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Research Article

Bifurcation Analysis and Spatiotemporal Patterns of Nonlinear Oscillations in a Ring Lattice of Identical Neurons with Delayed Coupling

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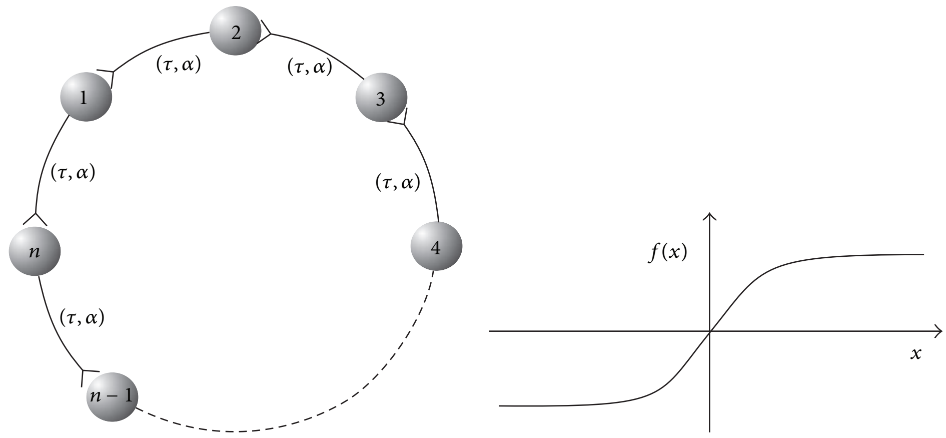
We investigate the dynamics of a delayed neural network model consisting of n identical neurons. We first analyze stability of the zero solution and then study the effect of time delay on the dynamics of the system. We also investigate the steady state bifurcations and their stability. The direction and stability of the Hopf bifurcation and the pitchfork bifurcation are analyzed by using the derived normal forms on center manifolds. Then, the spatiotemporal patterns of bifurcating periodic solutions are investigated by using the symmetric bifurcation theory, Lie group theory and S^1-equivariant degree theory. Finally, two neural network models with four or seven neurons are used to verify our theoretical results.

1. Introduction

Growing biological evidence suggests that propagation delay in axons and dendrites may play a key role in the processing of sensory information by the brain. For instance, spatiotemporal patterns of synchronous oscillations in γ and θ frequency bands emerge in the hippocampus when a rat goes through previously visited places. These oscillations and their interaction form a coding scheme that is used to readout from long-time memory (see, e.g., [1]). Although there are many experimental results pointing to the γ phase-locked firing of pyramidal cells (see, e.g., [2]), the mechanisms underlying this phenomenon are fairly unknown. The hippocampus has a laminar stratificated structure, where each pyramidal cell receives and integrates a large amount of spikes arriving at different time instants to different parts of the cell. Then their active integration by the cell, to produce an output spike, depends on the relative time delays of the incoming spikes. Thus the cell output is conditioned by delays occurring in incoming electric wave, which justifies the need of mathematical modeling of these complex phenomena [3].

On the other hand, artificial neural networks including delays have been known to be useful for mimicking various neuroprocesses like in the image processing (see, e.g., [4]). Their further expansion and exploitation are limited by the added complexity of the mathematical analysis brought by the delay (ordinary or partial differential equations become functional differential equations thus implying an infinite number of degrees of freedom). There are, however, some advantages. For instance, a harmonic oscillator augmented with time delay shows stable "robust" oscillations [5]. The "fragile" character of linear oscillations and linear waves when unfolded in space precludes their utility in reliably carrying information. An interesting point, however, is that a harmonic oscillator augmented with white noise permits transferring the latter into a colored noise thus creating a memory-like stochastic process and so a kind of delay process. In [6] this idea has been used to model quasiharmonic oscillations observed in inferior olive. The spiking propagation process in the brain comes indeed from robust oscillations as known by many authors. This fact permits modeling the dynamics using harmonic oscillators with appropriate nonlinearity which is like in the case of a van der Pol-Bonhöffer oscillator and



(a) (b)

FIGURE 1: Lattice ring. (a) Network architecture with n units unidirectionally coupled. Each unit receives a delayed output from the preceding unit. (b) Transfer (synaptic) function.

the Fitzhugh-Nagummo excitability dynamics underlying the Hodgkin-Huxley equation. An alternative to van der Pol's idea was earlier suggested by Lord Ragleigh who also proposed augmenting the harmonic oscillator with suitable active fiction thus allowing maintained oscillations. In both cases we have systems where there is an appropriate input-output energy balance and thus driven-dissipative systems.

Focusing on robots or neuron inspired artifacts, the use of ring lattice models has shown potential in different applications, for example, from electron transport to the dynamical decentralized gait control in robotics [7]. In the latter case movement of each robot leg is associated (coupled) with the behavior of the corresponding unit in the ring. Then standing waves excited in the ring can be used to reproduce the leg movement without the need of a computer, unlike what happens in nature, which uses central pattern generators (CPG) designing artificial locomotion for this purpose [8]. It has been shown in [9] for a hexapod robot that a ring of six units is able to reproduce metachronal, caterpillar, and tripod gaits. However, ripple gait, widely employed by insects for medium speed movement, is difficult to mimic. We foresee that the use of delays may help in CPGs to control the robot walking behavior.

Since Marcus and Westervelt [10] incorporated a single time delay into the connection term of Hopfield's model and observed sustained oscillations resulting from this time delay, there has been a growing interest to the dynamical properties of neural networks when the time delay cannot be neglected. It is known that delay can significantly alter the dynamics of neural networks [10–12], for example, leading to instability of a delay-free stable solution [13]. There have been lots of researches on the neural networks with delay (see, e.g., [14–18] and references therein).

A ring with identical elements often has a kind of topological symmetry. Considering these symmetries helps in the analysis of wave solutions in a ring lattice. Golubitsky et al. [19] have shown that systems with symmetry can exhibit different oscillatory patterns, which are predictable based on the theory of equivariant bifurcations. Later, in a series of papers, Wu and coworkers [20–22] have extended the theory of equivariant Hopf bifurcations to functional differential equations. These theoretical advances have led to a vast literature on the mechanisms of spatiotemporal activity in neural networks with symmetry and delays (see, e.g., [22–35]). The majority of these studies have focused on a ring structure with bidirectional couplings between the neighboring elements. From the view point of group theory, the symmetry in a ring with such bidirectional coupling if there is a reflection symmetry is given by a dihedral group D_n (the symmetry of a regular n-gon: n is the number of units). Another type of symmetry in a ring is a cyclic group Z_n (the symmetry of a direct n-gon) if one direction is preferred; that is, the coupling goes in one direction. The latter type of symmetry is more natural for the neural networks. In 1994, Baldi and Atiya [11] proposed a neural network consisting of n neurons connected in a ring by undirectional couplings with delays (Figure 1(a)):

$$\frac{dx_i}{dt} = -x_i(t) + \alpha_{i+1} f_{i+1}\left(x_{i+1}\left(t - \tau_{i+1}\right)\right), \quad i \,(\operatorname{mod} n), \quad (1)$$

where x_i is the state of the ith unit (related to the membrane potential), f is the transfer function, which describes chemical interaction between pre- and postsynaptic neurons, and α accounts for the coupling strength. For positive or negative α we can speak of *excitatory* or *inhibitory* coupling, respectively. The nonnegative constants τ_i represent the transmission delays between neighboring neurons.

Baldi and Atiya [11] investigated the effects of delays on the ring dynamics, in particular on the oscillatory properties, showing numerically that when the ring has *odd* number of units and the coupling is *inhibitory* ($\alpha < 0$), the origin (the zero solution) is the only fixed point, and if it is unstable, the dynamics of the network always converges to a stable limit cycle. They also argued that, for *even* n and $\alpha < 0$, there are no stable limit cycles. But a strict mathematical proof is still lacking. Model (1) has been studied by several researchers.

