

Research Article

Existence of Traveling Waves for a Delayed SIRS Epidemic Diffusion Model with Saturation Incidence Rate

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This paper is concerned with the existence of traveling waves for a delayed SIRS epidemic diffusion model with saturation incidence rate. By using the cross-iteration method and Schauder's fixed point theorem, we reduce the existence of traveling waves to the existence of a pair of upper-lower solutions. By careful analysis, we derive the existence of traveling waves connecting the disease-free steady state and the endemic steady state through the establishment of the suitable upper-lower solutions.

1. Introduction

Since Kermack and McKendrick [1] proposed an ordinary differential system to study epidemiology in 1927, various models have been used to describe various kinds of epidemics, and the dynamics of these systems have been investigated. Let $S(t)$ represent the number of individuals who are susceptible to the disease, let $I(t)$ represent the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals, and let $R(t)$ represent the number of individuals who have been infected and then removed from the possibility of being infected again. Mena-Lorca and Hethcote [2] considered the following SIRS epidemic model:

$$\begin{aligned}\dot{S}(t) &= A - dS(t) - \beta S(t)I(t) + \delta R(t), \\ \dot{I}(t) &= \beta S(t)I(t) - (\gamma + \mu + d)I(t), \\ \dot{R}(t) &= \gamma I(t) - (\delta + d)R(t),\end{aligned}\tag{1}$$

where the parameters A , d , β , δ , γ , μ are positive constants and A is the recruitment rate of the population, d is the natural death rate of the population, β is the transmission rate,

δ is the rate at which recovered individuals lose immunity and return to the susceptible class, γ is the recovery rate of the infective individuals, and μ is the death rate of the infective individuals due to disease. The SIRS model assumes that the recovered individuals have only temporary immunity, which is reasonable in the study of some communicable diseases.

However, due to the diseases latency or immunity, the presence of time delays in such models makes them more realistic. On the other hand, the environment in which an individual lives is actually heterogeneous and the mobility of people within a country or even worldwide is large; introducing the spatial diffusion in these epidemic models is unavoidable. In recent years, the dynamics of the delayed epidemic diffusion model have been widely studied by many researchers (see, e.g., [3–6]), and these studies are mainly focused on the global attractivity, basic reproductive number, and especially the epidemic waves. For example, Gan et al. [7] considered the following delayed SIRS epidemic model with spatial diffusion:

$$\begin{aligned}\frac{\partial S}{\partial t} &= D_S \frac{\partial^2 S}{\partial x^2} + A - dS(x, t) \\ &\quad - \beta S(x, t)I(x, t - \tau) + \delta R(x, t),\end{aligned}$$

$$\begin{aligned}\frac{\partial I}{\partial t} &= D_I \frac{\partial^2 I}{\partial x^2} + \beta S(x, t) I(x, t - \tau) \\ &\quad - (\gamma + \alpha + d) I(x, t), \\ \frac{\partial R}{\partial t} &= D_R \frac{\partial^2 R}{\partial x^2} + \gamma I(x, t) - (\delta + d) R(x, t),\end{aligned}\quad (2)$$

and obtained the existence of traveling wave solutions.

In systems (1) and (2), the terms $\beta S(t)I(t)$ and $\beta S(x, t)I(x, t - \tau)$ are called incidence rate and both of them are bilinear. However, as the number of susceptible individuals is large, it is reasonable to consider the saturation incidence rate (see [8]) instead of the bilinear incidence rate. Motivated by the works mentioned above, we will consider the following delayed SIRS epidemic diffusion model with nonlinear saturation rate

$$\begin{aligned}\frac{\partial S}{\partial t} &= D_S \frac{\partial^2 S}{\partial x^2} + A - dS(x, t) \\ &\quad - \frac{\beta S(x, t) I(x, t - \tau)}{1 + \alpha I(x, t - \tau)} + \delta R(x, t), \\ \frac{\partial I}{\partial t} &= D_I \frac{\partial^2 I}{\partial x^2} + \frac{\beta S(x, t) I(x, t - \tau)}{1 + \alpha I(x, t - \tau)} \\ &\quad - (\gamma + \mu + d) I(x, t), \\ \frac{\partial R}{\partial t} &= D_R \frac{\partial^2 R}{\partial x^2} + \gamma I(x, t) - (\delta + d) R(x, t)\end{aligned}\quad (3)$$

and study its traveling wave solutions. The main tool is the upper-lower solutions coupled with cross-iteration method established by Ma [9]. We point out that the nonlinear terms in (3) *do not* satisfy the common various (exponential) monotonicity conditions such as in [10–12]; thus the main difficulty is the construction and verification of the upper-lower solutions.

2. Preliminaries and Lemmas

Throughout this paper, we employ the usual notations for the standard ordering in \mathbb{R}^3 . That is, for $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, we denote $u \leq v$ if $u_i \leq v_i$, $i = 1, 2, 3$; $u < v$ if $u \leq v$ but $u \neq v$; and $u \ll v$ if $u \leq v$ but $u_i \neq v_i$, $i = 1, 2, 3$. Let $\|\cdot\|$ denote the Euclidean norm in \mathbb{R}^3 .

