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Research Article

2-Strict Convexity and Continuity of Set-Valued Metric Generalized Inverse in Banach Spaces

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Authors investigate the metric generalized inverses of linear operators in Banach spaces. Authors prove by the methods of geometry of Banach spaces that, if X is approximately compact and X is 2-strictly convex, then metric generalized inverses of bounded linear operators in X are upper semicontinuous. Moreover, authors also give criteria for metric generalized inverses of bounded linear operators to be lower semicontinuous. Finally, a sufficient condition for set-valued mapping T^{∂} to be continuous mapping is given.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. Let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball, respectively. By X^* we denote the dual space of X. Let N, R, and R^+ denote the set of natural numbers, reals, and nonnegative reals, respectively.

Let $A_f = \{x \in S(X) : f(x) = \|f\| = \|x\| = 1\}$ and $[x_1, x_2] = \{tx_1 + (1-t)x_2 : t \in [0,1]\}$. By $x_n \xrightarrow{w} x$ we denote that $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x. $C(\overline{C^w})$ denotes closed hull of C (weak closed hull) and $\operatorname{dist}(x, C)$ denotes the distance of x and C. Let $C \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_C : X \to C$

$$P_C(x) = \left\{ z \in C : \|x - z\| = \operatorname{dist}(x, C) := \inf_{y \in C} \|x - y\| \right\} \quad (1)$$

is called the metric projection operator from X onto C.

A subset C of X is said to be proximal if $P_C(x) \neq \emptyset$ for all $x \in X$ (see [1]). C is said to be semi-Chebyshev if $P_C(x)$ is at most a singleton for all $x \in X$. C is said to be Chebyshev if it is proximal and semi-Chebyshev. It is well known that (see [1]) X is reflexive if and only if each closed convex subset of X is proximal and that X is strictly convex if and only if each convex subset of X is semi-Chebyshev.

Definition 1 (see [2]). A nonempty subset C of X is said to be approximatively compact if, for any $\{y_n\}_{n=1}^{\infty} \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y \in C} \|x - y\|$ $(n \to \infty)$, the sequence $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C. X is called approximatively compact if every nonempty closed convex subset of X is approximatively compact.

Definition 2 (see [3]). Set-valued mapping $F : X \to Y$ is called upper semicontinuous at x_0, if, for each norm open set W with $F(x_0) \subset W$, there exists a norm neighborhood U of x_0 such that $F(x) \subset W$ for all x in U. F is called lower continuous at x_0, if, for any $y \in F(x_0)$ and any $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \to x_0$, there exists $y_n \in F(x_n)$ such that $y_n \to y$ as $n \to \infty$. F is called continuous at x_0, if F is upper semicontinuous and is lower continuous at x_0.

Let us present the history of the approximative compactness and related notions. This notion has been introduced by Jefimow and Stechkin in [2] as a property of Banach spaces, which guarantees the existence of the best approximation element in a nonempty closed convex set C for any $x \in X$. In 2007, Chen et al. (see [4]) proved that a nonempty closed convex C of a midpoint locally uniformly rotund space is approximately compact if and only if C is Chebyshev set and the metric projection operator P_C is continuous. In 1972, Oshman (see [5]) proved that the metric projection operator P_C is upper semicontinuous.

Definition 3 (see [6]). A Banach space X is called nearly dentable space if, for any $f \in S(X^*)$ and any open set $U_{A_f} \supset A_f$, we have $A_f \neq \emptyset$ and $A_f \cap \overline{\mathrm{co}}(B(X) \setminus U_{A_f}) = \emptyset$.

Definition 4 (see [7]). A Banach space X is said to be k-strictly convex if for any $k+1$ elements $x_1, x_2, \ldots, x_{k+1} \in S(X)$, if $\|x_1 + x_2 + \cdots + x_{k+1}\| = k+1$, then $x_1, x_2, \ldots, x_{k+1}$ are linearly dependent.

Definition 5 (see [8]). A Banach space X is said to be nearly strictly convex space if every convex subset of $S(X)$ is relatively compact.

I. Singer defined in [7] the k-strictly convex spaces and the dual notion (k-smooth spaces) was introduced by Sullivan. In 1988, Skowski and Stachura [8] introduced the notion of nearly strict convexity of Banach spaces by means of the Kuratowski measure of noncompactness. It is well known that if X is a k-strictly convex space, then X is a nearly strictly convex space. It is easy to see that, if X is a nearly dentable space, then X is a reflexive space. In 2011, Shang et al (see [6]) defined nearly dentable space and proved the following two results.

Theorem 6. *A Banach space X is approximatively compact if and only if*

(1) *X is a nearly dentable space,*

(2) *X is a nearly strictly convex space.*

Theorem 7. *Let X be nearly a dentable space. Then for any closed convex set C, the metric projection operator P_C is upper semicontinuous.*

Let T be a linear bounded operator from X into Y. Let $D(T)$, $R(T)$, and $N(T)$ denote the domain, range, and null space of T, respectively. If $N(T) \neq \{0\}$ or $R(T) \neq Y$, the operator equation $Tx = y$ is generally ill-posed. In applications, one usually looks for the best approximate solution (b.a.s.) to the equation $Tx = y$ (see [9]).

A point $x_0 \in D(T)$ is called the best approximate solution to the operator equation $Tx = y$, if

$$\|Tx_0 - y\| = \inf\{\|Tx - y\| : x \in D(T)\},$$
$$\|x_0\| = \min\left\{\|v\| : v \in D(T), \|Tv - y\| = \inf_{x \in D(T)} \|Tx - y\|\right\}, \tag{2}$$

where $y \in Y$(see [9]).

Nashed and Votruba [9] introduced the concept of the (set-valued) metric generalized inverse T as follows.