In [36, 37], the authors studied a ring with two and three units and obtained sufficient conditions for the asymptotic stability of the zero equilibrium and for the existence of Hopf bifurcations. Later Wei and Li investigated the global existence of multiple periodic solutions in a ring of three units [38]. The stability analysis and local Hopf bifurcation of the zero solution in a ring of four units have been given in [39]. In the above-mentioned studies the authors considered nonidentical elements, where the theory of equivariant bifurcations is not applicable. Besides, the properties of nonzero equilibria have not been addressed. To fill this gap Guo [40] studied a ring (1) consisting of four identical neurons coupled with the same delay. He provided conditions on the linear stability of zero solution, spatiotemporal patterns of nonlinear oscillations, albeit restricting consideration to the case of *excitatory* coupling ($\alpha > 1$). The direction and stability of bifurcating periodic solutions and steady state bifurcations were not considered. Thus there is no result on the case of the undirectional ring consisting of arbitrary number of units. In particular, little is known about the patterns of nonlinear oscillations and their stability even in a ring of more than four identical neurons. The steady state bifurcation patterns are also lacking. This paper is a generalization of the previous investigation for the case of the undirectional ring consisting of two or three neurons with discrete delay in [37]. We obtain not only the stability and delay-induced Hopf bifurcation but also the spatiotemporal patterns of periodic oscillations. We would also like to mention that the rings of neural networks with unidirectional coupling and distributed delays have recently been considered in [41–43]. The paper also can be considered the complement of the results in [41–43] for the case of distributed delays.

In this paper, we consider model (1) consisting of n (arbitrary) identical neurons; that is, $\alpha_i \equiv \alpha$, $f_i \equiv f$ coupled with the same delay $\tau_i \equiv \tau$. We assume that the synaptic coupling is described by a sufficiently smooth sigmoidal function (Figure 1(b)). A widely used example in the literature is $f(x) = \tanh(x)$. However, here we only assume $f(0) = 0$, $f \in C^1(\mathbb{R})$ for the stability analysis, and we require $f \in C^3(\mathbb{R})$, $f'(0)f'''(0) \neq 0$, $f(0) = f''(0) = 0$ for the bifurcation analysis. We investigate the global and local stability, Hopf bifurcations, pitchfork bifurcations, and spatiotemporal patterns of bifurcating periodic solutions. We also derive normal forms on center manifolds and determine direction and stability of the Hopf bifurcations.

2. Background

For convenience, we recall results for the model (1) for the case of a ring composed of n identical neurons

$$\frac{dx_i}{dt} = -x_i(t) + \alpha f\left(x_{i+1}(t-\tau)\right), \quad i \,(\mathrm{mod}\, n). \tag{2}$$

System (2) admits the zero as an equilibrium solution $x_0 = (0, 0, \dots, 0)$, which we will refer to as the rest state.

Every initial state $\varphi \in C$ uniquely defines a solution $x(\varphi, t)$ of system (2) for all $t \geq -\tau$. Due to the uniqueness of the Cauchy initial value problem of system (2), every initial state φ satisfying $\varphi_1 = \varphi_2 = \cdots = \varphi_n$ (called synchronous phase point) gives a synchronous solution $x(\varphi, t)$, that is, $x_1(\varphi, t) = x_2(\varphi, t) = \cdots = x_n(\varphi, t)$, and the solution $x_i(\varphi, t)$ of system (2) can be characterized by the scalar delay differential equation

$$\frac{dx}{dt} = -x(t) + \alpha f(x(t-\tau)). \tag{3}$$

System (3) has been studied by many researchers. For example, Mallet-Paret and Nussbaum [44] have obtained some results on global continuation and asymptotical behavior of periodic solutions, and Krisztin et al. [45] and Krisztin and Walther [46] give a complete description of the global attractor of (3) as a three-dimensional smooth solid spindle when τ is in a certain range. A few results from Mallet-Paret and Nussbaum [44] are needed for our subsequent work.

Lemma 1. *Assume that f is a smooth sigmoid-like function, x_* is a steady state of* (3), *and $\tau_{0,j}$ is defined by* (13) *with $j = 0, 1, \dots$.*

(i) *If $|\alpha f'(x_*)| < 1$, then x_* is asymptotically stable for any $\tau \geq 0$.*

(ii) *If $\alpha f'(x_*) < -1$, then x_* is asymptotically stable for $\tau \in [0, \tau_{0,0})$ and unstable for $\tau > \tau_{0,0}$.*

(iii) *If $\alpha f'(x_*) > 1$, then x_* is unstable for any $\tau \geq 0$.*

(iv) *If $|\alpha f'(x_*)| > 1$, then* (3) *undergoes a Hopf bifurcation at x_* when $\tau = \tau_{0,j}$.*

(v) *If $|\alpha f'(x_*)| > 1$, then* (3) *has at least $(j+1)$ periodic solution for $\alpha < -1$ with $\tau > \tau_{0,j}$ or $\alpha > 1$ with $\tau > \tau_{0,j+1}$.*

3. Stability and Hopf Bifurcations of the Rest State

In this section, we assume

(H1) $f \in C^1(\mathbb{R})$, $f(0) = 0$, $f'(0) = 1$.

Let us first study the global stability of the rest solution x_0.

Lemma 2. *If $|\alpha| < 1$, then the solution x_0 of system* (2) *is globally asymptotically stable for any $\tau \geq 0$.*

The proof is given in Appendix A.

Lemma 2 shows that the dynamics of the network (2) is simple when $|\alpha| < 1$. Any perturbation of the network state decays in time. Therefore, in the remainder of this paper, we will investigate the dynamics of (2) with $|\alpha| \geq 1$.

The linearization of system (2) around the rest state x_0 is given by

$$\frac{dx_i}{dt} = -x_i(t) + \alpha x_{i+1}(t-\tau), \quad i \,(\mathrm{mod}\, n). \tag{4}$$

The characteristic matrix associated with system (4) is

$$M_n(\tau, \lambda) = (\lambda + 1) I - \alpha e^{-\lambda\tau} M, \tag{5}$$

where I is $n\times n$ identity matrix and

$$M=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\1&0&0&\cdots&0\end{pmatrix}_{n\times n}. \tag{6}$$

Let $\chi=e^{(2\pi/n)i}$, $v_k=(1,\chi^k,\chi^{2k},\ldots,\chi^{(n-1)k})^T$, $k=0,1,\ldots,n-1$. Then, we get

$$M_n(\tau,\lambda)\,v_k=\left(\lambda+1-\alpha\chi^k e^{-\lambda\tau}\right)v_k. \tag{7}$$

It follows that the characteristic equation is

$$\det M_n(\tau,\lambda)=\prod_{k=0}^{n-1}\Delta_k(\tau,\lambda)=0, \tag{8}$$

where

$$\Delta_k(\tau,\lambda)=\lambda+1-\alpha e^{(2k\pi/n)i}e^{-\lambda\tau}. \tag{9}$$

An equilibrium solution of a delay differential equation is locally asymptotically stable if all roots of the corresponding characteristic equation have negative real parts and unstable if at least one root has positive real part (see, e.g., [47]). Thus, in order to investigate the stability of the rest state, it is necessary to study the distribution of roots of the characteristic equation (8) and when they cross the imaginary axis.

It is straightforward to see that if $|\alpha|>1$, then $\lambda=0$ is not a root of $\Delta_k(\tau,\lambda)=0$ for any $k\in\{0,1,2,\ldots,n-1\}$ and $\tau\ge 0$. Therefore, in what follows we assume that two complex roots cross the imaginary axis at some positive critical values of time delay; that is, $\lambda(\tau)=iv$. Then from (8) and (9) it follows that v satisfies

$$iv+1-\alpha e^{(2k\pi/n-v\tau)i}=0, \tag{10}$$

which gives

$$\begin{aligned}1&=\alpha\cos\left(\frac{2k\pi}{n}-v\tau\right),\\ v&=\alpha\sin\left(\frac{2k\pi}{n}-v\tau\right).\end{aligned} \tag{11}$$

