm imply

$$\| A\tilde{x} - B\tilde{y} \| \leq \liminf_{l\to\infty} \left\| Ax_{k_l} - By_{k_l} \right\| = 0; \tag{47}$$

hence, $(\tilde{x}, \tilde{y}) \in \Gamma$. So, $\omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$. Since $\lim_{k\to\infty} \alpha_k(\| f_1(x_k) - x^* \|^2 + \| f_2(y_k) - y^* \|^2) = 0$ and $\lim_{k\to\infty}(1 - (1-\alpha_k)\rho^2) = 1 - \rho^2$, to get (44), we only need to verify

$$\begin{aligned} \limsup_{l\to\infty} &\left(\left\langle f_1(x^*) - x^*, U\left(u_{k_l}\right) - x^* \right\rangle \right. \\ &\quad \left. + \left\langle f_2(y^*) - y^*, T\left(v_{k_l}\right) - y^* \right\rangle \right) \leq 0. \end{aligned} \tag{48}$$

Indeed, from (43) and (46), we have

$$\begin{aligned}
&\limsup_{l\to\infty}\left(\left\langle f_1(x^*)-x^*,U\left(u_{k_l}\right)-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle f_2(y^*)-y^*,T\left(v_{k_l}\right)-y^*\right\rangle\right)\\
&=\limsup_{l\to\infty}\left(\left\langle f_1(x^*)-x^*,u_{k_l}-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle f_2(y^*)-y^*,v_{k_l}-y^*\right\rangle\right)\\
&=\limsup_{l\to\infty}\left(\left\langle f_1(x^*)-x^*,x_{k_l}-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle f_2(y^*)-y^*,y_{k_l}-y^*\right\rangle\right)\\
&=-\liminf_{l\to\infty}\left(\left\langle (I-f_1)x^*,x_{k_l}-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle (I-f_2)y^*,y_{k_l}-y^*\right\rangle\right).
\end{aligned}\tag{49}$$

We can take subsequence $\{(x_{k_{l_j}}, y_{k_{l_j}})\}$ of $\{(x_{k_l}, y_{k_l})\}$ such that $(x_{k_{l_j}}, y_{k_{l_j}}) \rightharpoonup (\tilde{x}, \tilde{y})$ as $j \to \infty$ and

$$\begin{aligned}
&-\liminf_{l\to\infty}\left(\left\langle (I-f_1)x^*,x_{k_l}-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle (I-f_2)y^*,y_{k_l}-y^*\right\rangle\right)\\
&=-\lim_{j\to\infty}\left(\left\langle (I-f_1)x^*,x_{k_{l_j}}-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle (I-f_2)y^*,y_{k_{l_j}}-y^*\right\rangle\right)\\
&=-\left(\langle (I-f_1)x^*,\tilde{x}-x^*\rangle+\langle (I-f_2)y^*,\tilde{y}-y^*\rangle\right).
\end{aligned}\tag{50}$$

Since $\omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$ and (x^*, y^*) is the solution of the variational inequality problem (25), from (49) and (50), we obtain

$$\begin{aligned}
&\limsup_{l\to\infty}\left(\left\langle f_1(x^*)-x^*,U\left(u_{k_l}\right)-x^*\right\rangle\right.\\
&\qquad\left.+\left\langle f_2(y^*)-y^*,T\left(v_{k_l}\right)-y^*\right\rangle\right)\le 0.
\end{aligned}\tag{51}$$

From Lemma 8, it follows

$$\lim_{k\to\infty}\left(\|x_k-x^*\|^2+\|y_k-y^*\|^2\right)=0,\tag{52}$$

which implies that $x_k \to x^*$ and $y_k \to y^*$.

