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Research Article

The Regularity of Functions on Dual Split Quaternions in Clifford Analysis

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This paper shows some properties of dual split quaternion numbers and expressions of power series in dual split quaternions and provides differential operators in dual split quaternions and a dual split regular function on $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$ that has a dual split Cauchy-Riemann system in dual split quaternions.

1. Introduction

Hamilton introduced quaternions, extending complex numbers to higher spatial dimensions in differential geometry (see [1]). A set of quaternions can be represented as

$$\mathscr{H} = \{z = x_0 + x_1 i + x_2 j + x_3 k : x_m \in \mathbb{R},\ m = 0, 1, 2, 3\}, \tag{1}$$

where $i^2 = j^2 = k^2 = -1$, $ijk = -1$, and $\mathbb{R}$ denotes the set of real numbers. Cockle [2] introduced a set of split quaternions as

$$\mathscr{S} = \{z = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 : x_m \in \mathbb{R},\ m = 0, 1, 2, 3\}, \tag{2}$$

where $e_1^2 = -1$, $e_2^2 = e_3^2 = 1$, and $e_1 e_2 e_3 = 1$. A set of split quaternions is noncommutative and contains zero divisors, nilpotent elements, and nontrivial idempotents (see [3, 4]). Previous studies have examined the geometric and physical applications of split quaternions, which are required in solving split quaternionic equations (see [5, 6]). Inoguchi [7] reformulated the Gauss-Codazzi equations in forms consistent with the theory of integrable systems in the Minkowski 3-space for split quaternion numbers.

A dual quaternion can be represented in a form reflecting an ordinary quaternion and a dual symbol. Because dual-quaternion algebra is constructed from real eight-dimensional vector spaces and an ordered pair of quaternions, dual quaternions are used in computer vision applications. Kenwright [8] provided the characteristics of dual quaternions, and Pennestrì and Stefanelli [9] examined some properties by using dual quaternions. Son [10, 11] offered an extension problem for solutions of partial differential equations and generalized solutions for the Riesz system. By using properties of Hamilton operators, Kula and Yayli [4] defined dual split quaternions and gave some properties of the screw motion in the Minkowski 3-space, showing that $\mathscr{H}$ has a rotation with unit split quaternions in $\mathscr{H}$ and a scalar product that allows it to be identified with the semi-Euclidean space for split quaternion numbers.

It was shown (see [12, 13]) that any complex-valued harmonic function f_1 in a pseudoconvex domain D of $\mathbb{C}^2 \times \mathbb{C}^2$, $\mathbb{C}$ being the set of complex numbers, has a conjugate function f_2 in D such that the quaternion-valued function $f_1 + f_2 j$ is hyperholomorphic in D and gave a regeneration theorem in a quaternion analysis in view of complex and Clifford analysis. In addition, we [14, 15] provided a new expression of the quaternionic basis and a regular function on reduced quaternions by associating hypercomplex numbers e_1 and e_2. We [16] investigated the existence of hyperconjugate harmonic functions of an octonion number system, and we [17, 18] obtained some regular functions with values in dual quaternions and researched an extension problem for properties of regular functions with values in dual quaternions and some applications for such problems.

This paper provides a regular function and some properties of differential operators in dual split quaternions. In addition, we research some equivalent conditions for Cauchy-Riemann systems and expressions of power series in dual split quaternions from the definition of dual split regular on an open set $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$.

2. Preliminaries

A dual number A has the form $a + \varepsilon b$, where a and b are real numbers and ε is a dual symbol subject to the rules

$$\varepsilon \neq 0, \qquad 0\varepsilon = \varepsilon 0 = 0, \qquad 1\varepsilon = \varepsilon 1 = \varepsilon, \qquad \varepsilon^2 = 0, \quad (3)$$

and a split quaternion $q \in \mathcal{S}$ is an expression of the form

$$q = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, \quad (4)$$

where $x_m \in \mathbb{R}$ $(m = 0, 1, 2, 3)$ and e_r $(r = 1, 2, 3)$ are split quaternionic units satisfying noncommutative multiplication rules (for split quaternions, see [1]):

$$\begin{aligned} &e_1^2 = -1, \qquad e_2^2 = e_3^2 = 1, \\ &e_1 e_2 = -e_2 e_1 = e_3, \qquad e_2 e_3 = -e_3 e_2 = -e_1, \\ &e_3 e_1 = -e_1 e_3 = e_2. \end{aligned} \quad (5)$$

Similarly, a dual split quaternion z can be written as

$$\mathcal{D}(\mathcal{S}) = \{z \mid z = p_0 + \varepsilon p_1,\ p_r \in \mathcal{S},\ r = 0, 1\}, \quad (6)$$

which has elements of the following form:

$$\begin{aligned} z &= \{(x_0 + x_1 e_1) + (x_2 + x_3 e_1) e_2\} \\ &\quad + \varepsilon \{(y_0 + y_1 e_1) + (y_2 + y_3 e_1) e_2\} \\ &= (z_0 + z_1 e_2) + \varepsilon (z_2 + z_3 e_2) \\ &= p_0 + \varepsilon p_1, \end{aligned} \quad (7)$$