Then for $|\alpha|>1$, $v=\pm\omega_0$ with $\omega_0=\sqrt{\alpha^2-1}$. Let us first consider $\alpha<-1$. Set

$$\begin{aligned}\tilde{\tau}^+_{k,j}&=\frac{1}{\omega_0}\left\{\frac{2k\pi}{n}+\arccos\frac{1}{\alpha}+(j-1)\,2\pi\right\},\\ \tilde{\tau}^-_{k,j}&=\frac{1}{\omega_0}\left\{-\frac{2k\pi}{n}+\arccos\frac{1}{\alpha}+2j\pi\right\},\end{aligned} \tag{12}$$

where $k=0,1,\ldots,n-1$ and $j\in\{0,1,\ldots\}$. It is easy to verify that $\Delta_k(\tau,\lambda)$ has a purely imaginary root $\omega_0 i$ for $\tau=\tilde{\tau}^+_{k,j}$ and $-\omega_0 i$ for $\tau=\tilde{\tau}^-_{k,j}$. Since $\tilde{\tau}^-_{k,j}=\tilde{\tau}^+_{n-k,j}$, one can conclude that the characteristic equation (8) with $\alpha<-1$ has a pair of simple purely imaginary roots $\pm i\omega_0$ at $\tau=\tilde{\tau}^-_{k,j}$. Proceeding for the case $\alpha>1$ exactly as done above we can get similar results. In fact, we can set

$$\tau_{k,j}=\begin{cases}\dfrac{1}{\omega_0}\left\{-\dfrac{2k\pi}{n}+\arccos\dfrac{1}{\alpha}+2j\pi\right\}, & \text{for } \alpha<-1,\\ \dfrac{1}{\omega_0}\left\{\dfrac{2k\pi}{n}-\arccos\dfrac{1}{\alpha}+2j\pi\right\}, & \text{for } \alpha>1,\end{cases} \tag{13}$$

with $k=0,1,\ldots,n-1$ and $j\in\{0,1,\ldots\}$ such that $\tau_{k,j}>0$. Then the characteristic equation (8) has a pair of simple purely imaginary roots $\pm i\omega_0$ at $\tau=\tau_{k,j}$.

Differentiating $\Delta_k(\tau,\lambda)$ with respect to λ we obtain that

$$\begin{aligned}\frac{\partial}{\partial\lambda}\Delta_k(\tau,\lambda)\Big|_{\lambda=i\omega_0,\tau=\tau_{k,j}}&=1+\tau\alpha e^{(2k\pi/n)i}e^{-\lambda\tau}\Big|_{\lambda=i\omega_0,\tau=\tau_{k,j}}\\ &=1+\tau_{k,j}\left(1+i\omega_0\right)\neq 0.\end{aligned} \tag{14}$$

Thus, the implicit function theorem implies that there exist $\delta>0$ and a smooth curve $\lambda:(\tau_{k,j}-\delta,\tau_{k,j}+\delta)\to\mathbb{C}$ such that $\Delta_k(\tau,\lambda(\tau))=0$ and $\lambda(\tau_{k,j})=i\omega_0$. Differentiating $\Delta_k(\tau,\lambda)$ with respect to τ after some algebraic calculus we get

$$\operatorname{Re}\left\{\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_{k,j}}=\frac{\omega_0^2}{\tau_{k,j}^2\omega_0^2+\left(1+\tau_{k,j}\right)^2}>0. \tag{15}$$

Now we can state the following results about the distribution of roots of (8).

Lemma 3. *Assume that $\tau_{k,j}$ is defined by (13) with $k=0,1,\ldots,n-1$ and $j\in\{0,1,\ldots\}$ such that $\tau_{k,j}>0$.*

(I) *If n is even, then when $|\alpha|>1$ the characteristic equation (8) has at least one root with positive real part for all $\tau\ge 0$.*

(II) *If n is odd, then*

(i) *when $\alpha<\sec((n-1)\pi/n)$ or $\alpha>1$ the characteristic equation (8) has at least one root with positive real part for all $\tau\ge 0$;*

(ii) *when $\sec((n-1)\pi/n)<\alpha<-1$ all roots of the characteristic equation (8) have negative real parts for $\tau\in[0,\tau_{(n-1)/2,0})$ and all roots of the characteristic equation (8) except $\pm i\omega_0$ have negative real parts at $\tau=\tau_{(n-1)/2,0}$, but the characteristic equation (8) has at least two roots with positive real parts for $\tau>\tau_{(n-1)/2,0}$.*

(III) *If $|\alpha|>1$, then when $\tau=\tau_{k,j}$, the characteristic equation (8) has a pair of simple purely imaginary roots $\pm i\omega_0$.*

The proof is given in Appendix B.

From Lemma 3, the transversality condition (15), and the standard Hopf bifurcation theorem of delay differential equations (see, e.g., [47]), we can state the following theorem.

Theorem 4. *Assume that $|\alpha| > 1$ and $\tau_{k,j}$ is defined by* (13).

(I) *If n is even, then the rest state x_0 of system* (2) *is unstable for all $\tau \geq 0$.*

(II) *If n is odd, then*

(i) *when $\alpha < \sec((n-1)\pi/n)$ or $\alpha > 1$ the rest state x_0 of system* (2) *is unstable for all $\tau \geq 0$;*

(ii) *when $\sec((n-1)\pi/n) < \alpha < -1$ the rest state x_0 of system* (2) *is asymptotically stable for $\tau \in [0, \tau_{(n-1)/2,0})$ and unstable for $\tau > \tau_{(n-1)/2,0}$.*

(III) *The system* (2) *undergoes a Hopf bifurcation at $\tau = \tau_{k,j}$. That is, there exists a unique branch of periodic solutions $x^{(k,j)}(t,\tau)$ with period $p^{(k,j)}(\tau)$, and $x^{(k,j)}(t,\tau) \to 0$, $p^{(k,j)}(\tau) \to 2\pi/\omega_0$ as $\tau \to \tau_{k,j}$.*

Theorem 5. *(I) Assume that $\alpha = 1$. Then for all $\tau \geq 0$ the characteristic equation* (8) *has one simple root $\lambda = 0$, and other roots have negative real parts.*

(II) Assuming that $\alpha = -1$ and n is even, then, for all $\tau \geq 0$, the characteristic equation (8) *has one simple root $\lambda = 0$, and other roots have negative real parts.*

The proof is given in Appendix C.

4. Stability and Steady State Bifurcations

In this section, we study the properties of the equilibrium of the system (2) and assume

(H2) $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(0) = 1$, $f'(x) > 0$ $\forall x \in \mathbb{R}$, $xf''(x) < 0$ $\forall x \neq 0$, $\lim_{x\to\pm\infty}|f(x)| < +\infty$.

From system (2), we immediately have the following lemma.

Lemma 6. *If $(x_1, x_2, \ldots, x_n)$ is an equilibrium of system* (2) *and there exists one component $x_i = 0$, then it must be the zero equilibrium $x_0 = (0, 0, \ldots, 0)$.*

Theorem 7. *Assuming that $\alpha > 1$, every equilibrium of system* (2) *must be synchronous and system* (2) *has exactly three synchronous equilibria: the zero equilibrium x_0, the negative equilibrium $x_- = (u_-, u_-, \ldots, u_-)$, and the positive equilibrium $x_+ = (u_+, u_+, \ldots, u_+)$, where $u_\pm$ are nonzero solutions of the equation $u = \alpha f(u)$. Moreover, these two nonzero equilibria x_- and x_+ are both asymptotically stable for all $\tau \geq 0$.*

The proof is given in Appendix D.

Theorem 8. *Assuming that n is odd and $\alpha \leq 1$, system* (2) *has just one zero equilibrium x_0.*

The proof is given in Appendix E.

Theorem 9. *If n is even and $\alpha < -1$, then system* (2) *has exactly three equilibria: x_0, $x_1^* = (u_+, u_-, \ldots, u_+, u_-)$, $x_2^* = (u_-, u_+, \ldots, u_-, u_+)$, where $u_\pm$ are the nonzero solutions of $u = \alpha f(\alpha f(u))$. Moreover, these two nonzero equilibria are both asymptotically stable for all $\tau \geq 0$.*

The proof is given in Appendix F. From Theorems 7, 8, and 9, we have the following.

Theorem 10. *System* (2) *undergoes a pitchfork bifurcation at $|\alpha| = 1$ when n is even and at $\alpha = 1$ when n is odd.*

Remark 11. From Theorems 7, 8, and 9, we also know that the size of the network does not affect the position of the steady states (see Figures 2 and 3 obtained from numerical simulations) and their stability.

5. Center Manifold Reduction and Normal Forms

In this section, we will apply the method of Faria and Magalhães [48, 49] to obtain normal forms on center manifold. After that, we will study the properties of Hopf and steady state bifurcations. Without loss of generality, we assume

(H3) $f \in C^3(\mathbb{R})$, $f(0) = f''(0) = 0$, $f'(0) = 1$, $f'''(0) \neq 0$.