Definition 8. Let X, Y be Banach spaces and let T be a linear operator from X to Y. The set-valued mapping $T^\partial : Y \to X$ defined by

$$T^\partial(y) = \{x_0 \in D(T) : x_0 \text{ is a best approximation solution to } T(x) = y\} \tag{3}$$

for any $y \in D(T^\partial)$ is called the (set-valued) metric generalized inverse of T, where

$$D(T^\partial) = \{y \in Y : T(x) = y \text{ has a best approximation solution in } X\}. \tag{4}$$

During the last three decades, the linear generalized inverses of linear operators in Banach spaces and their applications have been investigated by many authors. In this paper, authors investigate the metric generalized inverses of linear operators in Banach spaces. Authors prove by the methods of geometry of Banach spaces that, if X is approximatively compact and X is 2-strictly convex space, then metric generalized inverse of a bounded linear operator is upper semicontinuous. Moreover, authors also give criteria for metric generalized inverses of bounded linear operators to be lower semicontinuous. Finally, authors give a sufficient condition for the set-valued mapping T^∂ to be continuous mapping. The topic of this paper is related to the topic of [10–15].

2. Main Results

Theorem 9. *Let X and Y be nearly dentable Banach spaces and X_1 a closed subspace of X. Then for any bounded linear operator T, if $D(T)$ is a closed subspace of X_1 and $R(T)$ is a Chebyshev subspace of Y, then (1) $\Leftrightarrow$ (2) + (3) and (1) $\Rightarrow$ (4), where*

(1) *X_1 is a 2-strictly convex Banach space;*

(2) *for any $y \in Y$, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that the set-valued mapping satisfies the equality $T^\partial(y) = [x_1, x_2]$;*

(3) *the set-valued mapping T^∂ is upper semicontinuous;*

(4) *for any $y \in Y$, the set-valued mapping T^∂ is lower semicontinuous at y if and only if the function $g(y) = \sup\{\|z_1 - z_2\| : z_1, z_2 \in T^\partial(y)\}$ is lower semicontinuous at y.*

In order to prove this theorem, we give a lemma.

Lemma 10. *Let X be a reflexive 2-strictly convex Banach space. Then for any closed convex set C and $x \in X$, there exist $y_1 \in C$ and $y_2 \in C$ such that $P_C(x) = [y_1, y_2]$.*

Proof. (a) We may assume without loss of generality that $x = 0$ and $\inf_{y\in C}\|0 - y\| = 1$. Hence, for any $x_1 \in P_C(0)$, $x_2 \in P_C(0)$, and $x_3 \in P_C(0)$, we have

$$\frac{x_1 + x_2 + x_3}{3} \in C \Longrightarrow \left\|\frac{x_1 + x_2 + x_3}{3}\right\| \geq 1 = \frac{\|x_1\| + \|x_2\| + \|x_3\|}{3}. \tag{5}$$

This implies that $\|x_1 + x_2 + x_3\| = \|x_1\| + \|x_2\| + \|x_3\|$. Since X is 2-strictly convex, we may assume that $x_3 = t_1x_1 + t_2x_2$. By the Hahn-Banach theorem, there exists $f \in S(X^*)$ such that $f(x_1 + x_2 + x_3) = 3$. Noticing that $x_1 \in S(X)$, $x_2 \in S(X)$, and $x_3 \in S(X)$, we have $f(x_1) = f(x_2) = f(x_3) = 1$. Thus

$$1 = f(x_3) = f(t_1x_1 + t_2x_2) = t_1f(x_1) + t_2f(x_2) = t_1 + t_2, \quad x_3 = t_1x_1 + t_2x_2. \tag{6}$$

(b) By $\inf_{y\in C}\|0-y\| = 1$, it is easy to see that $P_C(0) \subset S(X)$. Since X is a 2-strictly convex space, X is a nearly convex space. Since X is a nearly convex space and $P_C(0)$ is closed convex set, $P_C(0)$ is compact. Hence there exist $y_1 \subset P_C(0)$ and $y_1 \subset P_C(0)$ such that

$$d(0) = \sup\{\|x - y\| : x \in P_C(0), y \in P_C(0)\} = \|y_1 - y_2\|. \tag{7}$$

We may assume without loss of generality that $P_C(0)$ is not a singleton. Moreover, for any $y \in P_C(0)$, if $y_1 = ty + (1 - t)y_2$, then $t \neq 0$. Otherwise, we have $y_1 = y_2$. Therefore, by the proof of (a), we have $y = \alpha y_1 + (1 - \alpha)y_2$ for any $y \in P_C(0)$. Suppose that $\alpha < 0$. Then

$$y = \alpha y_1 + (1 - \alpha)y_2 \Longrightarrow y_2 = \frac{1}{1-\alpha}y + \frac{-\alpha}{1-\alpha}y_1. \tag{8}$$

Hence we have

$$\|y_1 - y_2\| = \left\|y_1 - \frac{1}{1-\alpha}y - \frac{-\alpha}{1-\alpha}y_1\right\| = \frac{-\alpha}{1-\alpha}\|y_1 - y\| < \|y_1 - y\|, \tag{9}$$

a contradiction. Hence $\alpha \geq 0$. Similarly, we have $1 - \alpha \geq 0$. Thus $\alpha \in [0, 1]$. This means that, for any $y \in P_C(0)$, we have $y \in [y_1, y_2]$. Hence we have the equation $P_C(x) = [y_1, y_2]$. This completes the proof. □

Proof of Theorem 9. Consider that (2) + (3) ⇒ (1). Suppose that X_1 is not a 2-strictly convex Banach space. Then there exist $x_1 \in S(X_1)$, $x_2 \in S(X_1)$, and $x_3 \in S(X_1)$ such that

$$\|x_1 + x_2 + x_3\| = \|x_1\| + \|x_2\| + \|x_3\| = 3 \tag{10}$$

and x_1, x_2, and x_3 are linearly independent. Therefore, by the Hahn-Banach theorem, there exists $f \in S(X_1^*)$ such that $f(x_1 + x_2 + x_3) = 3$. Noticing that $x_1 \in S(X_1)$, $x_2 \in S(X_1)$, and $x_3 \in S(X_1)$, we have $f(x_1) = f(x_2) = f(x_3) = 1$. Pick $y_0 \in Y$. Define the subspace $\{ty_0 : t \in R\}$ of Y. Since $\{ty_0 : t \in R\}$ is a one-dimensional subspace of Y, we obtain that $\{ty_0 : t \in R\}$ is a strictly convex Banach space. This implies that $\{ty_0 : t \in R\}$ is a Chebyshev subspace of Y. Define the bounded linear operator

$$Tx = f(x)y_0, \quad x \in X_1. \tag{11}$$

Since T is a bounded linear operator and $R(T)$ is a Chebyshev subspace of Y, there exist $z_1 \in D(T)$ and $z_2 \in D(T)$ such that $T^{\partial}(y_0) = [z_1, z_2]$. Moreover, it is easy to see that

$$\|x_1\| = \|x_2\| = \|x_3\| = \min\left\{v \in D(T) : \|Tv - y_0\| = \inf_{x\in D(T)}\|Tx - y_0\|\right\}. \tag{12}$$