First, we assume that $D_S = D_I = D_R = D$ for (3). Denoting $N = S + I + R$, then (3) reduces to the following system:

$$\begin{aligned}\frac{\partial N}{\partial t} &= D \frac{\partial^2 N}{\partial x^2} + A - dN(x, t) - \mu I(x, t), \\ \frac{\partial I}{\partial t} &= D \frac{\partial^2 I}{\partial x^2} + \frac{\beta(N - I - R) I(x, t - \tau)}{1 + \alpha I(x, t - \tau)} \\ &\quad - (\gamma + \mu + d) I(x, t), \\ \frac{\partial R}{\partial t} &= D \frac{\partial^2 R}{\partial x^2} + \gamma I(x, t) - (\delta + d) R(x, t).\end{aligned}\quad (4)$$

By making changes of variables $\tilde{N} = A/d - N$, $\tilde{I} = I$, $\tilde{R} = R$ and dropping the tildes, (4) is converted to the following system:

$$\begin{aligned}\frac{\partial N}{\partial t} &= D \frac{\partial^2 N}{\partial x^2} - dN(x, t) + \mu I(x, t), \\ \frac{\partial I}{\partial t} &= D \frac{\partial^2 I}{\partial x^2} + \beta \left(\frac{A}{d} - N - I - R \right) \frac{I(x, t - \tau)}{1 + \alpha I(x, t - \tau)} \\ &\quad - (\gamma + \mu + d) I(x, t), \\ \frac{\partial R}{\partial t} &= D \frac{\partial^2 R}{\partial x^2} + \gamma I(x, t) - (\delta + d) R(x, t).\end{aligned}\quad (5)$$

Consider the equilibrium equation of system (5):

$$\begin{aligned}\mu I - dN &= 0, \\ \beta \left(\frac{A}{d} - N - I - R \right) \frac{I}{1 + \alpha I} - (\gamma + \mu + d) I &= 0, \\ \gamma I - (\delta + d) R &= 0.\end{aligned}\quad (6)$$

Obviously, system (5) often has a trivial equilibrium $E^0(0, 0, 0)$. From the first and the third equation of (6), we know that $N = (\mu/d)I$, $R = (\gamma/(\delta + d))I$. Substituting the expressions into the second equation of (6), if $\mathfrak{R}_0 := A\beta/d(\gamma + \mu + d) > 1$, we get a positive equilibrium $E^*(k_1, k_2, k_3)$ of system (5), where

$$\begin{aligned}k_1 &= \mu(\delta + d) [A\beta - d(\gamma + \mu + d)] \\ &\quad \times (d\beta [\delta(\mu + d) + d(\gamma + \mu + d)] \\ &\quad + d^2\alpha(\gamma + \mu + d)(\delta + d))^{-1}, \\ k_2 &= \frac{(\delta + d) [A\beta - d(\gamma + \mu + d)]}{\beta [\delta(\mu + d) + d(\gamma + \mu + d)] + d\alpha(\gamma + \mu + d)(\delta + d)}, \\ k_3 &= \frac{\gamma [A\beta - d(\gamma + \mu + d)]}{\beta [\delta(\mu + d) + d(\gamma + \mu + d)] + d\alpha(\gamma + \mu + d)(\delta + d)}.\end{aligned}\quad (7)$$

By calculating, we can obtain that $k_1 + k_2 + k_3 < A/d$, which is important in the following text. In fact,

$$\begin{aligned}k_1 + k_2 + k_3 &= \frac{[A\beta - d(\gamma + \mu + d)]}{\beta [\delta(\mu + d) + d(\gamma + \mu + d)] + d\alpha(\gamma + \mu + d)(\delta + d)} \\ &\quad \times \left[\frac{\mu}{d}(\delta + d) + (\delta + d) + \gamma \right] \\ &< \frac{[A\beta - d(\gamma + \mu + d)]}{\beta(\mu\delta + \mu d + \delta d + \gamma d + d^2)} \\ &\quad \times \frac{1}{d}(\mu\delta + \mu d + \delta d + \gamma d + d^2) \\ &= \frac{[A\beta - d(\gamma + \mu + d)]}{\beta d} < \frac{A}{d}.\end{aligned}\quad (8)$$

Now, we study the existence of traveling wave solutions for system (5) connecting E^0 and E^* .

Substituting $N(x, t) = \phi(x + ct)$, $I(x, t) = \varphi(x + ct)$, $R(x, t) = \psi(x + ct)$ into (5), and denoting $x + ct$ still by t , we derive the following wave profile system from (5):

$$\begin{aligned}
 D\phi''(t) - c\phi'(t) - d\phi(t) + \mu\varphi(t) &= 0, \\
 D\varphi''(t) - c\varphi'(t) &+ \beta\left(\frac{A}{d} - \phi(t) - \varphi(t) - \psi(t)\right) \frac{\varphi(t - c\tau)}{1 + \alpha\varphi(t - c\tau)} \\
 - (\gamma + \mu + d)\varphi(t) &= 0, \\
 D\psi''(t) - c\psi'(t) + \gamma\varphi(t) - (\delta + d)\psi(t) &= 0.
 \end{aligned} \tag{9}$$

Note that $\mathfrak{R}_0 > 1$ imply $A\beta/d - (\gamma + \mu) > d$. Moreover, we have

$$\mu \frac{k_2}{k_1} = d, \quad \gamma \frac{k_2}{k_3} - \delta = d. \tag{10}$$

We can select suitable M_1, M_2, M_3 such that $M_i > k_i, i = 1, 2, 3$, which satisfy

$$\begin{aligned}
 \frac{A\beta}{d} - (\gamma + \mu) &> \mu \frac{M_2}{M_1} > d, \\
 \frac{A\beta}{d} - (\gamma + \mu) &> \gamma \frac{M_2}{M_3} - \delta > d, \\
 \frac{A}{d} &> M_1 + M_2 + M_3.
 \end{aligned} \tag{11}$$

Denote $C_{[0, \mathbf{M}]}(\mathbb{R}, \mathbb{R}^3) = \{(\phi, \varphi, \psi) \in C(\mathbb{R}, \mathbb{R}^3) : \mathbf{0} \leq (\phi(s), \varphi(s), \psi(s)) \leq \mathbf{M}\}$, where $\mathbf{M} = (M_1, M_2, M_3)$.