Firstly, we introduce a new time scale $t \mapsto t/\tau$ to normalize the delay and give universalities to our analysis. Then system (2) can be written as

$$\dot{z}(t) = F(z_t, \tau), \tag{16}$$

in the phase space $C = C([-1, 0], \mathbb{R}^n)$. For $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T \in C$, we have

$$(F(\varphi, \tau))_i = -\tau\varphi_i(0) + \tau\alpha f(\varphi_{i+1}(-1)), \quad i \,(\mathrm{mod}\, n). \tag{17}$$

Under the assumption (H3), we can expand f in the Taylor series

$$f(z) = z + \gamma z^3 + \text{h.o.t.}, \tag{18}$$

where $\gamma = (1/3!)f'''(0)$ and h.o.t. stands for the higher order terms.

5.1. Properties of Hopf Bifurcations: $|\alpha|>1$. From Lemma 3 and Theorem 4, we know that for $|\alpha| > 1$ the characteristic equation (8) has a pair of simple purely imaginary roots $\pm i\omega_0$ and system (2) undergoes a Hopf bifurcation at the critical delay values $\tau_{k,j}$. In this subsection, we will determine the direction and stability of Hopf bifurcations by calculating explicitly the normal form of system (2) on the associated two-dimensional (2D) center manifold. For a Hopf bifurcation, the normal form on the center manifold is given in polar coordinates ρ, ξ by

$$\begin{aligned} \dot{\rho} &= K_1\mu\rho + K_2\rho^3 + O\left(\mu^2\rho + |(\rho, \mu)|^4\right), \\ \dot{\xi} &= -\omega_* + O\left(|(\rho, \mu)|\right). \end{aligned} \tag{19}$$

Here, $\mu = \tau - \tau_{k,j}$ is the bifurcation parameter, $\omega_* = \omega_0\tau_{k,j}$, and the coefficients K_1 and K_2 are real numbers. The qualitative behavior of the asymptotic solutions of (16) (or equivalently of (2)) is the same as the behavior of solutions

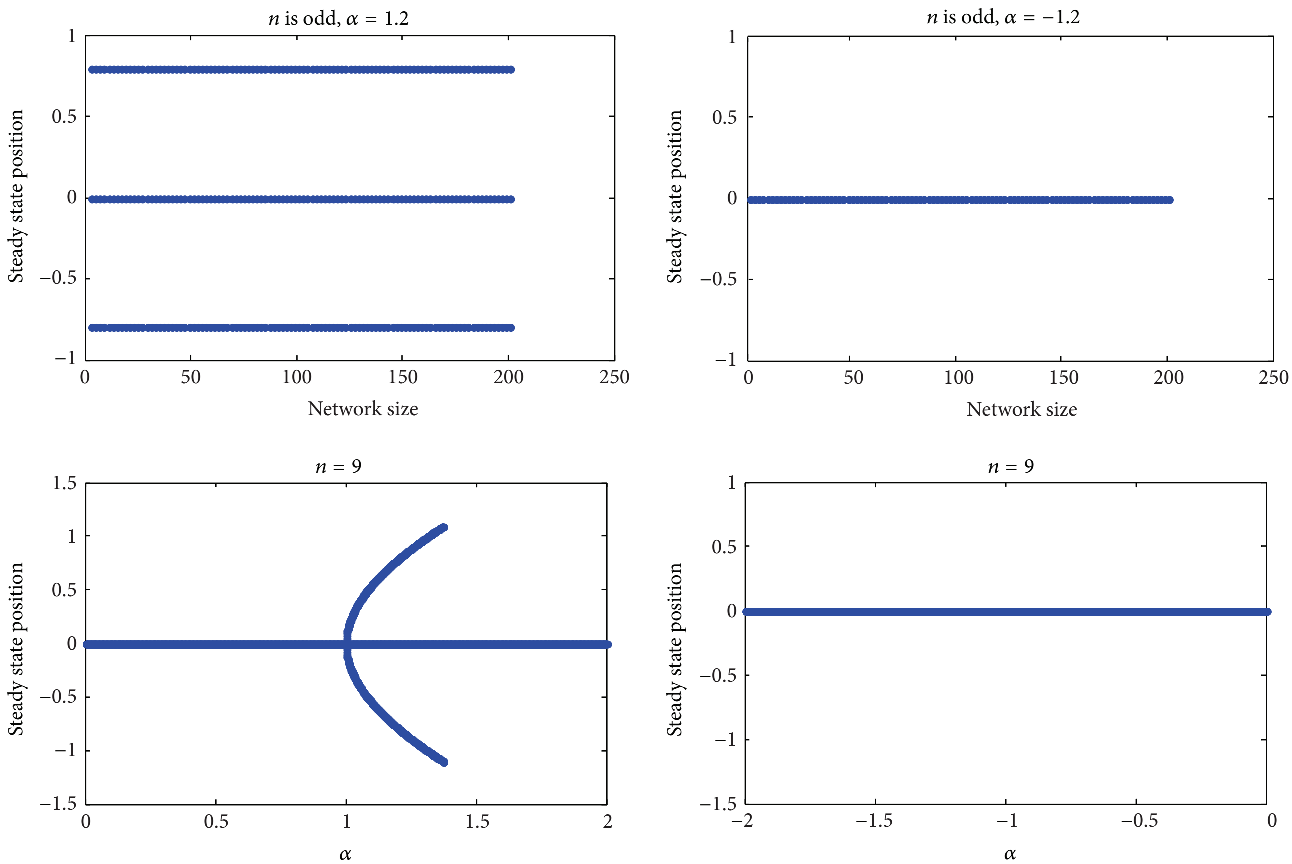


FIGURE 2: Steady state pitchfork bifurcation of system (2) for n odd.

of (19), which, in turn, only depends on the signs of the two coefficients K_1 and K_2. The sign of the product K_1K_2 determines the direction of the bifurcation (supercritical if $K_1K_2 < 0$ and subcritical if $K_1K_2 > 0$), and the sign of K_2 determines the stability of the nontrivial periodic orbits (stable if $K_2 < 0$ and unstable if $K_2 > 0$) (see, e.g., [50]).

The derivation of the normal form coefficients K_1 and K_2 are given in Appendix G. According to Appendix G, we have

$$K_1 = \frac{\omega_0^2\tau_{k,j}}{\left(1+\tau_{k,j}\right)^2 + \left(\omega_0\tau_{k,j}\right)} > 0,$$

$$K_2 = 3\gamma\tau_{k,j}\,\mathrm{Re}\left\{(\overline{a})^{-1}\left(1+i\omega_0\right)\right\} \quad (20)$$

$$= \frac{1}{2}f'''(0)\,\tau_{k,j}\frac{1+\tau_{k,j}+\omega_0^2\tau_{k,j}}{(1+\tau^*)^2+\left(\omega_0\tau_{k,j}\right)^2}.$$

Consequently, we have the following theorem.

Theorem 12. *If $f'''(0) < 0$ (resp., $f'''(0) > 0$), then the Hopf bifurcations occurring on the center manifold of system (2) at $x = 0, \tau = \tau_{k,j}$ are supercritical (resp., subcritical), with nontrivial periodic stable (resp., unstable) orbits on the center manifold.*

Remark 13. From Lemma 3, we know that the characteristic equation (8) has at least one root with positive real part if one of the following conditions is satisfied:

(i) n is even and $|\alpha| > 1$;

(ii) n is odd and either $\alpha < \sec((n-1)\pi/n)$ or $\alpha > 1$;

(iii) n is odd, $\tau > \tau_{(n-1)/2,0}$, and $\sec((n-1)\pi/n) < \alpha < -1$.

Therefore, if one of the above three conditions is satisfied, then the periodic solutions bifurcating from the rest state at the critical value of τ must be unstable in the whole phase space although they are stable on the center manifold.

However, when n is odd and $\sec((n-1)\pi/n) < \alpha < -1$, all roots of the characteristic equation (8) except $\pm i\omega_0$ have negative real parts at $\tau = \tau_{(n-1)/2,0}$. Thus in this case the stability of periodic solutions bifurcating at $\tau = \tau_{(n-1)/2,0}$ on the center manifold is equivalent to that of periodic solutions in the whole phase space.