where $p_0 = z_0 + z_1 e_2$ and $p_1 = z_2 + z_3 e_2$ are split quaternion components, $z_0 = x_0 + x_1 e_1$, $z_1 = x_2 + x_3 e_1$, $z_2 = y_0 + y_1 e_1$, and $z_3 = y_2 + y_3 e_1$ are usual complex numbers, and $x_m, y_m \in \mathbb{R}$ $(m = 0, 1, 2, 3)$. The multiplication of split quaternionic units with a dual symbol is commutative $\varepsilon e_r = e_r \varepsilon$ $(r = 1, 2, 3)$. However, by properties of split quaternionic unit,

$$\begin{aligned} &z_k e_r = e_r z_k \quad (k = 0, 1, 2, 3,\ r = 0, 1), \\ &z_k e_r = e_r \overline{z_k} \quad (k = 0, 1, 2, 3,\ r = 2, 3), \\ &e_r p_k \neq p_k e_r, \quad e_r p_k = p_{(kr)} e_r \quad (r = 1, 2, 3,\ k = 0, 1), \end{aligned} \quad (8)$$

where

$$\begin{aligned} p_{(01)} &= z_0 - z_1 e_2 = x_0 + x_1 e_1 - x_2 e_2 - x_3 e_3, \\ p_{(02)} &= \overline{z_0} + \overline{z_1} e_2 = x_0 - x_1 e_1 + x_2 e_2 - x_3 e_3, \\ p_{(03)} &= \overline{z_0} - \overline{z_1} e_2 = x_0 - x_1 e_1 - x_2 e_2 + x_3 e_3, \\ p_{(11)} &= z_2 - z_3 e_2 = y_0 + y_1 e_1 - y_2 e_2 - y_3 e_3, \\ p_{(12)} &= \overline{z_2} + \overline{z_3} e_2 = y_0 - y_1 e_1 + y_2 e_2 - y_3 e_3, \\ p_{(13)} &= \overline{z_2} - \overline{z_3} e_2 = y_0 - y_1 e_1 - y_2 e_2 + y_3 e_3, \end{aligned} \quad (9)$$

with $\overline{z_0} = x_0 - x_1 e_1$, $\overline{z_1} = x_2 - x_3 e_1$, $\overline{z_2} = y_0 - y_1 e_1$, and $\overline{z_3} = y_2 - y_3 e_1$. For instance,

$$\begin{aligned} e_2 p_0 &= e_2 (x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= (x_0 - x_1 e_1 + x_2 e_2 - x_3 e_3) e_2 = p_{(02)} e_2, \\ e_1 p_1 &= e_1 (y_0 + y_1 e_1 + y_2 e_2 + y_3 e_3) \\ &= (y_0 + y_1 e_1 - y_2 e_2 - y_3 e_3) e_1 = p_{(11)} e_1. \end{aligned} \quad (10)$$

Because of the properties of the eight-unit equality, the addition and subtraction of dual split quaternions are governed by the rules of ordinary algebra. Here the symbol $p_{(kr)}$ is used by just enumerating r and k, not r times k. For example, $p_{(22)} \neq p_4$ and $p_{22} = p_4$.

For any two elements $z = p_0 + \varepsilon p_1$ and $w = q_0 + \varepsilon q_1$ of $\mathcal{D}(\mathcal{S})$, where $q_0 = \sum_{r=0}^{3} s_r e_r$ and $q_1 = \sum_{r=0}^{3} t_r e_r$ are split quaternion components and $s_r, t_r \in \mathbb{R}$ $(r = 0, 1, 2, 3)$, their noncommutative product is given by

$$zw = (p_0 + \varepsilon p_1)(q_0 + \varepsilon q_1) = p_0 q_0 + \varepsilon (p_0 q_1 + p_1 q_0). \quad (11)$$

The conjugation z^* of z and the corresponding modulus zz^* in $\mathcal{D}(\mathcal{S})$ are defined by

$$\begin{aligned} z^* &= p_0^* + \varepsilon p_1^*, \\ zz^* &= z^* z = p_0 p_0^* + \varepsilon (p_0 p_1^* + p_1 p_0^*) \\ &= (z_0 \overline{z_0} - z_1 \overline{z_1}) + 2\varepsilon (z_0 \overline{z_2} - z_1 \overline{z_3}) \\ &= \sum_{r=0}^{1} \{(x_r^2 - x_{r+2}^2) + \varepsilon (x_r y_r - x_{r+2} y_{r+2})\}, \end{aligned} \quad (12)$$

where $p_0^* = \overline{z_0} - z_1 e_2$ and $p_1^* = \overline{z_2} - z_3 e_2$.

Lemma 1. *For all $z \in \mathcal{D}(\mathcal{S})$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, we have*

$$z^n = p_0^n + \varepsilon \sum_{k=1}^{n} p_0^{n-k} p_1 p_0^{k-1}. \quad (13)$$

Proof. If $n = 1$, then (13) is trivial. Now suppose that this holds for some $n \in \mathbb{N}$. Then, as desired,

$$\begin{aligned} z^{n+1} &= zz^n = z\left(p_0^n + \varepsilon\sum_{k=1}^{n} p_0^{n-k} p_1 p_0^{k-1}\right) \\ &= p_0^{n+1} + \varepsilon\sum_{k=1}^{n} p_0^{n-k+1} p_1 p_0^{k-1} + \varepsilon p_1 p_0^n \\ &= p_0^{n+1} + \varepsilon\sum_{k=1}^{n+1} p_0^{n+1-k} p_1 p_0^{k-1}. \end{aligned} \tag{14}$$