Denote $f = (f_1, f_2, f_3) : C_{[0, \mathbf{M}]}(\mathbb{R}, \mathbb{R}^3) \rightarrow C(\mathbb{R}, \mathbb{R}^3)$:

$$\begin{aligned}
 f_1(\phi, \varphi, \psi)(t) &= -d\phi(t) + \mu\varphi(t), \\
 f_2(\phi, \varphi, \psi)(t) &= \beta\left(\frac{A}{d} - \phi(t) - \varphi(t) - \psi(t)\right) \\
 &\times \frac{\varphi(t - c\tau)}{1 + \alpha\varphi(t - c\tau)} - (\gamma + \mu + d)\varphi(t), \\
 f_3(\phi, \varphi, \psi)(t) &= \gamma\varphi(t) - (\delta + d)\psi(t).
 \end{aligned} \tag{12}$$

For $(\phi, \varphi, \psi) \in C_{[0, \mathbf{M}]}(\mathbb{R}, \mathbb{R}^3)$, by a careful calculation, we have

$$\begin{aligned}
 &|f_1(\phi_1, \varphi_1, \psi_1)(t) - f_1(\phi_2, \varphi_2, \psi_2)(t)| \\
 &\leq (d + \mu) |\Phi(t) - \Psi(t)|_{\mathbb{R}^3}, \\
 &|f_2(\phi_1, \varphi_1, \psi_1)(t) - f_2(\phi_2, \varphi_2, \psi_2)(t)| \\
 &\leq L |\Phi(t) - \Psi(t)|_{\mathbb{R}^3}, \\
 &|f_3(\phi_1, \varphi_1, \psi_1)(t) - f_3(\phi_2, \varphi_2, \psi_2)(t)| \\
 &\leq (\gamma + \delta + d) |\Phi(t) - \Psi(t)|_{\mathbb{R}^3},
 \end{aligned} \tag{13}$$

where $L := A\beta/d + \beta(1 + \alpha M_2^2 + M_1 + M_3 + 2M_2 + 2M_2/(1 + \alpha M_2)) + (\gamma + \mu + d)$, $\Phi = (\phi_1, \varphi_1, \psi_1)$, $\Psi = (\phi_2, \varphi_2, \psi_2)$.

For the positive constants ρ_1, ρ_2, ρ_3 , we define $H : C_{[0, \mathbf{M}]}(\mathbb{R}, \mathbb{R}^3) \rightarrow C(\mathbb{R}, \mathbb{R}^3)$ by

$$\begin{aligned}
 H_1(\phi, \varphi, \psi)(t) &= f_1(\phi, \varphi, \psi)(t) + \rho_1\phi(t), \\
 H_2(\phi, \varphi, \psi)(t) &= f_2(\phi, \varphi, \psi)(t) + \rho_2\varphi(t), \\
 H_3(\phi, \varphi, \psi)(t) &= f_3(\phi, \varphi, \psi)(t) + \rho_3\psi(t).
 \end{aligned} \tag{14}$$

Then operators H_1, H_2, H_3 have the following properties.

Lemma 1. For $0 \leq \phi_2(t) \leq \phi_1(t) \leq M_1, 0 \leq \varphi_2(t) \leq \varphi_1(t) \leq M_2, 0 \leq \psi_2(t) \leq \psi_1(t) \leq M_3$, one has

$$\begin{aligned}
 &(i) \\
 &H_1(\phi_1, \varphi_1, \psi_1)(t) \geq H_1(\phi_2, \varphi_2, \psi_2)(t), \\
 &H_3(\phi_1, \varphi_1, \psi_1)(t) \geq H_3(\phi_2, \varphi_2, \psi_2)(t);
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 &(ii) \\
 &H_2(\phi_1, \varphi_1, \psi_1)(t) \geq H_2(\phi_1, \varphi_2, \psi_1)(t), \\
 &H_2(\phi_2, \varphi_1, \psi_1)(t) \geq H_1(\phi_1, \varphi_1, \psi_1)(t), \\
 &H_2(\phi_1, \varphi_1, \psi_2)(t) \geq H_2(\phi_1, \varphi_1, \psi_1)(t).
 \end{aligned} \tag{16}$$

Proof. According to the definitions of f and H , we have

$$\begin{aligned}
 &H_1(\phi_1, \varphi_1, \psi_1)(t) - H_1(\phi_2, \varphi_2, \psi_2)(t) \\
 &= (\rho_1 - d)(\phi_1(t) - \phi_2(t)) + \mu(\varphi_1(t) - \varphi_2(t)), \\
 &H_3(\phi_1, \varphi_1, \psi_1)(t) - H_3(\phi_2, \varphi_2, \psi_2)(t) \\
 &= (\rho_3 - d - \delta)(\psi_1(t) - \psi_2(t)) \\
 &\quad + \gamma(\phi_1(t) - \phi_2(t)).
 \end{aligned} \tag{17}$$

Let $\rho_1 = d, \rho_3 = d + \delta$; we obtain the properties for H_1 and H_3 .

For (ii), we have

$$\begin{aligned}
 &H_2(\phi_1, \varphi_1, \psi_1)(t) - H_2(\phi_1, \varphi_2, \psi_1)(t) \\
 &= \beta\left(\frac{A}{d} - \phi_1(t) - \psi_1(t)\right) \\
 &\times \left(\frac{\varphi_1(t - c\tau)}{1 + \alpha\varphi_1(t - c\tau)} - \frac{\varphi_1(t - c\tau)}{1 + \alpha\varphi_1(t - c\tau)}\right) \\
 &- \beta\left(\varphi_1(t) \frac{\varphi_1(t - c\tau)}{1 + \alpha\varphi_1(t - c\tau)} - \varphi_2(t) \frac{\varphi_2(t - c\tau)}{1 + \alpha\varphi_2(t - c\tau)}\right) \\
 &\quad + [\rho_2 - (\gamma + \mu + d)](\varphi_1(t) - \varphi_2(t)).
 \end{aligned} \tag{18}$$