5.2. The Steady State, Pitchfork Bifurcation: $|\alpha| = 1$. We first consider the case when n is even and $\alpha = -1$. From Theorem 5, we know that the characteristic equation (8) has a simple root $\lambda = 0$ for all $\tau \geq 0$. To study the bifurcation in

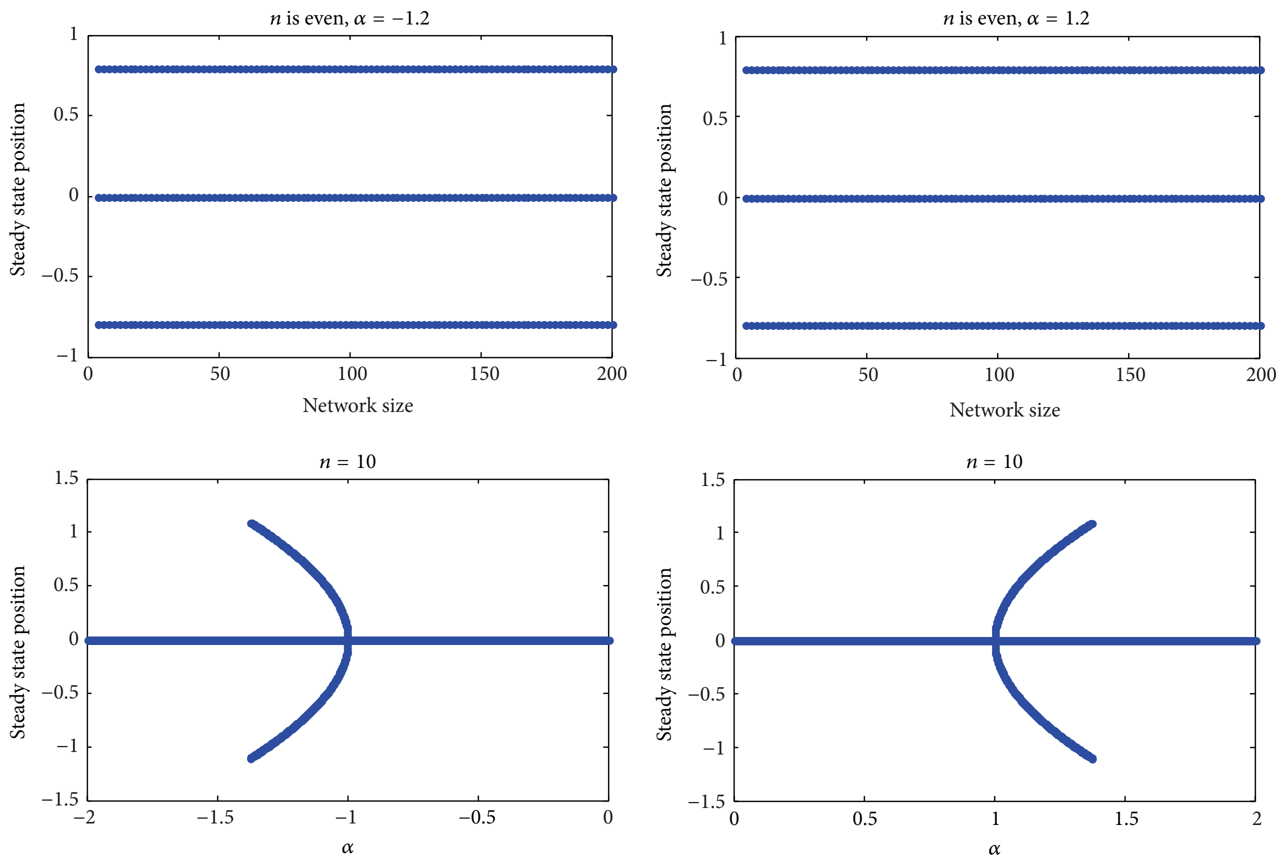


FIGURE 3: Steady state pitchfork bifurcation of system (2) for n even.

detail we introduce a new bifurcation parameter $\alpha = -1 + \nu$. Then the normal form (H.4) becomes (see Appendix H)

$$\dot{x} = -\frac{\tau}{1+\tau}\nu x + \frac{\gamma\tau}{1+\tau}x^3 + \cdots . \tag{21}$$

This normal form, together with the results in [51], permits the following theorem to hold.

Theorem 14. *Suppose that n is even and $\alpha = -1$. Then the rest state x_0 of system (2) is stable if $f'''(0) < 0$ and unstable if $f'''(0) > 0$ for any $\tau \geq 0$.*

Next if $\alpha = 1$, letting $\alpha = 1 + \nu$, one can obtain

$$\begin{aligned} \Phi(\theta) &= (1, 1, \ldots, 1)^T, \quad -1 \leq \theta \leq 0, \\ \Psi(s) &= \frac{1}{n(1+\tau)}(1, 1, \ldots, 1), \quad 0 \leq s \leq 1, \end{aligned} \tag{22}$$

and then get the same normal form (21). So, we can obtain the following theorem.

Theorem 15. *The rest state x_0 of system (2) is stable if $f'''(0) < 0$ and unstable if $f'''(0) > 0$ for any $\tau \geq 0$ and $\alpha = 1$.*

Now we can explicitly determine the stability of the zero solution of the system (2) on the (α, τ) plane. According to the results stated in Theorems 4, 5, 14, and 15, the stable regions of

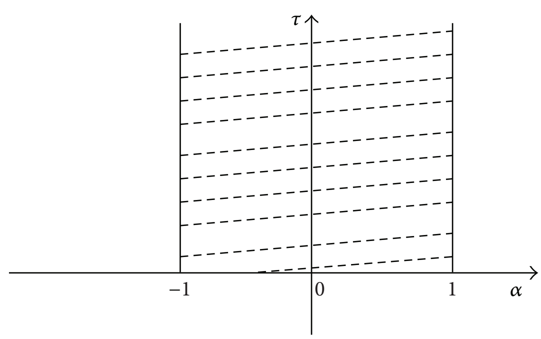


FIGURE 4: Stable region (shaded area) of the rest state x_0 of system (2) for n even.

the rest state of system (2) with $f'''(0) < 0$ can be illustrated graphically in Figures 4 and 5.

6. Spatiotemporal Patterns of Bifurcating Periodic Solutions

Earlier we have described the Hopf bifurcation at the critical value $\tau_{k,j}$ leading to a family of periodic solutions. In this section, we will investigate the spatiotemporal patterns of

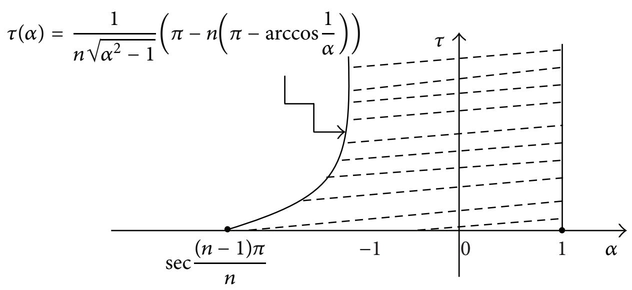


Figure 5: Stable region (shaded area) of the rest state x_0 of system (2) for n odd.

these bifurcating periodic solutions and refer to [22] for explanations of notations involved.

Let us first introduce notations from the theory of compact groups.

(i) Z_n is the cyclic group of order n, which corresponds to rotations of $2\pi/n$. Denoting the generator of this group by ρ, its action on $\mathbb{R}^n$ is given by

$$(\rho x_i) = x_{i+1}, \quad \forall i \,(\mathrm{mod}\, n), \quad x \in \mathbb{R}^n. \tag{23}$$

(ii) Let $G : C \to \mathbb{R}^n$ and Γ be a compact group. The system $\dot{x}(t) = G(x_t)$ is said to be Γ-equivariant if $G(\gamma x_t) = \gamma G(x_t)$ for all $\gamma \in \Gamma$.