By the principle of mathematical induction, (13) holds for all $n \in \mathbb{N}$. □

Let Ω be an open subset of $\mathbb{C}^2 \times \mathbb{C}^2$. Then the function $f : \Omega \to \mathscr{D}(\mathscr{S})$ can be expressed as

$$f(z) = f(p_0, p_1) = f_0(p_0, p_1) + \varepsilon f_1(p_0, p_1), \tag{15}$$

where the component functions $f_r : \Omega \to \mathscr{S}$ $(r = 0, 1)$ are split quaternionic-valued functions. The component functions f_r $(r = 0, 1)$ are

$$\begin{aligned} f_0(p_0, p_1) &= f_0(z_0, z_1, z_2, z_3) \\ &= g_0(z_0, z_1, z_2, z_3) + g_1(z_0, z_1, z_2, z_3) e_2, \\ f_1(p_0, p_1) &= f_1(z_0, z_1, z_2, z_3) \\ &= g_2(z_0, z_1, z_2, z_3) + g_3(z_0, z_1, z_2, z_3) e_2, \end{aligned} \tag{16}$$

where $g_k = u_{2k} + u_{2k+1}e_1$ $(k = 0, 1)$ and $g_k = v_{2k-4} + v_{2k-3}e_1$ $(k = 2, 3)$ are complex-valued functions, and u_r and v_r $(r = 0, 1, 2, 3)$ are real-valued functions.

Now, we let differential operators D_1 and D_2 be defined on $\mathscr{D}(\mathscr{S})$ as

$$D_1 := D_{(11)} + \varepsilon D_{(12)}, \qquad D_2 := D_{(21)} + \varepsilon D_{(22)}. \tag{17}$$

Then the conjugate operators D_1^* and D_2^* are

$$D_1^* = D_{(11)}^* + \varepsilon D_{(12)}^*, \qquad D_2^* = D_{(21)}^* + \varepsilon D_{(22)}^*, \tag{18}$$

where

$$\begin{aligned} D_{(11)} &= \frac{\partial}{\partial z_0} + \frac{\partial}{\partial \overline{z_1}} e_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3\right), \\ D_{(12)} &= \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \overline{z_3}} e_2 = \frac{1}{2}\left(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_1} e_1 + \frac{\partial}{\partial y_2} e_2 + \frac{\partial}{\partial y_3} e_3\right), \\ D_{(21)} &= \frac{\partial}{\partial z_0} + \frac{1}{2}\frac{\partial}{\partial z_1} e_2 \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} e_1 + \frac{1}{2}\frac{\partial}{\partial x_2} e_2 - \frac{1}{2}\frac{\partial}{\partial x_3} e_3\right), \\ D_{(22)} &= \frac{\partial}{\partial z_2} + \frac{1}{2}\frac{\partial}{\partial z_3} e_2 \\ &= \frac{1}{2}\left(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_1} e_1 + \frac{1}{2}\frac{\partial}{\partial y_2} e_2 - \frac{1}{2}\frac{\partial}{\partial y_3} e_3\right), \end{aligned} \tag{19}$$

$$\begin{aligned} D_{(11)}^* &= \frac{\partial}{\partial \overline{z_0}} - \frac{\partial}{\partial \overline{z_1}} e_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 - \frac{\partial}{\partial x_2} e_2 - \frac{\partial}{\partial x_3} e_3\right), \\ D_{(12)}^* &= \frac{\partial}{\partial \overline{z_2}} - \frac{\partial}{\partial \overline{z_3}} e_2 = \frac{1}{2}\left(\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1} e_1 - \frac{\partial}{\partial y_2} e_2 - \frac{\partial}{\partial y_3} e_3\right), \\ D_{(21)}^* &= \frac{\partial}{\partial \overline{z_0}} - \frac{1}{2}\frac{\partial}{\partial z_1} e_2 \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 - \frac{1}{2}\frac{\partial}{\partial x_2} e_2 + \frac{1}{2}\frac{\partial}{\partial x_3} e_3\right), \\ D_{(22)}^* &= \frac{\partial}{\partial \overline{z_2}} - \frac{1}{2}\frac{\partial}{\partial z_3} e_2 \\ &= \frac{1}{2}\left(\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1} e_1 - \frac{1}{2}\frac{\partial}{\partial y_2} e_2 + \frac{1}{2}\frac{\partial}{\partial y_3} e_3\right) \end{aligned} \tag{20}$$

act on $\mathscr{D}(\mathscr{S})$. These operators are called corresponding Cauchy-Riemann operators in $\mathscr{D}(\mathscr{S})$, where $\partial/\partial z_r$ and $\partial/\partial \overline{z_r}$ $(r = 0, 1, 2, 3)$ are usual differential operators used in the complex analysis.

Remark 2. From the definition of differential operators on $\mathscr{D}(\mathscr{S})$,

$$\begin{aligned} D_r f &= \left(D_{(r1)} + \varepsilon D_{(r2)}\right)(f_0 + \varepsilon f_1) \\ &= D_{(r1)} f_0 + \varepsilon\left(D_{(r1)} f_1 + D_{(r2)} f_0\right), \\ D_r^* f &= \left(D_{(r1)}^* + \varepsilon D_{(r2)}^*\right)(f_0 + \varepsilon f_1) \\ &= D_{(r1)}^* f_0 + \varepsilon\left(D_{(r1)}^* f_1 + D_{(r2)}^* f_0\right), \end{aligned} \tag{21}$$

where $r = 1, 2$.