Note that $M_1 + M_3 < A/d$, and $x/(1 + \alpha x)$ is nondecreasing; we have that the first term of the last formula is nonnegative,

and the second term is bigger than $-(\beta M_2/(1+\alpha M_2))(\varphi_1(t) - \varphi_2(t))$. Let $\rho_2 = \beta M_2/(1 + \alpha M_2) + \gamma + \mu + d$; we have $H_2(\phi_1, \varphi_1, \psi_1)(t) \geq H_2(\phi_1, \varphi_2, \psi_1)(t)$. Since

$$\begin{aligned} &H_2(\phi_2, \varphi_1, \psi_1)(t) - H_2(\phi_1, \varphi_1, \psi_1)(t) \\ &= \left[\beta \frac{\varphi_1(t - c\tau)}{1 + \alpha \varphi_1(t - c\tau)} + \rho_2 \right] (\varphi_1(t) - \varphi_2(t)), \\ &H_2(\phi_1, \varphi_1, \psi_2)(t) - H_2(\phi_1, \varphi_1, \psi_1)(t) \\ &= \left[\beta \frac{\varphi_1(t - c\tau)}{1 + \alpha \varphi_1(t - c\tau)} + \rho_2 \right] (\psi_1(t) - \psi_2(t)), \end{aligned} \tag{19}$$

then, for any positive constant ρ_2 , we have $H_2(\phi_2, \varphi_1, \psi_1)(t) \geq H_2(\phi_1, \varphi_1, \psi_1)(t)$, $H_2(\phi_1, \varphi_1, \psi_2)(t) \geq H_2(\phi_1, \varphi_1, \psi_1)(t)$. \square

Remark 2. For H_2 , we can further conclude that $H_2(\phi_2, \varphi_1, \psi_2)(t) \geq H_2(\phi_1, \varphi_2, \psi_1)(t)$ from Lemma 1(ii).

According to the definition of H , system (9) can be written as

$$\begin{aligned} D\phi''(t) - c\phi'(t) - \rho_1\phi(t) + H_1(\phi, \varphi, \psi)(t) &= 0, \\ D\varphi''(t) - c\varphi'(t) - \rho_2\varphi(t) + H_2(\phi, \varphi, \psi)(t) &= 0, \\ D\psi''(t) - c\psi'(t) - \rho_3\psi(t) + H_3(\phi, \varphi, \psi)(t) &= 0. \end{aligned} \tag{20}$$

Define

$$\begin{aligned} \lambda_1 &= \frac{c - \sqrt{c^2 + 4\rho_1 D}}{2D}, & \lambda_2 &= \frac{c + \sqrt{c^2 + 4\rho_1 D}}{2D}, \\ \lambda_3 &= \frac{c - \sqrt{c^2 + 4\rho_2 D}}{2D}, & \lambda_4 &= \frac{c + \sqrt{c^2 + 4\rho_2 D}}{2D}, \\ \lambda_5 &= \frac{c - \sqrt{c^2 + 4\rho_3 D}}{2D}, & \lambda_6 &= \frac{c + \sqrt{c^2 + 4\rho_3 D}}{2D}, \end{aligned} \tag{21}$$

and operator $F = (F_1, F_2, F_3) : C_{[0, M]}(\mathbb{R}, \mathbb{R}^3) \rightarrow C(\mathbb{R}, \mathbb{R}^3)$ by

$$\begin{aligned} &F_1(\phi, \varphi, \psi)(t) \\ &= \frac{1}{D(\lambda_2 - \lambda_1)} \left[\int_{-\infty}^t e^{\lambda_1(t-s)} H_1(\phi, \varphi, \psi)(s) ds \right. \\ &\quad \left. + \int_t^{+\infty} e^{\lambda_2(t-s)} H_1(\phi, \varphi, \psi)(s) ds \right], \\ &F_2(\phi, \varphi, \psi)(t) \\ &= \frac{1}{D(\lambda_4 - \lambda_3)} \left[\int_{-\infty}^t e^{\lambda_3(t-s)} H_2(\phi, \varphi, \psi)(s) ds \right. \\ &\quad \left. + \int_t^{+\infty} e^{\lambda_4(t-s)} H_2(\phi, \varphi, \psi)(s) ds \right], \end{aligned}$$

$$\begin{aligned} &F_3(\phi, \varphi, \psi)(t) \\ &= \frac{1}{D(\lambda_6 - \lambda_5)} \left[\int_{-\infty}^t e^{\lambda_5(t-s)} H_3(\phi, \varphi, \psi)(s) ds \right. \\ &\quad \left. + \int_t^{+\infty} e^{\lambda_6(t-s)} H_3(\phi, \varphi, \psi)(s) ds \right]. \end{aligned} \tag{22}$$

It is easy to see that $F_i(\phi, \varphi, \psi)(t)$ ($i = 1, 2, 3$) satisfy system (20); thus the fixed point of operator F satisfies (9), which is a traveling wave solution of system (5). Therefore, we will use Schauder's fixed point theorem to find the fixed point of F , where the continuity of F is required. For this purpose, let $\nu > 0$; we define a norm for $\Phi(t) = (\phi, \varphi, \psi)(t) \in C(\mathbb{R}, \mathbb{R}^3)$ by

$$|\Phi|_\nu = \sup_{t \in \mathbb{R}} e^{-\nu|t|} \|\Phi(t)\|. \tag{23}$$

Define

$$B_\nu(\mathbb{R}, \mathbb{R}^3) = \{ \Phi \in C(\mathbb{R}, \mathbb{R}^3) : |\Phi|_\nu < \infty \}. \tag{24}$$

Then it is obvious that $(B_\nu(\mathbb{R}, \mathbb{R}^3), |\cdot|_\nu)$ is a Banach space. We also need the following definition of upper and lower solutions for system (9).