It is clear that system (2) is Z_n equivariant. Let $\omega = 2\pi/\omega_0$. Denote by P_ω the Banach space of all continuous ω-periodic mappings from $\mathbb{R}$ into $\mathbb{R}^n$, equipped with the supremum norm. Then, for the circle group S^1, $Z_n \times S^1$ acts on P_ω by

$$\begin{gathered}\left(\rho, e^{i\theta}\right) x(t) = \rho x\left(t + \frac{\omega}{2\pi}\theta\right), \\ \left(\rho, e^{i\theta}\right) \in Z_n \times S^1, \ x \in P_\omega.\end{gathered} \tag{24}$$

Denote by SP_ω the subspace of P_ω of all ω-periodic solutions of (4) with $\tau = \tau_{k,j}$. Then

$$SP_\omega = \left\{y_1 \epsilon^1(t) + y_2 \epsilon^2(t), y_1, y_2 \in \mathbb{R}\right\}, \tag{25}$$

where $\epsilon^1(t)$ and $\epsilon^2(t)$ are n-dimensional vector functions with the jth components defined by $\epsilon^1_j(t) = \cos(\omega_0 t + 2(j-1)k\pi/n)$, $\epsilon^2_j(t) = \sin(\omega_0 t + 2(j-1)k\pi/n)$, respectively. For each subgroup $\Sigma \le Z_n \times S^1$, it is clear that the fixed point set

$$\mathrm{Fix}(\Sigma, SP_\omega) = \left\{x \in SP_\omega; \left(\rho, e^{i\theta}\right)x = x \quad \forall \left(\rho, e^{i\theta}\right) \in \Sigma\right\} \tag{26}$$

is a subspace.

Lemma 16. *Consider*

$$\rho\epsilon^1(t) = \epsilon^1\left(t + \frac{k\omega}{n}\right), \qquad \rho\epsilon^2(t) = \epsilon^2\left(t + \frac{k\omega}{n}\right). \tag{27}$$

The proof is given in Appendix I.

It is known from [19] that the subgroups of $Z_n \times S^1$, up to conjugacy, describe the symmetry of periodic solutions of system (2) which exhibit certain spatiotemporal patterns given by

$$\Sigma = \left\{\left(\rho, e^{(2k\pi/n)i}\right); 0 \le k \le n-1\right\}. \tag{28}$$

Form Lemma 16, it follows that $\mathrm{Fix}(\Sigma, SP_\omega) = SP_\omega$ which means that

$$\dim \mathrm{Fix}(\Sigma, SP_\omega) = 2. \tag{29}$$

This, together with Lemma 16, allows us to apply the symmetric Hopf bifurcation theorem for delay differential equations due to Wu [22] to obtain the following results.

Theorem 17. *Suppose that $|\alpha| > 1$ and $\tau_{k,j}$ is defined by* (13) *with $k = 0, 1, \ldots, n-1$ and $j \in \{0, 1, \ldots\}$ such that $\tau_{k,j} > 0$. Then near $\tau_{k,j}$ there exists a branch of small-amplitude periodic solutions of system* (2) *with period p near $2\pi/\omega_0$, satisfying*

$$x_{i+1}(t) = x_i\left(t - \frac{k}{n}p\right), \quad i\,(\mathrm{mod}\, n). \tag{30}$$

7. Examples and Numerical Simulations

In this section, we consider two examples with four neurons and seven neurons, respectively, to justify our theoretical results.

7.1. Four-Neural Network Model (n, Even). Consider a four-neuron network model

$$\frac{dx_i}{dt} = -x_i(t) + \alpha \tanh\left(x_{i+1}(t-\tau)\right), \quad i\,(\mathrm{mod}\, 4). \tag{31}$$

From (13), we get the Hopf bifurcation curves

$$\tau_{k,j} = \begin{cases} \dfrac{1}{\sqrt{\alpha^2-1}} \\ \quad \times \left\{-\dfrac{(k-2)\pi}{2} - \arccos\dfrac{1}{|\alpha|} + 2j\pi\right\}, & \text{for } \alpha < -1, \\ \dfrac{1}{\sqrt{\alpha^2-1}}\left\{\dfrac{k\pi}{2} - \arccos\dfrac{1}{\alpha} + 2j\pi\right\}, & \text{for } \alpha > 1, \end{cases} \tag{32}$$

with $k = 0, 1, 2$ and $j \in \{0, 1, \ldots\}$ such that $\tau_{k,j} > 0$. It follows that these Hopf bifurcation curves are symmetric about τ-axis (see Figure 6). From Sections 2 and 3, we can obtain the following results.

Corollary 18. *(i) If $-1 \le \alpha \le 1$, then the rest state x_0 of system* (31) *is stable for all $\tau \ge 0$.*

(ii) If $|\alpha| > 1$, then the rest state x_0 of system (31) *is unstable but the other two nonzero steady states are both stable for all $\tau \ge 0$.*

(iii) System (31) *undergoes a Hopf bifurcation at $\tau_{k,j}$ (as shown in Figure 6) and bifurcating periodic solutions are all unstable in the phase space (although stable on the center manifold).*

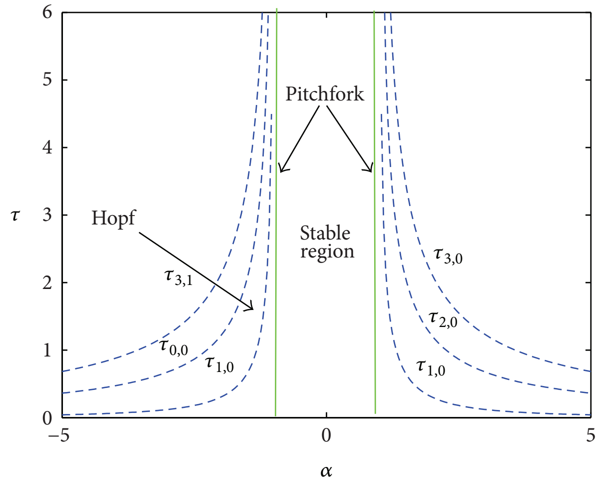


FIGURE 6: The curves of Hopf and pitchfork bifurcations of system (31).

For $\alpha = -1.5$, the system (31) has three equilibria x_0, $x_1^* = (1.2878, -1.2878, 1.2878, -1.2878)$ and $x_2^* = (-1.2878, 1.2878, -1.2878, 1.2878)$. Figure 7 shows the evolution of system (31) starting with the initial value $(0.3, 0.5, -0.5, 1)$ and $\tau = 2$.

Taking $\alpha = 1.5$, system (31) has three equilibria x_0, $x_1^* = (1.2878, 1.2878, 1.2878, 1.2878)$ and $x_2^* = (-1.2878, -1.2878, -1.2878, -1.2878)$. The zero equilibrium is unstable and the other two nonzero equilibria are both stable (see Figure 8).

7.2. Seven-Neuron Network Model (n, Odd). Consider a seven-neuron network model

$$\frac{dx_i}{dt} = -x_i(t) + \alpha \tanh(x_{i+1}(t-\tau)), \quad i \,(\mathrm{mod}\, 7). \tag{33}$$

Let

$$\tau_{k,j} = \begin{cases} \dfrac{1}{\sqrt{\alpha^2-1}} \left\{ -\dfrac{2k\pi}{7} + \arccos\dfrac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha < -15 \\ \dfrac{1}{\sqrt{\alpha^2-1}} \left\{ \dfrac{2k\pi}{7} - \arccos\dfrac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha > 1, \end{cases} \tag{34}$$

with $k = 0, 1, \ldots, 6$ and $j \in \{0, 1, \ldots\}$ such that $\tau_{k,j} > 0$. Then from Sections 2 and 3, we have the following results.

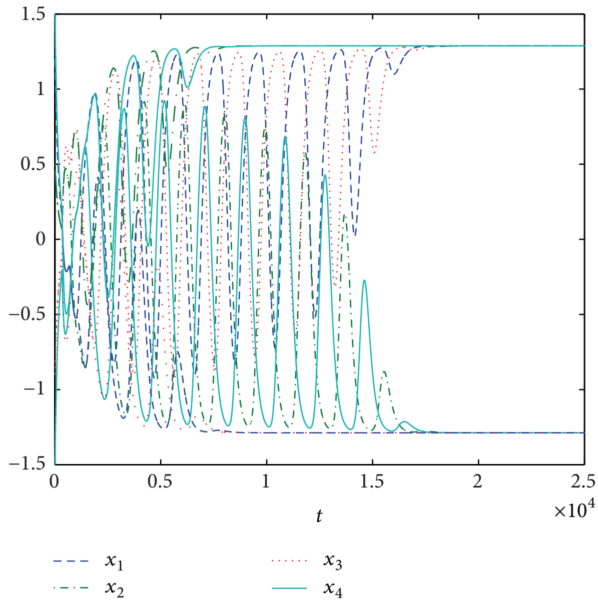
Corollary 19. *(i) If $-1 \le \alpha \le 1$, then the rest state x_0 of system* (33) *is stable for all $\tau \ge 0$.*