This implies that $x_1, x_2, x_3 \in T^{\partial}(y_0) = [z_1, z_2]$. Hence there exist $t_1 \in (0, 1)$, $t_2 \in (0, 1)$, and $t_3 \in (0, 1)$ such that

$$x_1 = t_1z_1 + (1 - t_1)z_2, \quad x_2 = t_2z_1 + (1 - t_2)z_2, \quad x_3 = t_3z_1 + (1 - t_3)z_2. \tag{13}$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t_1 & 1-t_1 \\ t_2 & 1-t_2 \\ t_3 & 1-t_3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{14}$$

Hence there exists $(a, b, c) \neq (0, 0, 0)$ such that

$$a(t_1, 1 - t_1) + b(t_2, 1 - t_2) + c(t_3, 1 - t_3) = (0, 0). \tag{15}$$

Then

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= a(t_1, 1 - t_1)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + b(t_2, 1 - t_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + c(t_3, 1 - t_3)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= [a(t_1, 1 - t_1) + b(t_2, 1 - t_2) + c(t_3, 1 - t_3)]\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= (0, 0)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0. \end{aligned} \tag{16}$$

This implies that x_1, x_2, and x_3 are linearly dependent, a contradiction. Hence we obtain that X_1 is a 2-strictly convex Banach space.

Consider that (1) ⇒ (2) + (3). (a) Since $R(T)$ is a Chebyshev subspace of Y, we obtain that for any $y \in Y$, $P_{R(T)}(y)$ is single-point set. Hence, for any $y \in Y$, there exists $x_0 \in D(T)$ such that $T^{-1}(P_{R(T)}(y)) = x_0 - N(T)$. Moreover, by Lemma 10, there exist $z_1 \in P_{N(T)}(x_0)$ and $z_2 \in P_{N(T)}(x_0)$ such that $P_{N(T)}(x_0) = [z_1, z_2]$. Thus

$$T^{\partial}(y) = x_0 - P_{N(T)}(x_0) = x_0 - [z_1, z_2] = [x_0 - z_1, x_0 - z_2]. \tag{17}$$

(b) By Theorem 7, the metric projector operator $P_{R(T)}$ is upper semicontinuous. Since $R(T)$ is a Chebyshev subspace, we obtain that $P_{R(T)}$ is a single-valued operator. This means that the metric projector operator $P_{R(T)}$ is continuous. Next we will prove that T^{∂} is upper semicontinuous; that is, for any $\{y_n\}_{n=1}^{\infty} \subset Y$, $y_n \to y \in Y$ and any norm open set W with $T^{\partial}(y) \subset W$, there exists a natural number N_0 such that $T^{\partial}(y_n) \subset W$ whenever $n > N_0$. Otherwise, there exists $x_n \in T^{\partial}(y_n)$ such that $\{x_n\}_{n=1}^{\infty} \cap W = \emptyset$. Since the metric projector operator $P_{R(T)}$ is continuous, we obtain that $P_{R(T)}(y_n) \to P_{R(T)}(y)$ as $n \to \infty$. Noticing that $Tx_n = P_{R(T)}(y_n)$, we have $Tx_n \to P_{R(T)}(y)$ as $n \to \infty$. Since T is a bounded linear operator, we obtain that $N(T)$ is a closed subspace of $D(T)$. Put

$$\overline{T} : \frac{D(T)}{N(T)} \longrightarrow R(T), \qquad \overline{T}[x] = Tx, \tag{18}$$

where $[x] \in D(T)/N(T)$ and $x \in D(T)$. It is easy to see that $R(\overline{T}) = R(T)$. Moreover, $\overline{R(T)} = R(T)$. In fact, suppose that $\overline{R(T)} \neq R(T)$. Then there exists $y' \in \overline{R(T)}$ such that $y' \notin R(T)$. It is easy to see that $\{y \in R(T) : \|y' - y\| = \operatorname{dist}(y', R(T))\} = \emptyset$. This implies that $R(T)$ is not a Chebyshev subspace of Y, a contradiction. By $\overline{R(T)} = R(T)$, we obtain that $R(T)$ is a Banach space. Moreover, it is easy to see that $\overline{T}$ is a bounded linear operator and $N(\overline{T}) = \{0\}$. This implies that the bounded linear operator T is both injective and surjective. By the inverse operator theorem, $\overline{T}^{-1}$ is a bounded linear operator. Hence we have

$$[x_n] = \overline{T}^{-1}(P_{R(T)}(y_n)) \longrightarrow \overline{T}^{-1}(P_{R(T)}(y)) = [x]. \tag{19}$$

This means that $\|[x_n]\| \to \|[x]\|$ as $n \to \infty$. Noticing that

$$\begin{gathered} x_n \in T^{\partial}(y_n), \\ \|[x_n]\| = \inf_{z \in N(T)} \|x_n + z\|, \\ \overline{T}[x_n] = T(x_n + z) = P_{R(T)}(y_n), \\ x \in T^{\partial}(y), \\ \|[x]\| = \inf_{z \in N(T)} \|x + z\|, \\ \overline{T}[x] = T(x + z) = P_{R(T)}(y), \end{gathered} \tag{20}$$

it is easy to see that $\|[x_n]\| = \|x_n\|$ and $\|[x]\| = \|x\|$. Since $\|[x_n]\| \to \|[x]\|$, $\|[x_n]\| = \|x_n\|$, and $\|[x]\| = \|x\|$, we have $\|x_n\| \to \|x\|$ as $n \to \infty$. We will derive a contradiction for each of the following two cases.