Definition 3. A pair of continuous functions $\bar{\Phi} = (\bar{\phi}, \bar{\varphi}, \bar{\psi})$ and $\underline{\Phi} = (\underline{\phi}, \underline{\varphi}, \underline{\psi})$ are called an upper solution and a lower solution of (9), respectively, if there exist finite points T_1, T_2, \dots, T_m such that $\bar{\Phi}, \underline{\Phi}$ are twice differentiable and bounded on $\mathbb{R} \setminus \{T_i\}$, $i = 1, 2, \dots, m$, and satisfy

$$\begin{aligned} D\bar{\phi}''(t) - c\bar{\phi}'(t) + f_1(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t) &\leq 0, \\ D\bar{\varphi}''(t) - c\bar{\varphi}'(t) + f_2(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t) &\leq 0, \\ D\bar{\psi}''(t) - c\bar{\psi}'(t) + f_3(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t) &\leq 0, \\ D\underline{\phi}''(t) - c\underline{\phi}'(t) + f_1(\underline{\phi}, \underline{\varphi}, \underline{\psi})(t) &\geq 0, \\ D\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_2(\underline{\phi}, \underline{\varphi}, \underline{\psi})(t) &\geq 0, \\ D\underline{\psi}''(t) - c\underline{\psi}'(t) + f_3(\underline{\phi}, \underline{\varphi}, \underline{\psi})(t) &\geq 0, \end{aligned} \tag{25}$$

for $t \in \mathbb{R} \setminus \{T_i\}$, respectively.

We assume that a pair of upper-lower solutions $\bar{\Phi} = (\bar{\phi}, \bar{\varphi}, \bar{\psi})$ and $\underline{\Phi} = (\underline{\phi}, \underline{\varphi}, \underline{\psi})$ are given such that

$$\begin{aligned} &(P1) \\ &(0, 0, 0) \leq (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) \leq (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) \\ &\leq (M_1, M_2, M_3); \end{aligned} \tag{26}$$

(P2)

$$\begin{aligned} & \lim_{t \rightarrow -\infty} (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) \\ &= \lim_{t \rightarrow -\infty} (\underline{\bar{\phi}}(t), \underline{\bar{\varphi}}(t), \underline{\bar{\psi}}(t)) = (0, 0, 0), \end{aligned} \tag{27}$$

$$\begin{aligned} & \lim_{t \rightarrow +\infty} (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) \\ &= \lim_{t \rightarrow +\infty} (\underline{\bar{\phi}}(t), \underline{\bar{\varphi}}(t), \underline{\bar{\psi}}(t)) = (k_1, k_2, k_3); \end{aligned}$$

(P3)

$$\underline{\Phi}'(t+) \leq \underline{\Phi}'(t-), \quad \underline{\Phi}'(t+) \geq \underline{\Phi}'(t-) \quad \text{for } t \in \mathbb{R}. \tag{28}$$

Define the set

$$\begin{aligned} \Gamma = \{ & (\phi, \varphi, \psi)(t) \in C_{[0, M]}(\mathbb{R}, \mathbb{R}^3) : \underline{\phi}(t) \leq \phi(t) \leq \bar{\phi}(t), \\ & \underline{\varphi}(t) \leq \varphi(t) \leq \bar{\varphi}(t), \underline{\psi}(t) \leq \psi(t) \leq \bar{\psi}(t)\}. \end{aligned} \tag{29}$$

Then by the property of H , we have the property of F .

Lemma 4. For $0 \leq \phi_2(t) \leq \phi_1(t) \leq M_1, 0 \leq \varphi_2(t) \leq \varphi_1(t) \leq M_2, 0 \leq \psi_2(t) \leq \psi_1(t) \leq M_3$, one has

$$\begin{aligned} F_1(\phi_1, \varphi_1, \psi_1)(t) & \geq F_1(\phi_2, \varphi_2, \psi_2)(t), \\ F_2(\phi_2, \varphi_1, \psi_2)(t) & \geq F_2(\phi_1, \varphi_2, \psi_1)(t), \\ F_3(\phi_1, \varphi_1, \psi_1)(t) & \geq F_3(\phi_2, \varphi_2, \psi_2)(t). \end{aligned} \tag{30}$$

Similar to Lemmas 4.4–4.6 in [7], we have the following lemmas and omit their proofs.

Lemma 5. $F = (F_1, F_2, F_3)$ is continuous with respect to the norm $|\cdot|_v$ in $(B_v(\mathbb{R}, \mathbb{R}^3))$.

Lemma 6. Consider $F : \Gamma \rightarrow \Gamma$.

Lemma 7. $F : \Gamma \rightarrow \Gamma$ is compact with respect to the norm $|\cdot|_v$.

Theorem 8. Assume that $A\beta/d(\gamma + \mu + d) > 1$. If (9) has a pair of upper-lower solutions $\bar{\Phi} = (\bar{\phi}, \bar{\varphi}, \bar{\psi})$ and $\underline{\Phi} = (\underline{\phi}, \underline{\varphi}, \underline{\psi})$ and satisfies (P1), (P2), and (P3), then system (5) has a traveling wave solution.

Proof. According to Lemmas 5–7 and applying Schauder’s fixed point theorem, operator F has a fixed point $\Phi^* = (\phi^*, \varphi^*, \psi^*) \in \Gamma$, which is traveling wave for system (5). Furthermore, by (P2) we have

$$\begin{aligned} & \lim_{t \rightarrow -\infty} (\phi^*, \varphi^*, \psi^*) = (0, 0, 0), \\ & \lim_{t \rightarrow +\infty} (\phi^*, \varphi^*, \psi^*) = (k_1, k_2, k_3). \end{aligned} \tag{31}$$

Therefore, the fixed point is a traveling wave solution for (5) connecting E^0 and E^* . The proof is complete. \square

3. Existence of Traveling Waves

To prove the existence of traveling wave solutions for (5), we only need to construct a pair of upper-lower solutions.