(ii) If $\alpha < \sec(6\pi/7)$ or $\alpha > 1$, then the rest state x_0 of system (33) *is unstable for all $\tau \ge 0$.*

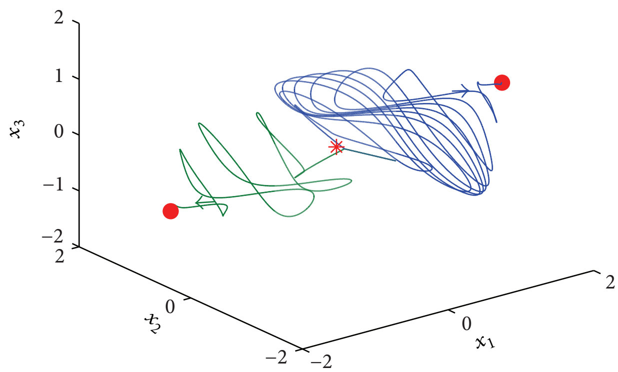
(iii) If $\sec(6\pi/7) < \alpha < -1$, then the rest state x_0 of system (33) *is asymptotically stable for $\tau \in [0, \tau_{3,0}]$ and unstable for $\tau > \tau_{3,0}$.*

(iv) System (33) *undergoes a Hopf bifurcation at $\tau_{k,j}$ (as shown in Figure 9) and the bifurcating periodic solutions satisfy*

$$x_{i+1} = x_i\left(t - \frac{k}{7}p\right), \quad i \,(\mathrm{mod}\, 7). \tag{35}$$



(a)



(b)

FIGURE 7: Evolution of system (31) with $\alpha = -1.5, \tau = 2$. The rest state x_0 of system (2) is unstable and the other two nonzero equilibria are stable. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3 of system (2).

Moreover, the bifurcating periodic solutions at the first bifurcating point $\tau_{3,0}$ are stable and others unstable in the phase space.

To perform the numerical calculation, we consider the parameter value $\sec(6\pi/7) < \alpha = -1.08 < -1$. It follows from (34) that

$$0 < \tau_{3,0} \approx 0.1507 < \tau_{2,0} \approx 2.3511 < \tau_{1,0} \approx 4.5516 < \cdots. \tag{36}$$

Taking $\tau = 0.1 < \tau_{3,0}$, Figure 10 shows that the rest state x_0 of system (33) is asymptotically stable.

For $\tau_{3,0} < \tau = 0.2 < \tau_{2,0}$, it follows from Corollary 19 that the rest state x_0 of system (33) becomes unstable and there exists a small-amplitude phase-locked oscillation with

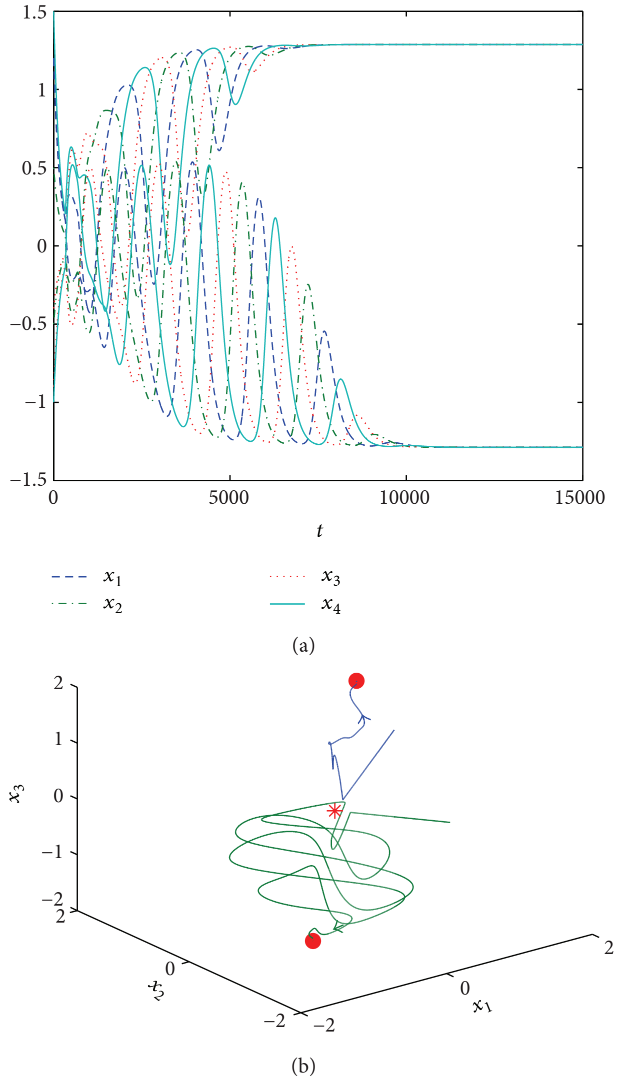


FIGURE 8: Evolution of (31) with $\alpha = 1.5, \tau = 2$. The rest state x_0 of system (2) is unstable and the other two nonzero equilibria are stable. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3 of system (2).

period p satisfying $x_{i+1} = x_i(t-(3/7)p), i(\text{mod } 7)$. This result is illustrated in Figure 11.

Taking $\tau = 4.5$ far away from the first critical value $\tau_{3,0}$, the numerical simulation shows the existence of phase-locked oscillations with period p satisfying $x_{i+1} = x_i(t-(3/7)p)$, $i(\text{mod}7)$ (see Figure 12). The amplitude of this periodic oscillation is larger than that of periodic oscillation shown in Figure 11. This large-amplitude phase-locked oscillation is very similar to the square waves reported in [52, 53] for singularly perturbed delay equations. This phase-locked oscillation with period p satisfying $x_{i+1} = x_i(t-(3/7)p), i(\text{mod } 7)$ also shows that this stable bifurcating periodic solution comes from the first critical value $\tau_{3,0}$ not from other critical values.

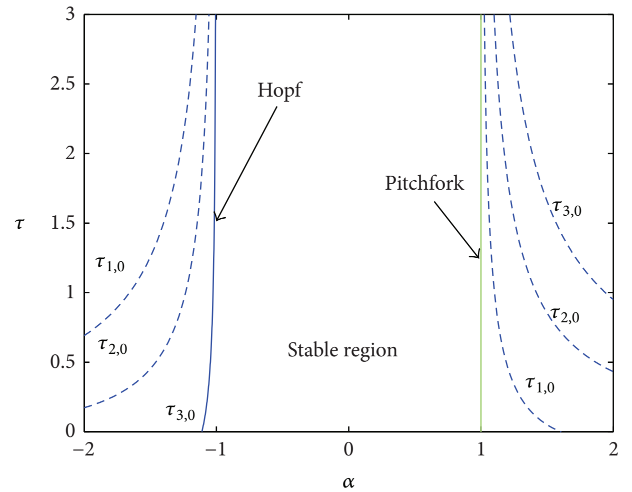


FIGURE 9: Stable region and curves of Hopf and pitchfork bifurcations of system (33) for the rest state x_0.

Taking $\alpha = 1.5$, system (33) has three synchronous equilibria x_0, $x_1^* = (1.2878, 1.2878, \ldots, 1.2878)$ and $x_2^* = (-1.2878, -1.2878, \ldots, -1.2878)$. The zero equilibrium is unstable and the other two nonzero equilibria are both stable (see Figure 13).

8. Conclusions

In this paper, we have studied a lattice ring of n identical neurons coupled with the same delay. We have analyzed the global stability and delay dependent local stability regions and conditions and explored the different types of bifurcation after the stability is lost. We also have shown that the stability of the zero equilibrium state depends not only on the synaptic strength α and time delay τ but also on the parity of the network size. However, if the nonzero equilibrium exists, it is always stable.

We have also studied the stability of the zero equilibrium on the pitchfork values $|\alpha| = 1$ and have given easy-to-check conditions on the stability and direction of Hopf bifurcations. For example, if $f'''(0) < 0$, then when n is *odd* the periodic orbits bifurcating from the first critical value $\tau_{(n-1)/2,0}$ are stable in the phase space, but when n is *even* all bifurcating periodic orbits are unstable in the phase space.

On the other hand, we have analyzed the spatiotemporal patterns of nonlinear oscillations by using the symmetric bifurcation theory of delay differential equations coupled with representation theory of cyclic groups. Finally, using MATLAB software, we have done some numerical simulations showing the existence of the stable equilibrium and phase-locked periodic waves arising from Hopf bifurcations.