Case 1 ($x = 0$). By (19), we have $[x_n] \to [x] = 0$ as $n \to \infty$. This implies that $\|[x_n]\| \to 0$ as $n \to \infty$. Thus $\|x_n\| \to 0$ as $n \to \infty$. By $x = 0$, we have $0 \in T^{\partial}(y) \subset W$. Moreover, by $\|x_n\| \to 0$, we have $x_n \to 0$ as $n \to \infty$, which contradicts the equation $\{x_n\}_{n=1}^{\infty} \cap W = \emptyset$.

Case 2 ($x \neq 0$). By $y_n \to y$, we obtain that $\{y_n\}_{n=1}^{\infty}$ is a bounded sequence. Since the distance function is continuous, we have

$$\|y_n - P_{R(T)}(y_n)\| = \operatorname{dist}(y_n, R(T)) \longrightarrow \operatorname{dist}(y, R(T)) \quad \text{as } n \longrightarrow \infty. \tag{21}$$

This implies that $\{y_n - P_{R(T)}(y_n)\}_{n=1}^{\infty}$ is a bounded sequence. Hence $\{P_{R(T)}(y_n)\}_{n=1}^{\infty}$ is a bounded sequence. Since $\overline{T}^{-1}$ is a bounded linear operator, we obtain that $\{[x_n]\}_{n=1}^{\infty}$ is a bounded sequence. By $\|[x_n]\| = \|x_n\|$, we obtain that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Since X is a nearly dentable Banach space, X is reflexive. Hence, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{w} x'$. Since $D(T)$ is a closed convex set, $D(T)$ is a weakly closed convex set. Hence we obtain that $x' \in D(T)$. By $x_{n_k} \xrightarrow{w} x'$ and $x' \in D(T)$, we have $Tx_{n_k} \xrightarrow{w} Tx'$. Noticing that $Tx_n \to Tx$, we have $Tx' = Tx$. Since $x \in T^{\partial}(y)$, we have $\|x'\| \geq \|x\|$. By the Hahn-Banach theorem, there exists $f \in S(X^*)$ such that $f(x') = \|x'\|$. Hence

$$\begin{aligned} \|x'\| &= f(x') = \lim_{k \to \infty} f(x_{n_k}) \leq \lim_{k \to \infty} \|x_{n_k}\| \\ &= \lim_{k \to \infty} \|[x_{n_k}]\| = \|[x]\| = \|x\|. \end{aligned} \tag{22}$$

This implies that $\|x'\| = \|x\|$. By $\{x_{n_k}\}_{k=1}^{\infty} \subset D(T)$ and $\|x'\| = \|x\|$, we have $x' \in T^{\partial}(y)$. Define

$$z_{n_k} = \frac{x_{n_k}}{\|x_{n_k}\|}, \qquad z_0 = \frac{x'}{\|x'\|}. \tag{23}$$

By $\|x_{n_k}\| \to \|x'\|$ and $x_{n_k} \xrightarrow{w} x'$, we have $z_{n_k} \xrightarrow{w} z_0$. Since $\{x_{n_k}\}_{k=1}^{\infty} \subset D(T)$, $x \in D(T)$, $z_{n_k} = x_{n_k}/\|x_{n_k}\|$ and $z_0 = x'/\|x'\|$, we have $\{z_{n_k}\}_{k=1}^{\infty} \subset D(T)$, and $z_0 \in D(T)$.

Next we will prove that there exists a subsequence $\{z_{n_l}\}_{l=1}^{\infty}$ of $\{z_{n_k}\}_{k=1}^{\infty}$ such that $z_{n_l} \to z_0$ as $l \to \infty$. Pick $f \in A(z_0)$. Then for $x \in A_f \setminus (A_f \cap D(T))$, we have $x \notin D(T)$. Since $D(T)$ is a closed set, there exists $\varepsilon_x > 0$ such that $\operatorname{dist}(x, D(T)) > 2\varepsilon_x$ for any $x \in A_f \setminus (A_f \cap D(T))$. Noticing that $\{z_{n_k}\}_{k=1}^{\infty} \subset D(T)$, we have $\operatorname{dist}(x, \{z_{n_k}\}_{k=1}^{\infty}) > 2\varepsilon_x$. Then $\operatorname{dist}(A_f \cap D(T), \{z_{n_k}\}_{k=1}^{\infty}) = 0$. In fact, suppose that $\operatorname{dist}(A_f \cap D(T), \{z_{n_k}\}_{k=1}^{\infty}) > 0$. Then for any $x \in A_f \cap D(T)$, there exists $\varepsilon_x > 0$ such that $\operatorname{dist}(x, \{z_{n_k}\}_{k=1}^{\infty}) > 2\varepsilon_x$. Hence, for any $x \in A_f$, there exists $\varepsilon_x > 0$ such that $\operatorname{dist}(x, \{z_{n_k}\}_{k=1}^{\infty}) > 2\varepsilon_x$. We define the open set

$$U_{A_f} = \bigcup_{x \in A_f} \{y \in X : \|x - y\| < \varepsilon_x\}. \tag{24}$$

It is easy to see that $A_f \subset U_{A_f}$ and $U_{A_f} \cap \{z_{n_{k_i}}\}_{i=1}^{\infty} = \emptyset$. Since X is a nearly dentable space, we obtain that X is a reflexive space. Hence $\overline{\mathrm{co}}(B(X) \setminus U_{A_f}) = \overline{\mathrm{co}}^{w}(B(X) \setminus U_{A_f})$ is a weakly compact set and $A_f \cap \overline{\text{co}}(B(X) \setminus U_{A_f}) = \emptyset$. By the separation theorem of locally convex space, there exists $g \in X^* = (X, w)^*$ and $r > 0$ such that

$$\inf \{g(z) : z \in A_f\} - r > \sup \left\{ g(z) : z \in \overline{\text{co}}^w \left(\frac{B(X)}{U_{A_f}} \right) = \overline{\text{co}} \left(\frac{B(X)}{U_{A_f}} \right) \right\}. \quad (25)$$