Definition 3. Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$. A function $f = f_0 + \varepsilon f_1$ is called an L_r (resp., R_r)-regular function $(r = 1, 2)$ on Ω if the following two conditions are satisfied:

(i) f_k $(k = 0, 1)$ are continuously differential functions on Ω, and

(ii) $D_r^* f(z) = 0$ (resp., $f(z) D_r^* = 0$) on Ω $(r = 1, 2)$.

In particular, the equation $D_1^* f(z) = 0$ of Definition 3 is equivalent to

$$D_{(11)}^* f_0 = 0, \qquad D_{(12)}^* f_0 + D_{(11)}^* f_1 = 0. \tag{22}$$

In addition,

$$\frac{\partial g_0}{\partial \overline{z_0}} - \frac{\partial \overline{g_1}}{\partial \overline{z_1}} = 0, \qquad \frac{\partial g_1}{\partial \overline{z_0}} - \frac{\partial \overline{g_0}}{\partial \overline{z_1}} = 0,$$
$$\frac{\partial g_2}{\partial \overline{z_0}} + \frac{\partial g_0}{\partial \overline{z_2}} - \frac{\partial \overline{g_3}}{\partial \overline{z_1}} - \frac{\partial \overline{g_1}}{\partial \overline{z_3}} = 0, \tag{23}$$
$$\frac{\partial g_3}{\partial \overline{z_0}} + \frac{\partial g_1}{\partial \overline{z_2}} - \frac{\partial \overline{g_2}}{\partial \overline{z_1}} - \frac{\partial \overline{g_0}}{\partial \overline{z_3}} = 0.$$

Concretely, the following system is obtained:

$$\begin{aligned}
&\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} = 0,\\
&\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} - \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0,\\
&\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} - \frac{\partial u_0}{\partial x_2} - \frac{\partial u_1}{\partial x_3} = 0,\\
&\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_0}{\partial x_3} + \frac{\partial u_1}{\partial x_2} = 0,\\
&\frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} - \frac{\partial u_3}{\partial y_3} + \frac{\partial v_0}{\partial x_0}\\
&\qquad - \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} - \frac{\partial v_3}{\partial x_3} = 0,\\
&\frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} - \frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2} + \frac{\partial v_1}{\partial x_0}\\
&\qquad + \frac{\partial v_0}{\partial x_1} - \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} = 0,\\
&\frac{\partial u_2}{\partial y_0} - \frac{\partial u_3}{\partial y_1} - \frac{\partial u_0}{\partial y_2} - \frac{\partial u_1}{\partial y_3} + \frac{\partial v_2}{\partial x_0}\\
&\qquad - \frac{\partial v_3}{\partial x_1} - \frac{\partial v_0}{\partial x_2} - \frac{\partial v_1}{\partial x_3} = 0,\\
&\frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} - \frac{\partial u_0}{\partial y_3} + \frac{\partial u_1}{\partial y_2} + \frac{\partial v_3}{\partial x_0}\\
&\qquad + \frac{\partial v_2}{\partial x_1} - \frac{\partial v_0}{\partial x_3} + \frac{\partial v_1}{\partial x_2} = 0.
\end{aligned} \tag{24}$$

The above systems (23) and (24) are corresponding Cauchy-Riemann systems in $\mathcal{D}(\mathcal{S})$. Similarly, the equation $D_2^* f(z) = 0$ of Definition 3 is equivalent to

$$D^*_{(21)} f_0 = 0, \qquad D^*_{(22)} f_0 + D^*_{(21)} f_1 = 0. \tag{25}$$

Then,

$$\frac{\partial g_0}{\partial \overline{z_0}} - \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_1} = 0, \qquad \frac{\partial g_1}{\partial \overline{z_0}} - \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_1} = 0,$$
$$\frac{\partial g_2}{\partial \overline{z_0}} + \frac{\partial g_0}{\partial \overline{z_2}} - \frac{1}{2}\frac{\partial \overline{g_3}}{\partial z_1} - \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_3} = 0, \tag{26}$$
$$\frac{\partial g_3}{\partial \overline{z_0}} + \frac{\partial g_1}{\partial \overline{z_2}} - \frac{1}{2}\frac{\partial \overline{g_2}}{\partial z_1} - \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_3} = 0.$$

Concretely, the following system is obtained:

$$\begin{aligned}
&\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\frac{\partial u_2}{\partial x_2} + \frac{1}{2}\frac{\partial u_3}{\partial x_3} = 0,\\
&\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{1}{2}\frac{\partial u_2}{\partial x_3} + \frac{1}{2}\frac{\partial u_3}{\partial x_2} = 0,\\
&\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} - \frac{1}{2}\frac{\partial u_0}{\partial x_2} + \frac{1}{2}\frac{\partial u_1}{\partial x_3} = 0,\\
&\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{1}{2}\frac{\partial u_0}{\partial x_3} + \frac{1}{2}\frac{\partial u_1}{\partial x_2} = 0,\\
&\frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{1}{2}\frac{\partial u_2}{\partial y_2} + \frac{1}{2}\frac{\partial u_3}{\partial y_3} + \frac{\partial v_0}{\partial x_0}\\
&\qquad - \frac{\partial v_1}{\partial x_1} - \frac{1}{2}\frac{\partial v_2}{\partial x_2} + \frac{1}{2}\frac{\partial v_3}{\partial x_3} = 0,\\
&\frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{1}{2}\frac{\partial u_2}{\partial y_3} + \frac{1}{2}\frac{\partial u_3}{\partial y_2} + \frac{\partial v_1}{\partial x_0}\\
&\qquad + \frac{\partial v_0}{\partial x_1} + \frac{1}{2}\frac{\partial v_2}{\partial x_3} + \frac{1}{2}\frac{\partial v_3}{\partial x_2} = 0,\\
&\frac{\partial u_2}{\partial y_0} - \frac{\partial u_3}{\partial y_1} - \frac{1}{2}\frac{\partial u_0}{\partial y_2} + \frac{1}{2}\frac{\partial u_1}{\partial y_3} + \frac{\partial v_2}{\partial x_0}\\
&\qquad - \frac{\partial v_3}{\partial x_1} - \frac{1}{2}\frac{\partial v_0}{\partial x_2} + \frac{1}{2}\frac{\partial v_1}{\partial x_3} = 0,\\
&\frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} + \frac{1}{2}\frac{\partial u_0}{\partial y_3} + \frac{1}{2}\frac{\partial u_1}{\partial y_2}\\
&\qquad + \frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} + \frac{1}{2}\frac{\partial v_0}{\partial x_3} + \frac{1}{2}\frac{\partial v_1}{\partial x_2} = 0.
\end{aligned} \tag{27}$$

The above systems (26) and (27) are corresponding Cauchy-Riemann systems in $\mathcal{D}(\mathcal{S})$.

On the other hand, the equation $f(z) D_1^* = 0$ of Definition 3 is equivalent to

$$f_0 D^*_{(11)} = 0, \qquad f_0 D^*_{(12)} = -f_1 D^*_{(11)}. \tag{28}$$

Then,

$$g_0\frac{\partial}{\partial\overline{z_0}}=g_1\frac{\partial}{\partial z_1},\qquad g_1\frac{\partial}{\partial z_0}=g_0\frac{\partial}{\partial\overline{z_1}},$$
$$g_0\frac{\partial}{\partial\overline{z_2}}-g_1\frac{\partial}{\partial z_3}=-g_2\frac{\partial}{\partial\overline{z_0}}+g_3\frac{\partial}{\partial z_1},\qquad(29)$$
$$g_1\frac{\partial}{\partial z_2}-g_0\frac{\partial}{\partial\overline{z_3}}=-g_3\frac{\partial}{\partial z_0}+g_2\frac{\partial}{\partial\overline{z_1}},$$

where

$$g_k\frac{\partial}{\partial\overline{z_m}}=\frac{\partial g_k}{\partial\overline{z_m}},\quad g_k\frac{\partial}{\partial z_m}=\frac{\partial g_k}{\partial z_m}\quad(k,m=0,1,2,3).\qquad(30)$$

Concretely, the following system is obtained:

$$\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}=\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3},\qquad\frac{\partial u_1}{\partial x_0}+\frac{\partial u_0}{\partial x_1}=-\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2},$$
$$\frac{\partial u_2}{\partial x_0}+\frac{\partial u_3}{\partial x_1}=\frac{\partial u_0}{\partial x_2}-\frac{\partial u_1}{\partial x_3},\qquad\frac{\partial u_3}{\partial x_0}-\frac{\partial u_2}{\partial x_1}=\frac{\partial u_0}{\partial x_3}+\frac{\partial u_1}{\partial x_2},$$
$$\frac{\partial u_0}{\partial y_0}-\frac{\partial u_1}{\partial y_1}-\frac{\partial u_2}{\partial y_2}-\frac{\partial u_3}{\partial y_3}=-\frac{\partial v_0}{\partial x_0}+\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3},$$
$$\frac{\partial u_1}{\partial y_0}+\frac{\partial u_0}{\partial y_1}+\frac{\partial u_2}{\partial y_3}-\frac{\partial u_3}{\partial y_2}=-\frac{\partial v_1}{\partial x_0}-\frac{\partial v_0}{\partial x_1}-\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2},$$
$$\frac{\partial u_2}{\partial y_0}+\frac{\partial u_3}{\partial y_1}-\frac{\partial u_0}{\partial y_2}+\frac{\partial u_1}{\partial y_3}=-\frac{\partial v_2}{\partial x_0}-\frac{\partial v_3}{\partial x_1}+\frac{\partial v_0}{\partial x_2}-\frac{\partial v_1}{\partial x_3},$$
$$\frac{\partial u_3}{\partial y_0}-\frac{\partial u_2}{\partial y_1}-\frac{\partial u_0}{\partial y_3}-\frac{\partial u_1}{\partial y_2}=-\frac{\partial v_3}{\partial x_0}+\frac{\partial v_2}{\partial x_1}+\frac{\partial v_0}{\partial x_3}+\frac{\partial v_1}{\partial x_2}.\qquad(31)$$