Consider the following functions:

$$\begin{aligned} \Delta_1(\eta, c) &= D\eta^2 - c\eta - d + \mu \frac{M_2}{M_1}, \\ \Delta_2(\eta, c) &= D\eta^2 - c\eta + \frac{A\beta}{d} - (\gamma + \mu + d), \\ \Delta_3(\eta, c) &= D\eta^2 - c\eta - (\delta + d) + \gamma \frac{M_2}{M_3}. \end{aligned} \tag{32}$$

Note that (11) and $\mathfrak{R}_0 > 1$; we know that there exist positive numbers c_1^*, c_2^*, c_3^* such that

$$\begin{aligned} \Delta_1(\eta, c) = 0 & \text{ has two zeros } 0 < \eta_1 < \eta_2, \text{ for } c > c_1^*, \\ \Delta_2(\eta, c) = 0 & \text{ has two zeros } 0 < \eta_3 < \eta_4, \text{ for } c > c_2^*, \\ \Delta_3(\eta, c) = 0 & \text{ has two zeros } 0 < \eta_5 < \eta_6, \text{ for } c > c_3^*. \end{aligned} \tag{33}$$

Denote $c^* = \max\{c_1^*, c_2^*, c_3^*\}$. According to [5, Lemma 3.8], we have $\eta_1 < \eta_3$ and $\eta_5 < \eta_3$.

Assume that $\mu/d + \gamma/(\delta + d) < 1$; we can select $\varepsilon_i > 0$ ($i = 1, 2, \dots, 6$), $\varepsilon_1, \varepsilon_2 \in (0, k_1)$, $\varepsilon_3, \varepsilon_4 \in (0, k_2)$, $\varepsilon_5, \varepsilon_6 \in (0, k_3)$ satisfying the following inequalities:

$$\begin{aligned} & \mu(k_2 + \varepsilon_3) - d(k_1 + \varepsilon_1) < 0, \\ & \beta \left(\frac{A}{d} - k_1 + \varepsilon_2 - k_2 - \varepsilon_3 - k_3 + \varepsilon_6 \right) \\ & \quad - (\gamma + \mu + d) < 0, \\ & \gamma(k_2 + \varepsilon_3) - (\delta + d)(k_3 + \varepsilon_5) < 0, \\ & d(k_1 - \varepsilon_2) - \mu(k_2 - \varepsilon_4) < 0, \\ & (\gamma + \mu + d) - \frac{\beta}{1 + \alpha(k_2 - \varepsilon_4)} \\ & \quad \times \left(\frac{A}{d} - k_1 - \varepsilon_1 - k_2 + \varepsilon_4 - k_3 - \varepsilon_5 \right) < 0, \\ & (\delta + d)(k_3 - \varepsilon_6) - \gamma(k_2 - \varepsilon_4) < 0. \end{aligned} \tag{34}$$

In fact, we first choose $\varepsilon_3, \varepsilon_4 \in (0, k_2)$ such that

$$\left(\frac{\mu}{d} + \frac{\gamma}{\delta + d} \right) \varepsilon_3 < \varepsilon_4, \quad \left(\frac{\mu}{d} + \frac{\gamma}{\delta + d} \right) \varepsilon_4 < \varepsilon_3. \tag{35}$$

For $\varepsilon_3 \in (0, k_2)$, noting that $k_1 = (\mu/d)k_2$ and $k_3 = (\gamma/(\delta + d))k_2$, we can find $\varepsilon_1 \in (0, k_1)$, $\varepsilon_5 \in (0, k_3)$ such that

$$\begin{aligned} k_1 &> \varepsilon_1 > \frac{\mu}{d}\varepsilon_3 = \frac{\mu}{d}(k_2 + \varepsilon_3) - k_1 \\ &\implies \mu(k_2 + \varepsilon_3) - d(k_1 + \varepsilon_1) < 0, \\ k_3 &> \varepsilon_5 > \frac{\gamma}{\delta + d}\varepsilon_3 = \frac{\gamma}{\delta + d}(k_2 + \varepsilon_3) - k_3 \\ &\implies \gamma(k_2 + \varepsilon_3) - (\delta + d)(k_3 + \varepsilon_5) < 0. \end{aligned} \tag{36}$$

For $\varepsilon_4 \in (0, k_2)$, we can find $\varepsilon_2 \in (0, k_1)$, $\varepsilon_6 \in (0, k_3)$ such that

$$\begin{aligned} k_1 &> \varepsilon_2 > \frac{\mu}{d}\varepsilon_4 = k_1 - \frac{\mu}{d}(k_2 - \varepsilon_4) \\ &\implies d(k_1 - \varepsilon_2) - \mu(k_2 - \varepsilon_4) < 0, \\ k_3 &> \varepsilon_6 > \frac{\gamma}{\delta + d}\varepsilon_4 = k_3 - \frac{\gamma}{\delta + d}(k_2 - \varepsilon_4) \\ &\implies (\delta + d)(k_3 - \varepsilon_6) - \gamma(k_2 - \varepsilon_4) < 0. \end{aligned} \tag{37}$$

Furthermore, for $\varepsilon_1 > (\mu/d)\varepsilon_3$, $\varepsilon_5 > (\gamma/(\delta + d))\varepsilon_3$ and (35), we can find suitable $\varepsilon_1, \varepsilon_5$ satisfying $\varepsilon_1 + \varepsilon_5 < \varepsilon_4$. Similarly, we can find suitable $\varepsilon_2, \varepsilon_6$ satisfying $\varepsilon_2 + \varepsilon_6 < \varepsilon_3$. Thus we have

$$\begin{aligned} &\beta \left(\frac{A}{d} - k_1 + \varepsilon_2 - k_2 - \varepsilon_3 - k_3 + \varepsilon_6 \right) - (\gamma + \mu + d) < 0, \\ &(\gamma + \mu + d) - \frac{\beta}{1 + \alpha(k_2 - \varepsilon_4)} \\ &\times \left(\frac{A}{d} - k_1 - \varepsilon_1 - k_2 + \varepsilon_4 - k_3 - \varepsilon_5 \right) < 0. \end{aligned} \tag{38}$$