Noticing that $\{z_{n_{k_i}}\}_{i=1}^{\infty} \subset \overline{\text{co}}(B(X)/U_{A_f})$ and $z_0 \in A_f$, we have $g(z_0) - r > \sup\{g(z) : z \in \{z_{n_{k_i}}\}_{i=1}^{\infty}\}$. This means that $z_{n_k} \xrightarrow{w} z_0$ is impossible, a contradiction. Hence $\text{dist}(A_f \cap D(T), \{z_{n_k}\}_{k=1}^{\infty}) = 0$. Then there exists a subsequence $\{z_{n_l}\}_{l=1}^{\infty}$ of $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\text{dist}(z_{n_l}, A_f \cap D(T)) \to 0$ as $l \to \infty$. Then there exists a sequence $\{h_{n_l}\}_{l=1}^{\infty} \subset A_f \cap D(T)$ such that $\|z_{n_l} - h_{n_l}\| \to 0$ as $l \to \infty$. Since X_1 is a 2-strictly convex space, we obtain that X_1 is a nearly strictly convex space. This means that $A_f \cap D(T)$ is compact. Hence the sequence $\{h_{n_l}\}_{l=1}^{\infty}$ has a Cauchy subsequence. By $\|z_{n_l} - h_{n_l}\| \to 0$, we have that $\{z_{n_l}\}_{l=1}^{\infty}$ has a Cauchy subsequence. Noticing that $z_{n_k} \xrightarrow{w} z_0$, we obtain that there exists a subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ of $\{z_{n_l}\}_{l=1}^{\infty}$ such that $z_{n_j} \to z_0$ as $j \to \infty$. By $\|x_{n_k}\| \to \|x'\|$ and $z_{n_j} \to z_0$, we have

$$\begin{aligned}
\left\|x_{n_j} - x'\right\| &= \left\| \left\|x_{n_j}\right\| \cdot z_{n_j} - \left\|x'\right\| \cdot z_0 \right\| \\
&= \left\| \left\|x_{n_j}\right\| \cdot z_{n_j} - \left\|x'\right\| \cdot z_{n_j} \right\| \\
&\quad + \left\| \left\|x'\right\| \cdot z_{n_j} - \left\|x'\right\| \cdot z_0 \right\| \\
&= \left| \left\|x_{n_j}\right\| - \left\|x'\right\| \right| \cdot \left\|z_{n_j}\right\| + \left\|x'\right\| \cdot \left\|z_{n_j} - z_0\right\| \\
&\longrightarrow 0 \text{ as } j \longrightarrow \infty.
\end{aligned} \quad (26)$$

Moreover, we have $x' \in T^{\partial}(y) \subset W$, which contradicts the equation $\{x_n\}_{n=1}^{\infty} \cap W = \emptyset$. Hence, for any bounded linear operator T and $y \in Y$, if $D(T)$ is a closed subspace of X_1 and $R(T)$ is Chebyshev subspace of Y, then there exist $x_1, x_2 \in D(T)$ such that the set-valued mapping $T^{\partial}(y) = [x_1, x_2]$ is upper semicontinuous.

Consider that (1)⇒(4). Let the function $g(y) = \sup\{\|z_1 - z_2\| : x_1, x_2 \in T^{\partial}(y)\}$ be lower semicontinuous at y. Since X_1 is a 2-strictly convex Banach space, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that set-valued mapping $T^{\partial}(y) = [x_1, x_2]$. Hence $g(y) = \|x_1 - x_2\|$. Let $y_n \to y$ as $n \to \infty$. Then $\text{dist}(x_1, T^{\partial}(y_n)) = 0$ as $n \to \infty$. Otherwise, there exist $\varepsilon_0 > 0$ and a subsequence $\{T^{\partial}(y_{n_k})\}_{k=1}^{\infty}$ of $\{T^{\partial}(y_n)\}_{n=1}^{\infty}$ such that $\text{dist}(x_1, T^{\partial}(y_{n_k})) \geq 8\varepsilon_0$ for any $k \in N$. We define the open set

$$W(y) = \bigcup_{x \in [x_1, x_2]} \text{int}\, B(x, \varepsilon_0). \quad (27)$$

Since X_1 is a 2-strictly convex Banach space, by the implication (1)⇒(2)+(3), the set-valued mapping T^{∂} is upper semicontinuous. Hence there exists a natural number k_0 such that $T^{\partial}(y_{n_k}) \subset W(y)$ whenever $k > k_0$. Let $T^{\partial}(y_{n_k}) = [x_{1,n_k}, x_{2,n_k}]$. Then there exist $y_{1,n_k} \in T^{\partial}(y) = [x_1, x_2]$ and $y_{2,n_k} \in T^{\partial}(y) = [x_1, x_2]$ such that

$$\left\|x_{1,n_k} - y_{1,n_k}\right\| < \varepsilon_0, \qquad \left\|x_{2,n_k} - y_{2,n_k}\right\| < \varepsilon_0. \quad (28)$$

Then

$$\begin{aligned}
\left\|x_1 - y_{1,n_k}\right\| &\geq \left\|x_{1,n_k} - x_1\right\| - \left\|x_{1,n_k} - y_{1,n_k}\right\| \\
&\geq 8\varepsilon_0 - \varepsilon_0 = 7\varepsilon_0, \\
\left\|x_1 - y_{2,n_k}\right\| &\geq \left\|x_{2,n_k} - x_1\right\| - \left\|x_{2,n_k} - y_{2,n_k}\right\| \\
&\geq 8\varepsilon_0 - \varepsilon_0 = 7\varepsilon_0.
\end{aligned} \quad (29)$$