Similarly, the equation $f(z)D_2^*=0$ of Definition 3 is equivalent to

$$f_0D_{(21)}^*=0,\qquad f_0D_{(22)}^*=-f_1D_{(21)}^*.\qquad(32)$$

Then,

$$g_0\frac{\partial}{\partial\overline{z_0}}=\frac{1}{2}g_1\frac{\partial}{\partial\overline{z_1}},\qquad g_1\frac{\partial}{\partial z_0}=\frac{1}{2}g_0\frac{\partial}{\partial z_1},$$
$$g_0\frac{\partial}{\partial\overline{z_2}}-\frac{1}{2}g_1\frac{\partial}{\partial\overline{z_3}}=-g_2\frac{\partial}{\partial\overline{z_0}}+\frac{1}{2}g_3\frac{\partial}{\partial\overline{z_1}},\qquad(33)$$
$$g_1\frac{\partial}{\partial z_2}-\frac{1}{2}g_0\frac{\partial}{\partial z_3}=-g_3\frac{\partial}{\partial z_0}+\frac{1}{2}g_2\frac{\partial}{\partial z_1},$$

where

$$g_k\frac{\partial}{\partial\overline{z_m}}=\frac{\partial g_k}{\partial\overline{z_m}},\quad g_k\frac{\partial}{\partial z_m}=\frac{\partial g_k}{\partial z_m}\quad(k,m=0,1,2,3).\qquad(34)$$

Concretely, the system is obtained as follows:

$$\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{1}{2}\frac{\partial u_2}{\partial x_2}+\frac{1}{2}\frac{\partial u_3}{\partial x_3}=0,$$
$$\frac{\partial u_1}{\partial x_0}+\frac{\partial u_0}{\partial x_1}-\frac{1}{2}\frac{\partial u_2}{\partial x_3}-\frac{1}{2}\frac{\partial u_3}{\partial x_2}=0,$$
$$\frac{\partial u_2}{\partial x_0}+\frac{\partial u_3}{\partial x_1}-\frac{1}{2}\frac{\partial u_0}{\partial x_2}-\frac{1}{2}\frac{\partial u_1}{\partial x_3}=0,$$
$$\frac{\partial u_3}{\partial x_0}-\frac{\partial u_2}{\partial x_1}+\frac{1}{2}\frac{\partial u_0}{\partial x_3}-\frac{1}{2}\frac{\partial u_1}{\partial x_2}=0,$$
$$\frac{\partial u_0}{\partial y_0}-\frac{\partial u_1}{\partial y_1}-\frac{1}{2}\frac{\partial u_2}{\partial y_2}+\frac{1}{2}\frac{\partial u_3}{\partial y_3}+\frac{\partial v_0}{\partial x_0}-\frac{\partial v_1}{\partial x_1}-\frac{1}{2}\frac{\partial v_2}{\partial x_2}+\frac{1}{2}\frac{\partial v_3}{\partial x_3}=0,$$
$$\frac{\partial u_1}{\partial y_0}+\frac{\partial u_0}{\partial y_1}-\frac{1}{2}\frac{\partial u_2}{\partial y_3}-\frac{1}{2}\frac{\partial u_3}{\partial y_2}+\frac{\partial v_1}{\partial x_0}+\frac{\partial v_0}{\partial x_1}-\frac{1}{2}\frac{\partial v_2}{\partial x_3}-\frac{1}{2}\frac{\partial v_3}{\partial x_2}=0,\qquad(35)$$
$$\frac{\partial u_2}{\partial y_0}+\frac{\partial u_3}{\partial y_1}-\frac{1}{2}\frac{\partial u_0}{\partial y_2}+\frac{1}{2}\frac{\partial u_1}{\partial y_3}+\frac{\partial v_2}{\partial x_0}+\frac{\partial v_3}{\partial x_1}-\frac{1}{2}\frac{\partial v_0}{\partial x_2}+\frac{1}{2}\frac{\partial v_1}{\partial x_3}=0,$$
$$\frac{\partial u_3}{\partial y_0}-\frac{\partial u_2}{\partial y_1}-\frac{1}{2}\frac{\partial u_0}{\partial y_3}-\frac{1}{2}\frac{\partial u_1}{\partial y_2}+\frac{\partial v_3}{\partial x_0}-\frac{\partial v_2}{\partial x_1}-\frac{1}{2}\frac{\partial v_0}{\partial x_3}-\frac{1}{2}\frac{\partial v_1}{\partial x_2}=0.$$

From the systems (24), (27), (31), and (35), the equations $D_r^*f(z)=0$ and $f(z)D_r^*=0$ $(r=1,2)$ are different. Therefore, the equations $D_r^*f(z)=0$ and $f(z)D_r^*=0$ $(r=1,2)$ should be distinguished as L_r-regular functions $(r=1,2)$ and R_r-regular functions $(r=1,2)$ on Ω, respectively. Now the properties of the L_r-regular function $(r=1,2)$ with values in $\mathcal{D}(\mathcal{S})$ are considered.

3. Properties of L_r-Regular Functions $(r=1,2)$ with Values in $\mathcal{D}(\mathcal{S})$

We consider properties of a L_r-regular functions $(r=1,2)$ with values in $\mathcal{D}(\mathcal{S})$.