We define continuous functions $\Phi(t) = (\phi_1(t), \varphi_1(t), \psi_1(t))$ and $\Psi(t) = (\phi_2(t), \varphi_2(t), \psi_2(t))$ as follows:

$$\begin{aligned} \phi_1(t) &= \begin{cases} k_1 e^{\eta t}, & t \leq t_1, \\ k_1 + \varepsilon_1 e^{-\eta t}, & t > t_1, \end{cases} \\ \phi_2(t) &= \begin{cases} 0, & t \leq t_2, \\ k_1 - \varepsilon_2 e^{-\eta t}, & t > t_2, \end{cases} \\ \varphi_1(t) &= \begin{cases} k_2 e^{\eta_3 t}, & t \leq t_3, \\ k_2 + \varepsilon_3 e^{-\eta t}, & t > t_3, \end{cases} \\ \varphi_2(t) &= \begin{cases} 0, & t \leq t_4, \\ k_2 - \varepsilon_4 e^{-\eta t}, & t > t_4, \end{cases} \\ \psi_1(t) &= \begin{cases} k_3 e^{\eta_5 t}, & t \leq t_5, \\ k_3 + \varepsilon_5 e^{-\eta t}, & t > t_5, \end{cases} \\ \psi_2(t) &= \begin{cases} 0, & t \leq t_6, \\ k_3 - \varepsilon_6 e^{-\eta t}, & t > t_6, \end{cases} \end{aligned} \tag{39}$$

where $t_1, t_3, t_5 > 0$, $t_2, t_4, t_6 < 0$, and $\eta > 0$ is a proper constant to be chosen later.

Furthermore, we can conclude that $t_3 \geq \max\{t_1, t_5\}$ and $t_4 \leq \min\{t_2, t_6\}$, which can help us verify the upper-lower solution for system (9). We point out that $\Phi(t)$ and $\Psi(t)$ satisfy (P1), (P2), and (P3) for proper parameters.

Lemma 9. *Suppose $\mu/d + \gamma/(\delta + d) < 1$. Then the functions $\Phi(t)$ and $\Psi(t)$ defined above are upper and lower solutions of (9), respectively.*

Proof. If $t \leq t_1$, $\phi_1(t) = k_1 e^{\eta t}$, $\varphi_1(t) = k_2 e^{\eta_3 t}$, we have

$$\begin{aligned} p_1(t) &:= D\phi_1''(t) - c\phi_1'(t) - d\phi_1(t) + \mu\varphi_1(t) \\ &\leq \left(D\eta_1^2 - c\eta_1 - d + \mu \frac{M_2}{M_1} \right) k_1 e^{\eta t} = 0. \end{aligned} \tag{40}$$

If $t > t_1$, $\phi_1(t) = k_1 + \varepsilon_1 e^{-\eta t}$, $\varphi_1(t) \leq k_2 + \varepsilon_3 e^{-\eta t}$, we know

$$p_1(t) \leq I_1(\eta), \tag{41}$$

where $I_1(\eta) = (D\varepsilon_1\eta^2 + c\varepsilon_1\eta)e^{-\eta t} - d(k_1 + \varepsilon_1 e^{-\eta t}) + \mu(k_2 + \varepsilon_3 e^{-\eta t})$. Then $I_1(0) = \mu(k_2 + \varepsilon_3) - d(k_1 + \varepsilon_1)$. It follows from (34) that $I_1(0) < 0$ and there exists $\eta_1^* > 0$ such that $p_1(t) < 0$ for all $\eta \in (0, \eta_1^*)$.

If $t \leq t_3$, $\varphi_1(t) = k_2 e^{\eta_3 t}$, we obtain that

$$\begin{aligned} p_2(t) &:= D\varphi_1''(t) - c\varphi_1'(t) \\ &+ \beta \left(\frac{A}{d} - \phi_2(t) - \varphi_1(t) - \psi_2(t) \right) \frac{\varphi_1(t - c\tau)}{1 + \alpha\varphi_1(t - c\tau)} \\ &- (\gamma + \mu + d)\varphi_1(t) \\ &\leq (D\eta_2^3 - c\eta_3 - \gamma - \mu - d)k_2 e^{\eta_3 t} + \frac{A\beta}{d} k_2 e^{\eta_3(t - c\tau)} \\ &\leq \left[D\eta_2^3 - c\eta_3 + \frac{A\beta}{d} - \gamma - \mu - d \right] k_2 e^{\eta_3 t} = 0. \end{aligned} \tag{42}$$

If $t > t_3$, $\varphi_1(t) = k_2 + \varepsilon_3 e^{-\eta t}$, $\phi_2(t) = k_1 - \varepsilon_2 e^{-\eta t}$, $\psi_2(t) = k_3 - \varepsilon_6 e^{-\eta t}$, we have

$$p_2(t) \leq I_2(\eta), \tag{43}$$

where

$$\begin{aligned} I_2(\eta) &= (D\varepsilon_3\eta^2 + c\varepsilon_3\eta)e^{-\eta t} \\ &- (\gamma + \mu + d)(k_2 + \varepsilon_3 e^{-\eta t}) \\ &+ \beta \left[\frac{A}{d} - k_1 + \varepsilon_2 e^{-\eta t} - k_2 - \varepsilon_3 e^{-\eta t} - k_3 + \varepsilon_6 e^{-\eta t} \right] \\ &\times (k_2 + \varepsilon_3 e^{-\eta(t - c\tau)}). \end{aligned} \tag{44}$$

It follows from (34) that $I_2(0) < 0$ and there exists $\eta_2^* > 0$ such that $p_2(t) < 0$ for all $\eta \in (0, \eta_2^*)$.