Case 2. Suppose there exists a subsequence $\{s_{k_j}\}_{j\ge 0}$ of $\{s_k\}$ such that $s_{k_j} < s_{k_j+1}$ for all $j \ge 0$. In this situation, we consider the sequence of indices $\{\tau(k)\}$ as defined in Lemma 8. It follows that $s_{\tau(k)+1} - s_{\tau(k)} > 0$. From (42), we have

$$0\le \eta_{\tau(k)}\le s_{\tau(k)}-s_{\tau(k)+1}+\mu_{\tau(k)}<\mu_{\tau(k)},\quad k\ge 0.\tag{53}$$

So, by $\lim_{k\to\infty}\mu_k = 0$, we obtain

$$\lim_{k\to\infty}\eta_{\tau(k)}=0.\tag{54}$$

Again from (42), we get

$$0\le\theta_{\tau(k)}\le\mu_{\tau(k)}-\eta_{\tau(k)};\tag{55}$$

hence,

$$\begin{aligned}
&\lim_{k\to\infty}\theta_{\tau(k)}\\
&=\lim_{k\to\infty}\left(1-\alpha_{\tau(k)}\right)\\
&\quad\times\left(\left\|U\left(u_{\tau(k)}\right)-u_{\tau(k)}\right\|^2+\left\|T\left(v_{\tau(k)}\right)-v_{\tau(k)}\right\|^2\right)=0.
\end{aligned}\tag{56}$$

In light of $\alpha_k \to 0$, we obtain

$$\lim_{k\to\infty}\left\|u_{\tau(k)}-U\left(u_{\tau(k)}\right)\right\|=\lim_{k\to\infty}\left\|v_{\tau(k)}-T\left(v_{\tau(k)}\right)\right\|=0.\tag{57}$$

From $y_{\tau(k)} \to 0$, similar to Case 1, we have

$$\begin{aligned}
\lim_{k\to\infty}\left\|Ax_{\tau(k)}-By_{\tau(k)}\right\|&=\lim_{k\to\infty}\left\|u_{\tau(k)}-x_{\tau(k)}\right\|\\
&=\lim_{k\to\infty}\left\|v_{\tau(k)}-y_{\tau(k)}\right\|=0,
\end{aligned}\tag{58}$$

$\omega_w(x_{\tau(k)}, y_{\tau(k)}) \subset \Gamma$, and

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\langle f_1(x^*)-x^*,U\left(u_{\tau(k)}\right)-x^*\rangle\right.\\
&\qquad\left.+\langle f_2(y^*)-y^*,T\left(v_{\tau(k)}\right)-y^*\rangle\right)\le 0,
\end{aligned}\tag{59}$$

which implies

$$\limsup_{k\to\infty}\delta_{\tau(k)}\le 0.\tag{60}$$

From $s_{\tau(k)+1} - s_{\tau(k)} > 0$ and (38), it follows that

$$\lambda_{\tau(k)}s_{\tau(k)}\le\lambda_{\tau(k)}\delta_{\tau(k)}.\tag{61}$$

Since $s_{\tau(k)+1} - s_{\tau(k)} > 0$, again from (38), we may assume $\lambda_{\tau(k)} > 0$ for all $k \ge 0$. It follows from (60) and (61) that $\lim_{k\to\infty}s_{\tau(k)} = 0$ and hence

$$\lim_{k\to\infty}\left\|x_{\tau(k)}-x^*\right\|=\lim_{k\to\infty}\left\|y_{\tau(k)}-y^*\right\|=0.\tag{62}$$

On the other hand, it follows that

$$\begin{aligned}
&\left\|x_{\tau(k)+1}-x_{\tau(k)}\right\|\\
&=\left\|\alpha_{\tau(k)}\left(f\left(x_{\tau(k)}\right)-x_{\tau(k)}\right)\right.\\
&\qquad\left.+\left(1-\alpha_{\tau(k)}\right)\left(U\left(u_{\tau(k)}\right)-x_{\tau(k)}\right)\right\|\\
&\le\alpha_{\tau(k)}\left\|f\left(x_{\tau(k)}\right)-x_{\tau(k)}\right\|+\left(1-\alpha_{\tau(k)}\right)\\
&\qquad\times\left[\left\|U\left(u_{\tau(k)}\right)-u_{\tau(k)}\right\|+\left\|u_{\tau(k)}-x_{\tau(k)}\right\|\right],
\end{aligned}\tag{63}$$

which, by $\alpha_k \to 0$, (57), and (58), implies that

$$\lim_{k\to\infty}\left\|x_{\tau(k)+1}-x_{\tau(k)}\right\|=0.\tag{64}$$

By (62), we obtain

$$\lim_{k\to\infty}\left\|x_{\tau(k)+1}-x^*\right\|=0.\tag{65}$$

Similarly, we have $\lim_{k\to\infty}\|y_{\tau(k)+1}-y^*\|=0$; hence,

$$\lim_{k\to\infty}s_{\tau(k)+1} = \lim_{k\to\infty}\left(\|x_{\tau(k)+1}-x^*\|^2+\|y_{\tau(k)+1}-y^*\|^2\right)=0. \tag{66}$$

Then, recalling that $s_k \le s_{\tau(k)+1}$ (by Lemma 8), we get $\lim_{k\to\infty}s_k=0$.