Noticing that $y_{1,n_k} \in T^{\partial}(y) = [x_1, x_2]$ and $y_{2,n_k} \in T^{\partial}(y) = [x_1, x_2]$, we have $\|y_{1,n_k} - x_2\| = \|x_1 - x_2\| - \|x_1 - y_{1,n_k}\|$. Moreover, we may assume without loss of generality that $\|x_1 - y_{1,n_k}\| \leq \|x_1 - y_{2,n_k}\|$. Therefore, by (29), we have

$$\begin{aligned}
\left\|y_{1,n_k} - y_{2,n_k}\right\| &\leq \left\|y_{1,n_k} - x_2\right\| = \left\|x_1 - x_2\right\| - \left\|x_1 - y_{1,n_k}\right\| \\
&\leq \left\|x_1 - x_2\right\| - 7\varepsilon_0.
\end{aligned} \quad (30)$$

Then

$$\begin{aligned}
\left\|x_{1,n_k} - x_{2,n_k}\right\| &\leq \left\|x_{1,n_k} - y_{1,n_k}\right\| + \left\|y_{1,n_k} - x_{2,n_k}\right\| \\
&\leq \left\|x_{1,n_k} - y_{1,n_k}\right\| + \left\|y_{1,n_k} - y_{2,n_k}\right\| \\
&\quad + \left\|y_{2,n_k} - x_{2,n_k}\right\| \\
&\leq \varepsilon_0 + \left\|x_1 - x_2\right\| - 7\varepsilon_0 + \varepsilon_0 \\
&\leq \left\|x_1 - x_2\right\| - 5\varepsilon_0.
\end{aligned} \quad (31)$$

This implies that

$$\begin{aligned}
\liminf_{n \to \infty} g\left(y_{n_k}\right) &= \liminf_{n \to \infty} \left\|x_{1,n_k} - x_{2,n_k}\right\| \\
&\leq \left\|x_1 - x_2\right\| - 5\varepsilon_0 \\
&< \left\|x_1 - x_2\right\| = g\left(y\right),
\end{aligned} \quad (32)$$

a contradiction. Then $\text{dist}(x_1, T^{\partial}(y_n)) = 0$ as $n \to \infty$. Hence there exists $x_{n,1} \in T^{\partial}(y_n)$ such that $\|x_{n,1} - x_1\| \to 0$ as $n \to \infty$. Similarly, there exists $x_{n,2} \in T^{\partial}(y_n)$ such that $\|x_{n,2} - x_2\| \to 0$ as $n \to \infty$. For any $x \in T^{\partial}(y) = [x_1, x_2]$, there exists $t \in [0, 1]$ such that $x = tx_1 + (1-t)x_2$. Moreover, it is easy to see that $tx_{n,1} + (1-t)x_{n,2} \in T^{\partial}(y_n) = [x_{n,1}, x_{n,2}]$. Hence, for any $x \in T^{\partial}(y) = [x_1, x_2]$, we have

$$\begin{aligned}
&\left\|tx_{n,1} + (1-t)\,x_{n,2} - x\right\| \\
&\quad = \left\|tx_{n,1} + (1-t)\,x_{n,2} - \left(tx_1 + (1-t)\,x_2\right)\right\| \\
&\quad \leq \left\|tx_{n,1} - tx_1\right\| + \left\|(1-t)\,x_{n,2} - (1-t)\,x_2\right\| \\
&\quad = t\left\|x_{n,1} - x_1\right\| + (1-t)\left\|x_{n,2} - x_2\right\| \\
&\quad \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \quad (33)$$

This means that, if the function $g(y) = \sup\{\|z_1 - z_2\| : x_1, x_2 \in T^\partial(y)\}$ is lower semicontinuous at y, then the set-valued mapping T^∂ is lower semicontinuous at y.

Let the set-valued mapping T^∂ be lower semicontinuous at y. Since X_1 is a 2-strictly convex Banach space, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that the set-valued mapping satisfies the equality $T^\partial(y) = [x_1, x_2]$. Hence we have $g(y) = \|x_1 - x_2\|$. Let $y_n \to y$ as $n \to \infty$ and $T^\partial(y_n) = [z_{n,1}, z_{n,2}]$. Then $g(y_n) = \|z_{n,1} - z_{n,2}\|$. Since the set-valued mapping T^∂ is lower semicontinuous at y, there exist $x_{n,1} \in T^\partial(y_n)$ and $x_{n,2} \in T^\partial(y_n)$ such that $\|x_{n,1} - x_1\| \to 0$ and $\|x_{n,2} - x_2\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned} g(y_n) = & \left\|z_{n,1} - z_{n,2}\right\| \geq \left\|x_{n,1} - x_{n,2}\right\| \\ \geq & \left\|x_1 - x_{n,2}\right\| - \left\|x_1 - x_{n,1}\right\| \\ \geq & \left\|x_1 - x_2\right\| - \left\|x_2 - x_{n,2}\right\| - \left\|x_1 - x_{n,1}\right\|, \end{aligned} \tag{34}$$

we have

$$\begin{aligned} & \liminf_{n\to\infty} g(y_n) \\ & \quad \geq \liminf_{n\to\infty} \left(\left\|x_1 - x_2\right\| - \left\|x_2 - x_{n,2}\right\| - \left\|x_1 - x_{n,1}\right\|\right) \\ & \quad = \lim_{n\to\infty} \left\|x_1 - x_2\right\| - \lim_{n\to\infty} \left\|x_2 - x_{n,2}\right\| - \lim_{n\to\infty} \left\|x_1 - x_{n,1}\right\| \\ & \quad = \left\|x_1 - x_2\right\| = g(y). \end{aligned} \tag{35}$$

Hence the function $g(y) = \sup\{\|z_1 - z_2\| : z_1, z_2 \in T^\partial(y)\}$ is lower semicontinuous at y. This completes the proof. □

Theorem 11. *Let X and Y be nearly dentable Banach spaces and X_1 a closed subspace of X. Then for any bounded linear operator T, if $D(T)$ is a closed subspace of X_1, $N(T)$ is a hyperplane of $D(T)$, and $R(T)$ is a Chebyshev subspace of Y, then (1) ⇔ (2) + (3), where*

(1) *X_1 is a 2-strictly convex Banach space;*
(2) *for any $y \in Y$, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that the set-valued mapping satisfies the equality $T^\partial(y) = [x_1, x_2]$;*
(3) *the set-valued mapping T^∂ is continuous.*

In order to prove the theorem, we first give a lemma.