If $t \leq t_5$, $\psi_1(t) = k_3 e^{\eta_5 t}$, $\varphi_1(t) = k_2 e^{\eta_5 t}$, we have

$$p_3(t) := D\psi_1''(t) - c\psi_1'(t) + \gamma\varphi_1(t) - (\delta + d)\psi_1(t) \leq \left[D\eta_5^2 - c\eta_5 - (\delta + d) + \gamma \frac{M_2}{M_3} \right] k_3 e^{\eta_5 t} = 0. \tag{45}$$

If $t > t_5$, $\psi_1(t) = k_3 + \varepsilon_5 e^{-\eta t}$, $\varphi_1(t) \leq k_2 + \varepsilon_3 e^{-\eta t}$,

$$p_3(t) \leq I_3(\eta), \tag{46}$$

where $I_3(\eta) = (D\varepsilon_5\eta^2 + c\varepsilon_5\eta)e^{-\eta t} + \gamma(k_2 + \varepsilon_3 e^{-\eta t}) - (\delta + d)(k_3 + \varepsilon_5 e^{-\eta t})$. We can derive from (34) that there exists $\eta_3^* > 0$ such that $p_3(t) < 0$ for all $\eta \in (0, \eta_3^*)$.

If $t \leq t_2$, $\phi_2(t) = 0$, we have

$$q_1(t) := D\phi_2''(t) - c\phi_2'(t) - d\phi_2(t) + \mu\phi_2(t) = \mu\phi_2(t) \geq 0. \tag{47}$$

If $t > t_2$, $\phi_2(t) = k_1 - \varepsilon_2 e^{-\eta t}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\eta t}$, we have

$$q_1(t) \geq I_4(\eta), \tag{48}$$

where $I_4(\eta) = -(D\varepsilon_2\eta^2 + c\varepsilon_2\eta)e^{-\eta t} - d(k_1 - \varepsilon_2 e^{-\eta t}) + \mu(k_2 - \varepsilon_4 e^{-\eta t})$. It follows from (34) that $I_4(0) > 0$ and there exists $\eta_4^* > 0$ such that $q_1(t) > 0$ for all $\eta \in (0, \eta_4^*)$.

If $t \leq t_4$, $\varphi_2(t) = 0$, we obtain that

$$q_2(t) := D\varphi_2''(t) - c\varphi_2'(t) + \beta \left(\frac{A}{d} - \phi_1(t) - \varphi_2(t) - \psi_1(t) \right) \frac{\varphi_2(t - c\tau)}{1 + \alpha\varphi_2(t - c\tau)} - (\gamma + \mu + d)\varphi_2(t) = 0. \tag{49}$$

If $t > t_4$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\eta t}$, $\varphi_2(t - c\tau) \geq k_2 - \varepsilon_4 e^{-\eta(t-c\tau)}$, $\phi_1(t) \leq k_1 + \varepsilon_1 e^{-\eta t}$, $\psi_1(t) \leq k_3 + \varepsilon_5 e^{-\eta t}$, we have

$$q_2(t) \geq I_5(\eta), \tag{50}$$

where

$$I_5(\eta) = - (D\varepsilon_4\eta^2 + c\varepsilon_4\eta) e^{-\eta t} - (\gamma + \mu + d)(k_2 - \varepsilon_4 e^{-\eta t}) + \beta \left(\frac{A}{d} - k_1 - \varepsilon_1 e^{-\eta t} - k_2 + \varepsilon_4 e^{-\eta t} - k_3 - \varepsilon_5 e^{-\eta t} \right) \frac{k_2 - \varepsilon_4 e^{-\eta(t-c\tau)}}{1 + \alpha k_2 - \alpha \varepsilon_4 e^{-\eta(t-c\tau)}}. \tag{51}$$

It follows from (34) that $I_5(0) > 0$ and there exists $\eta_5^* > 0$ such that $q_2(t) > 0$ for all $\eta \in (0, \eta_5^*)$.

If $t \leq t_6$, $\psi_2(t) = 0$, we have

$$q_3(t) := D\psi_2''(t) - c\psi_2'(t) + \gamma\varphi_2(t) - (\delta + d)\psi_2(t) \geq 0. \tag{52}$$

If $t > t_6$, $\psi_2(t) = k_3 - \varepsilon_6 e^{-\eta t}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\eta t}$, we have

$$q_3(t) \geq I_6(\eta), \tag{53}$$

where $I_6(\eta) = -(D\varepsilon_6\eta^2 + c\varepsilon_6\eta)e^{-\eta t} + \gamma(k_2 - \varepsilon_4 e^{-\eta t}) - (\delta + d)(k_3 - \varepsilon_6 e^{-\eta t})$. It follows from (34) that $I_6(0) > 0$ and there exists $\eta_6^* > 0$ such that $q_3(t) > 0$ for all $\eta \in (0, \eta_6^*)$.

Thus, taking $\eta \in (0, \min_{1 \leq i \leq 6} \{\eta_i^*\})$, we prove that $\Phi(t)$ and $\Psi(t)$ are upper and lower solutions of (9). \square

Now we obtain and state the main result in this paper.

Theorem 10. *Assume that $A\beta/d(\gamma + \mu + d) > 1$ and $\mu/d + \gamma/(\delta + d) < 1$; then, for any $c > c^*$, (5) has a traveling wave solution connecting two equilibria E^0 and E^* . Furthermore, system (4) has a traveling wave solution with speed c , which connects two states $(A/d, 0, 0)$ and $(A/d - k_1, k_2, k_3)$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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