Theorem 4. *Let Ω be an open set in $\mathbb{C}^2\times\mathbb{C}^2$ and let $f=f_0+\varepsilon f_1=(g_0+g_1e_2)+\varepsilon(g_2+g_3e_2)$ be an L_1-regular function defined on Ω. Then*

$$D_1f=\left\{2\left(\frac{\partial}{\partial\overline{z_1}}+\varepsilon\frac{\partial}{\partial\overline{z_3}}\right)e_2-\left(\frac{\partial}{\partial x_1}+\varepsilon\frac{\partial}{\partial y_1}\right)e_1\right\}f.\qquad(36)$$

Proof. By the system (23), we have

$$\begin{aligned}
D_1 f &= D_{(11)} f_0 + \varepsilon \left(D_{(12)} f_0 + D_{(11)} f_1\right) \\
&= \left(\frac{\partial g_0}{\partial z_0} + \frac{\partial \overline{g_1}}{\partial \overline{z_1}}\right) + \left(\frac{\partial g_1}{\partial z_0} + \frac{\partial \overline{g_0}}{\partial \overline{z_1}}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial g_0}{\partial z_2} + \frac{\partial \overline{g_1}}{\partial \overline{z_3}} + \frac{\partial g_2}{\partial z_0} + \frac{\partial \overline{g_3}}{\partial \overline{z_1}}\right) \\
&\quad + \varepsilon \left(\frac{\partial g_1}{\partial z_2} + \frac{\partial \overline{g_0}}{\partial \overline{z_3}} + \frac{\partial g_3}{\partial z_0} + \frac{\partial \overline{g_2}}{\partial \overline{z_1}}\right) e_2 \\
&= \left(\frac{\partial g_0}{\partial \overline{z_0}} + \frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1} e_1 + \frac{\partial \overline{g_1}}{\partial \overline{z_1}}\right) \\
&\quad + \left(\frac{\partial g_1}{\partial \overline{z_0}} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1} e_1 + \frac{\partial \overline{g_0}}{\partial \overline{z_1}}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial g_0}{\partial \overline{z_2}} + \frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1} e_1 + \frac{\partial \overline{g_1}}{\partial \overline{z_3}}\right. \\
&\qquad \left. + \frac{\partial g_2}{\partial \overline{z_0}} + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1} e_1 + \frac{\partial \overline{g_3}}{\partial \overline{z_1}}\right) \\
&\quad + \varepsilon \left(\frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1} e_1 + \frac{\partial \overline{g_0}}{\partial \overline{z_3}}\right. \\
&\qquad \left. + \frac{\partial g_3}{\partial \overline{z_0}} + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1} e_1 + \frac{\partial \overline{g_2}}{\partial \overline{z_1}}\right) e_2 \\
&= \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1} e_1 + 2\frac{\partial \overline{g_1}}{\partial \overline{z_1}}\right) \\
&\quad + \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1} e_1 + 2\frac{\partial \overline{g_0}}{\partial \overline{z_1}}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1} e_1 + 2\frac{\partial \overline{g_1}}{\partial \overline{z_3}}\right. \\
&\qquad \left. + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1} e_1 + 2\frac{\partial \overline{g_3}}{\partial \overline{z_1}}\right) \\
&\quad + \varepsilon \left(\frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1} e_1 + 2\frac{\partial \overline{g_0}}{\partial \overline{z_3}}\right. \\
&\qquad \left. + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1} e_1 + 2\frac{\partial \overline{g_2}}{\partial \overline{z_1}}\right) e_2 \\
&= \left\{2\left(\frac{\partial}{\partial \overline{z_1}} + \varepsilon \frac{\partial}{\partial \overline{z_3}}\right) e_2 - \left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial y_1}\right) e_1\right\} f.
\end{aligned} \tag{37}$$

Therefore, we obtain

$$D_1 f = \left\{2\left(\frac{\partial}{\partial \overline{z_1}} + \varepsilon \frac{\partial}{\partial \overline{z_3}}\right) e_2 - \left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial y_1}\right) e_1\right\} f. \tag{38}$$

□

Theorem 5. *Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and $f = f_0 + \varepsilon f_1 = (g_0 + g_1 e_2) + \varepsilon (g_2 + g_3 e_2)$ be an L_2-regular function defined on Ω. Then*

$$D_2 f = \left\{\left(\frac{\partial}{\partial z_1} + \varepsilon \frac{\partial}{\partial z_3}\right) e_2 - \left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial y_1}\right) e_1\right\} f. \tag{39}$$