Lemma 12. *Let X be a reflexive 2-strictly convex space and $H = \{x \in X : f(x) = 0\}$ be a hyperplane of X. Then the function $g(y) = \sup\{\|z_1 - z_2\| : z_1, z_2 \in P_H(y)\}$ is lower semicontinuous.*

Proof. (a) We will prove that, if $y_1, y_2 \in \{x \in X : f(x) = \lambda\}$, then $g(y_1) = g(y_2)$. It is easy to see that $y_1 - y_2 \in H$ and $\mathrm{dist}(y_1, H) = \mathrm{dist}(y_2, H)$. Hence, for any $z \in P_H(y_2)$, we have

$$\begin{aligned} \mathrm{dist}(y_1, H) = \mathrm{dist}(y_2, H) = \left\|y_2 - z\right\| \\ = \left\|y_1 - (y_1 - y_2 + z)\right\|. \end{aligned} \tag{36}$$

This implies that $y_1 - y_2 + z \in P_H(y_1)$. Hence we have $y_1 - y_2 + P_H(y_2) \subset P_H(y_1)$. Similarly, we have $y_2 - y_1 + P_H(y_1) \subset P_H(y_2)$. By $y_1 - y_2 + P_H(y_2) \subset P_H(y_1)$ and $y_2 - y_1 + P_H(y_1) \subset P_H(y_2)$, we have $y_1 - y_2 + P_H(y_2) = P_H(y_1)$. This implies that $g(y_1) = g(y_2)$.

(b) Let $f(x) = \lambda_1$, $f(y) = \lambda_2$, and $\lambda_1 > \lambda_2 > 0$. Next we will prove that $g(x) \geq g(y)$. Pick $x_H \in P_H(x)$. Then

$$\begin{aligned} f\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right) &= f\left(\frac{\lambda_2}{\lambda_1}x\right) + f\left(\frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right) \\ &= f\left(\frac{\lambda_2}{\lambda_1}x\right) = \lambda_2. \end{aligned} \tag{37}$$

Moreover, we have

$$P_H(x) \supset P_H\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right). \tag{38}$$

In fact, noticing that $f(x) = \mathrm{dist}(x, H)$ and $f(y) = \mathrm{dist}(y, H)$, we have

$$\begin{aligned} & \left\|\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right) - z\right\| \\ & \quad \geq \|x - z\| - \left\|x - \left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right)\right\| \\ & \quad > \|x - x_H\| - \frac{\lambda_1 - \lambda_2}{\lambda_1}\|x - x_H\| \\ & \quad = \frac{\lambda_2}{\lambda_1}\|x - x_H\| = \frac{\lambda_2}{\lambda_1}f(x) = f(y), \end{aligned} \tag{39}$$

for any $z \notin P_H(x)$. By

$$\begin{aligned} & \left(\left\|\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right) - z\right\|\right) \\ & \quad > f(y) = \mathrm{dist}(y, H) \\ & \quad = \mathrm{dist}\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H, H\right), \end{aligned} \tag{40}$$

we have

$$z \notin P_H\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right). \tag{41}$$

This implies that

$$P_H(x) \supset P_H\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right). \tag{42}$$

By (a), there exists $z \in X$ such that

$$P_H\left(\frac{\lambda_2}{\lambda_1}x + \frac{\lambda_1 - \lambda_2}{\lambda_1}x_H\right) = P_H(y) + z. \tag{43}$$

By (42) and (43), we have $P_H(x) \supset P_H(y) + z$. Hence $g(x) \geq g(y)$.

(c) We will prove that, if $f(y) = \lambda \geq 0$ and $P_H(y) = [z_1, z_2]$, then $P_{H_t}(y) \supset [ty + (1-t)z_1, ty + (1-t)z_2]$, where $H_t = \{x \in X : f(x) = t\lambda\}$ and $t \in (0,1)$. Let $x \in H_t$ and $x \notin P_{H_t}(y)$. Then

$$\begin{aligned} \operatorname{dist}(y, H_t) &= \operatorname{dist}(y - x, H_t - x) = \operatorname{dist}(y - x, H) \\ &= f(y) - f(x) < \|y - x\|. \end{aligned} \tag{44}$$

Let $(1-t)z = x - ty$. Therefore, by (44), we have

$$\begin{aligned} f(z) &= f\left(\frac{1}{1-t}x\right) - f\left(\frac{t}{1-t}y\right) = \frac{1}{1-t}\cdot t\lambda - \frac{t}{1-t}\lambda = 0, \\ \|z - y\| &= \left\|\frac{1}{1-t}x - \frac{t}{1-t}y - y\right\| \\ &= \frac{1}{1-t}\|x - y\| > \frac{f(y) - f(x)}{1-t} = \lambda. \end{aligned} \tag{45}$$

This means that $z \in H$ and $\|z - y\| > \lambda = f(y) = \operatorname{dist}(y, H)$. Hence we have $z \notin P_H(y) = [z_1, z_2]$. Then $x \notin [ty + (1-t)z_1, ty + (1-t)z_2]$. Otherwise, there exists $h \in [0,1]$ such that $x = h[ty + (1-t)z_1] + (1-h)[ty + (1-t)z_2]$. By $x = ty + (1-t)z$, we have $z = hz_1 + (1-h)z_2$, a contradiction. This implies that $P_{H_t}(y) \supset [ty + (1-t)z_1, ty + (1-t)z_2]$.