So, sequence $\{(x_k, y_k)\}$ strongly converges to the solution (x^*, y^*) of (6) which solves the variational inequality problem (25). □

4. Another Split Problem Deduced from SCFP

We now turn our attention to providing some algorithms for solving another split problem without prior knowledge of operator norms.

4.1. Split Feasibility Problem. Taking $U=P_C$ and $T=P_Q$, we have that the following viscosity iterative algorithm for split feasibility problem (SFP) under consideration is nothing but

$$\text{find } x\in C,\quad y\in Q,\quad \text{such that } Ax=By. \tag{67}$$

Algorithm 13. Let $x_0\in H_1$, $y_0\in H_2$ be arbitrary. Consider

$$\begin{aligned} u_k &= x_k-\gamma_k A^*\left(Ax_k-By_k\right),\\ x_{k+1} &= \alpha_k f_1\left(x_k\right)+\left(1-\alpha_k\right)P_C\left(u_k\right),\\ v_k &= y_k+\gamma_k B^*\left(Ax_k-By_k\right),\\ y_{k+1} &= \alpha_k f_2\left(x_k\right)+\left(1-\alpha_k\right)P_Q\left(v_k\right), \end{aligned} \tag{68}$$

where the stepsize γ_k is chosen by (22) in Algorithm 9.

In [16], Dong et al. introduced Algorithm 13 for SFP (67) without prior knowledge of operator norms. The stepsize γ_k is chosen in such a way that

$$\gamma_k\in\left(\epsilon,\min\left\{\frac{\|Ax_k-By_k\|^2}{\|A^*\left(Ax_k-By_k\right)\|^2},\frac{\|Ax_k-By_k\|^2}{\|B^*\left(Ax_k-By_k\right)\|^2}\right\}-\epsilon\right). \tag{69}$$

It is easy to see that the results of this paper improve and extend the corresponding results of [16].

4.2. Split Common Null Point Problem. Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well known that its associated resolvent mapping, $J_\mu^M(x) := (I+\mu M)^{-1}$, is firmly quasi-nonexpansive and $0\in M(x) \Leftrightarrow x=J_\mu^M(x)$. In other words, zeroes of M are exactly fixed-points of its resolvent mapping. By taking $U=J_\mu^M$, $T=J_\nu^N$, where $N : H_2\to 2^{H_2}$ is another maximal monotone operator, the problem under consideration is nothing but

$$\text{find } x^*\in M^{-1}(0),\quad y^*\in N^{-1}(0)\quad \text{such that } Ax^*=By^*, \tag{70}$$

and the algorithms take the following equivalent form:

$$\begin{aligned} &\forall x_0\in H_1,\quad y_0\in H_2,\\ u_k &= x_k-\gamma_k A^*\left(Ax_k-By_k\right),\\ x_{k+1} &= \alpha_k f_1\left(x_k\right)+\left(1-\alpha_k\right)J_\mu^M\left(u_k\right),\\ v_k &= y_k+\gamma_k B^*\left(Ax_k-By_k\right),\\ y_{k+1} &= \beta_k f_2\left(x_k\right)+\left(1-\beta_k\right)J_\nu^N\left(v_k\right),\quad k\ge 1, \end{aligned} \tag{71}$$

The stepsize γ_k is chosen as follows:

$$\gamma_k\in\left(\epsilon,\left(2\|Ax_k-By_k\|^2\right)\times\left(\|A^*\left(Ax_k-By_k\right)\|^2+\|B^*\left(Ax_k-By_k\right)\|^2\right)^{-1}-\epsilon\right),\quad k\in\Omega; \tag{72}$$

otherwise, $\gamma_k=\gamma$ (γ being any nonnegative value), where the set of indexes $\Omega=\{k : Ax_k-By_k\neq 0\}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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