Proof. By the system (26), we have

$$\begin{aligned}
D_2 f &= D_{(21)} f_0 + \varepsilon \left(D_{(22)} f_0 + D_{(21)} f_1\right) \\
&= \left(\frac{\partial g_0}{\partial z_0} + \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_1}\right) + \left(\frac{\partial g_1}{\partial z_0} + \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_1}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial g_0}{\partial z_2} + \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_3} + \frac{\partial g_2}{\partial z_0} + \frac{1}{2}\frac{\partial \overline{g_3}}{\partial z_1}\right) \\
&\quad + \varepsilon \left(\frac{\partial g_1}{\partial z_2} + \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_3} + \frac{\partial g_3}{\partial z_0} + \frac{1}{2}\frac{\partial \overline{g_2}}{\partial z_1}\right) e_2 \\
&= \left(\frac{\partial g_0}{\partial \overline{z_0}} + \frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_1}\right) \\
&\quad + \left(\frac{\partial g_1}{\partial \overline{z_0}} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_1}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial g_0}{\partial \overline{z_2}} + \frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_1}}{\partial z_3}\right. \\
&\qquad \left. + \frac{\partial g_2}{\partial \overline{z_0}} + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_3}}{\partial z_1}\right) \\
&\quad + \varepsilon \left(\frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_0}}{\partial z_3}\right. \\
&\qquad \left. + \frac{\partial g_3}{\partial \overline{z_0}} + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1} e_1 + \frac{1}{2}\frac{\partial \overline{g_2}}{\partial z_1}\right) e_2 \\
&= \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1} e_1 + \frac{\partial \overline{g_1}}{\partial z_1}\right) \\
&\quad + \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1} e_1 + \frac{\partial \overline{g_0}}{\partial z_1}\right) e_2 \\
&\quad + \varepsilon \left(\frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1} e_1 + \frac{\partial \overline{g_1}}{\partial z_3}\right. \\
&\qquad \left. + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1} e_1 + \frac{\partial \overline{g_3}}{\partial z_1}\right) \\
&\quad + \varepsilon \left(\frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1} e_1 + \frac{\partial \overline{g_0}}{\partial z_3}\right. \\
&\qquad \left. + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1} e_1 + \frac{\partial \overline{g_2}}{\partial z_1}\right) e_2 \\
&= \left\{\left(\frac{\partial}{\partial z_1} + \varepsilon \frac{\partial}{\partial z_3}\right) e_2 - \left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial y_1}\right) e_1\right\} f.
\end{aligned} \tag{40}$$

Therefore, we obtain the following equation:

$$D_2 f = \left\{\left(\frac{\partial}{\partial z_1} + \varepsilon \frac{\partial}{\partial z_3}\right) e_2 - \left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial y_1}\right) e_1\right\} f. \tag{41}$$

□

Proposition 6. *From properties of differential operators, the following equations are obtained:*

$$\begin{aligned}
&D_{(1r)}p_{r-1} = 2, \qquad D_{(2r)}p_{r-1} = 1,\\
&D^*_{(1r)}p_{r-1} = -1, \qquad D^*_{(2r)}p_{r-1} = 0,\\
&D^*_{(1r)}p^*_{r-1} = 2, \qquad D^*_{(2r)}p^*_{r-1} = 1,\\
&D_{(r1)}p_1 = D^*_{(r1)}p_1 = D_{(r1)}p^*_1 = D^*_{(r1)}p^*_1\\
&\quad = D_{(r2)}p_0 = D^*_{(r2)}p_0 = D_{(r2)}p^*_0\\
&\quad = D^*_{(r2)}p^*_0 = 0 \quad (r = 1, 2).
\end{aligned} \tag{42}$$

Proof. By properties of the power of dual split quaternions and derivatives on $\mathscr{D}(\mathscr{S})$, the following derivatives are obtained:

$$\begin{aligned}
D_{(11)}p_0 &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)\\
&\quad \times (x_0 + x_1e_1 + x_2e_2 + x_3e_3) = 2,\\
D^*_{(22)}p_1 &= \frac{1}{2}\left(\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1}e_1 - \frac{1}{2}\frac{\partial}{\partial y_2}e_2 + \frac{1}{2}\frac{\partial}{\partial y_3}e_3\right)\\
&\quad \times (y_0 + y_1e_1 + y_2e_2 + y_3e_3) = 0,\\
D_{(11)}p^*_0 &= \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)\\
&\quad \times (x_0 - x_1e_1 - x_2e_2 - x_3e_3) = -1.
\end{aligned} \tag{43}$$

The other equations are calculated using a similar method, and the above equations are obtained. □

Theorem 7. *Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and let $f(z)$ be a function on Ω with values in $\mathscr{D}(\mathscr{S})$. Then the power z^n of z in $\mathscr{D}(\mathscr{S})$ is not an L_1-regular function but an L_2-regular function on Ω, where $n \in \mathbb{N}$.*

Proof. From the definition of the L_r-regular function ($r = 1, 2$) on Ω and Proposition 6, we may consider whether the power z^n of z in $\mathscr{D}(\mathscr{S})$ satisfies the equation $D^*_r z^n = 0$ ($r = 1, 2$). Since $D^*_{(11)}p_0 = 2$,

$$\begin{aligned}
D^*_1 z^n &= \left(D^*_{(11)} + \varepsilon D^*_{(12)}\right)\left(p^n_0 + \varepsilon\sum_{k=1}^{n} p^{n-k}_0 p_1 p^{k-1}_0\right)\\
&= D^*_{(11)}p^n_0 + \varepsilon\left(\sum_{k=1}^{n} D^*_{(11)}p^{n-k}_0 p_1 p^{k-1}_0 + D^*_{(12)}p^n_0\right) \neq 0.
\end{aligned} \tag{44}$$

Hence, the power z^n of z is not L_1-regular on Ω. On the other hand, from the equations in Proposition 6, we have $D^*_{(21)}p_0 = 0$, $D^*_{(21)}p_1 = 0$, and $D^*_{(22)}p_0 = 0$. Then,

$$D^*_2 z^n = D^*_{(21)}p^n_0 + \varepsilon\left(\sum_{k=1}^{n} D^*_{(21)}p^{n-k}_0 p_1 p^{k-1}_0 + D^*_{(22)}p^n_0\right) = 0. \tag{45}$$

Therefore, by the definition of the L_r-regular function ($r = 1, 2$) on Ω, a power z^n of z is L_2-regular on Ω. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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