(d) Suppose that there exists $y \in Y$ such that function $g(y)$ is not lower semicontinuous at y. Then there exists $\varepsilon > 0$ and $\{y_n\}_{n=1}^{\infty}$ such that $g(y) > g(y_n) + \varepsilon$ and $y_n \to y$ as $n \to \infty$. Moreover, we may assume without loss of generality that $f(y) > 0$. Otherwise, let $f = -f$. Pick $y_H \in P_H(y)$. By (a) and (b), there exists $\{t_n\}_{n=1}^{\infty} \subset [0,1]$ such that $g(t_n y + (1-t_n)y_H) = g(y_n)$ and $t_n \to 1$ as $n \to \infty$. Put

$$\begin{aligned} z_n &= t_n y + (1-t_n)(y - y_H), \\ H_{z_n} &= \{x \in X : f(x) = (1-t_n) f(y - y_H)\}. \end{aligned} \tag{46}$$

Hence, for any $z \in P_{H_{z_n}}(y)$, we have

$$\begin{aligned} \operatorname{dist}(z_n, H) &= \operatorname{dist}(z_n + (y - z_n), H + (y - z_n)) \\ &= \operatorname{dist}(y, H_{z_n}) \\ &= \|y - z\| = \|z_n - (z_n - y + z)\|, \\ f(z_n - y + z)& \\ &= f(z_n - y) + f(z) \\ &= (1-t_n) f(y - y_H) - (1-t_n) f(y - y_H) = 0. \end{aligned} \tag{47}$$

This implies that $z_n - y + z \in P_H(z_n)$. Hence we have

$$\begin{aligned} g(z_n) &= \sup\{\|z_1 - z_2\| : z_1, z_2 \in P_H(z_n)\} \\ &\geq \sup\{\|x_1 - x_2\| : x_1, x_2 \in P_{H_{z_n}}(y)\}. \end{aligned} \tag{48}$$

Let $P_H(y) = [z_1, z_2]$. Therefore, by (c) and $H_{z_n} = \{x \in X : f(x) = (1-t_n)f(y - y_H)\}$, we have $P_{H_{z_n}}(y) \supset [t_n y + (1-t_n)z_1, t_n y + (1-t_n)z_2]$. Hence

$$\begin{aligned} \sup&\{\|x_1 - x_2\| : x_1, x_2 \in P_{H_{z_n}}(y)\} \\ &\geq (1-t_n)\|z_1 - z_2\| = (1-t_n) g(y) \longrightarrow g(y). \end{aligned} \tag{49}$$

By $g(z_n) \geq \sup\{\|x_1 - x_2\| : x_1, x_2 \in P_{H_{z_n}}(y)\}$ and $g(z_n) = g(y_n)$, we have $\liminf_{n\to\infty} g(y_n) = \liminf_{n\to\infty} g(z_n) \geq g(y)$, a contradiction. This completes the proof. □

Proof of Theorem 11. By Theorem 9, we just need to prove that, for any $y \in Y$, the function $g(y) = \sup\{\|z_1 - z_2\| : z_1, z_2 \in T^{\partial}(y)\}$ is lower semicontinuous on y. Let $y_n \to y$ as $n \to \infty$. Since Y is a nearly dentable Banach space and $R(T)$ is a Chebyshev subspace of Y, by Theorem 7, we have $P_{R(T)}(y_n) \to P_{R(T)}(y)$ as $n \to \infty$. Put

$$\overline{T} : \frac{D(T)}{N(T)} \longrightarrow R(T), \qquad \overline{T}[x] = Tx, \tag{50}$$

where $[x] \in D(T)/N(T)$ and $x \in D(T)$. By the proof of Theorem 9, we obtain that $\overline{T}^{-1}$ is a bounded linear operator. Then

$$\begin{aligned} [x_n] = \overline{T}^{-1}(P_{R(T)}(y_n)) &\longrightarrow [x] = \overline{T}^{-1}(P_{R(T)}(y)) \\ &\text{as } n \longrightarrow \infty. \end{aligned} \tag{51}$$

Hence there exist $x \in [x]$ and $x_n \in [x_n]$ such that $x_n \to x$ as $n \to \infty$. By Lemma 12, we obtain that $\liminf_{n\to\infty} h(x_n) \geq h(x)$, where $h(x) = \sup\{\|x_1 - x_2\| : x_1, x_2 \in P_{N(T)}(x)\}$. Noticing that $h(x) = g(y)$ and $h(x_n) = g(y_n)$, we have $\liminf_{n\to\infty} g(y_n) \geq g(y)$. This completes the proof. □

By Theorems 6 and 9, we have the following.

Theorem 13. *Let X and Y be approximatively compact Banach spaces and X_1 a closed subspace of X. Then for any bounded linear operator T, if $D(T)$ is a closed subspace of X_1 and $R(T)$ is a Chebyshev subspace of Y, then (1) $\Leftrightarrow$ (2)+(3) and (1) $\Rightarrow$ (4), where*

(1) *X_1 is a 2-strictly convex Banach space;*

(2) *for any $y \in Y$, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that the set-valued mapping satisfies the equality $T^{\partial}(y) = [x_1, x_2]$;*

(3) *the set-valued mapping T^{∂} is upper semicontinuous;*

(4) *for any $y \in Y$, the set-valued mapping T^{∂} is lower semicontinuous at y if and only if the function $g(y) = \sup\{\|z_1 - z_2\| : z_1, z_2 \in T^{\partial}(y)\}$ is lower semicontinuous at y.*

By Theorems 6 and 11, we have the following.

Theorem 14. *Let X and Y be approximatively compact Banach spaces and X_1 a closed subspace of X. Then for any bounded linear operator T, if $D(T)$ is a closed subspace of X_1, $N(T)$ is a hyperplane of $D(T)$, and $R(T)$ is a Chebyshev subspace of Y, then* (1) $\Leftrightarrow$ (2) + (3)*, where*

(1) *X_1 is a 2-strictly convex Banach space;*

(2) *for any $y \in Y$, there exist $x_1 \in D(T)$ and $x_2 \in D(T)$ such that the set-valued mapping satisfies the equality $T^{\partial}(y) = [x_1, x_2]$;*

(3) *the set-valued mapping T^{∂} is continuous.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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