We have generalized recently results for lattice rings with a few units to an arbitrary number. The results obtained are expected to be of interest to neurodynamics and to scientists interested in robots. Indeed we have shown that there are significant features different for n even at qualitative level and odd which do not disappear when n goes to large. A finding of

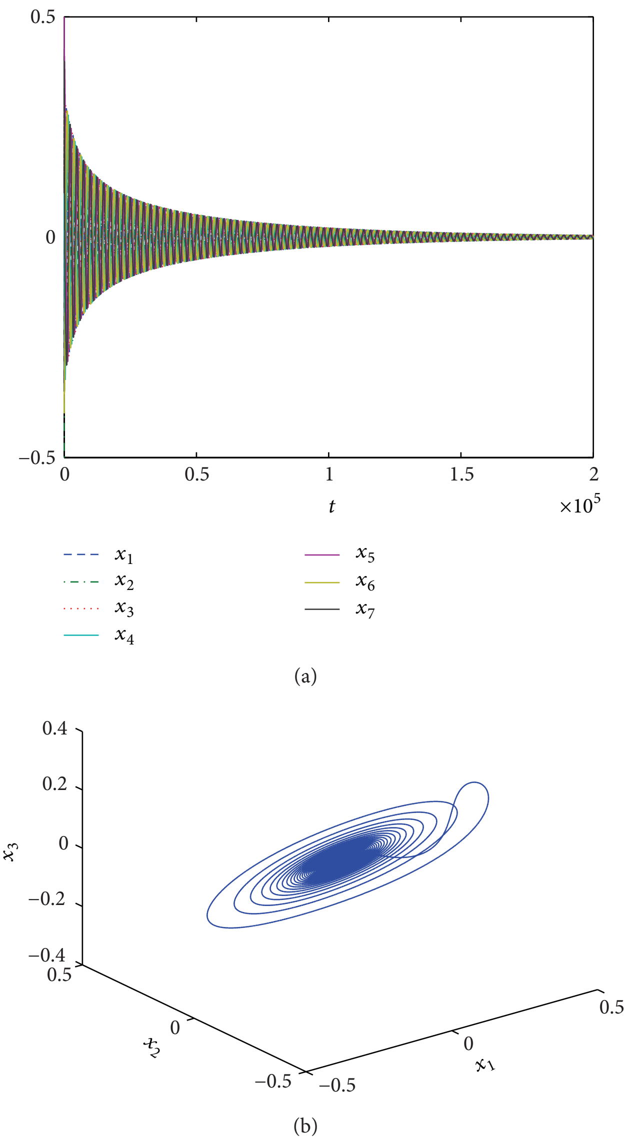


FIGURE 10: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3 of system (33) at $\tau = 0.1 < \tau_{3,0}$.

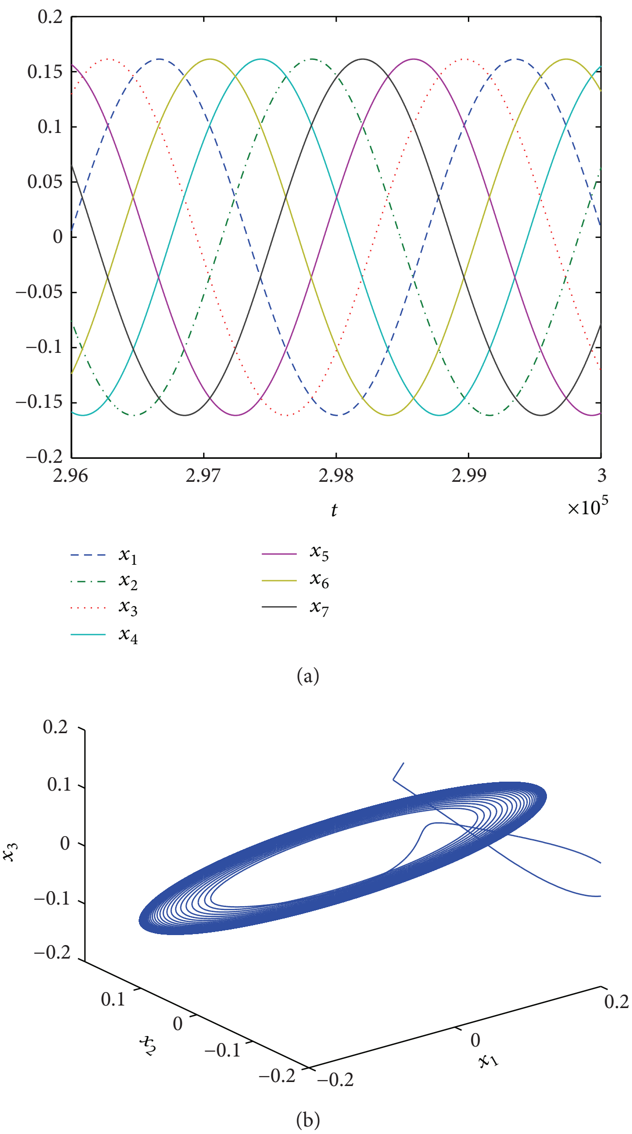


FIGURE 11: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3 of system (33) at $\tau = 0.2 \in (\tau_{3,0}, \tau_{2,0})$.

practical interest is that for what enough time intervals these differences are not so relevant.

Appendices

A. Proof of Lemma 2

Using the assumption (H1), we can write $f(x_i(t-\tau))$ as

$$f\left(x_i\left(t-\tau\right)\right) = p_i\left(t\right) x_i\left(t-\tau\right), \tag{A.1}$$

where

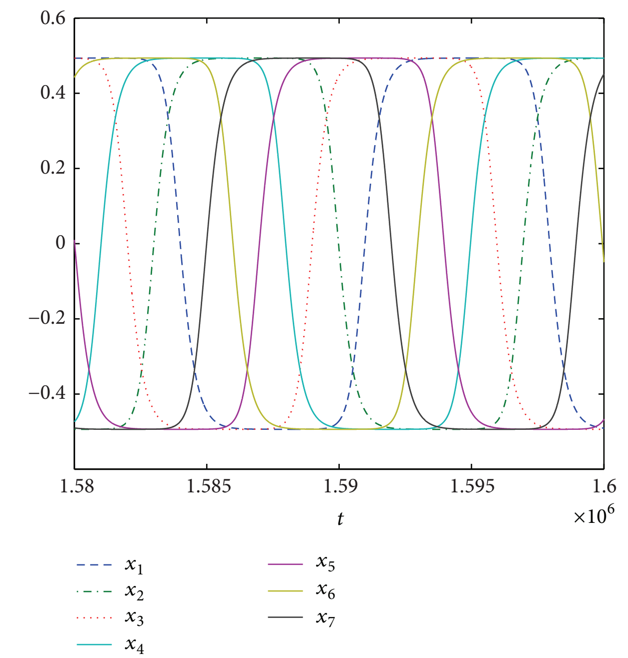
$$p_i\left(t\right) = \int_0^1 f'\left(s x_i\left(t-\tau\right)\right) ds, \tag{A.2}$$

and find $p^* \in (0,1]$ such that $p_i(t) \leq p^*$ for all $t \geq 0$ and i. Thus, system (2) becomes

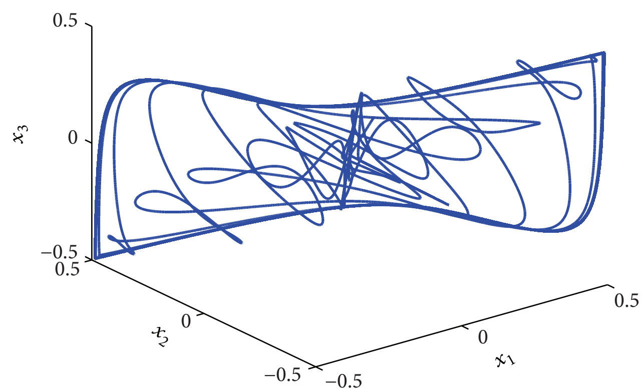
$$\frac{dx_i}{dt} = -x_i\left(t\right) + \alpha p_{i+1}\left(t\right) x_{i+1}\left(t-\tau\right), \quad i\left(\operatorname{mod} n\right). \tag{A.3}$$

Using the Lyapunov functional

$$V\left(t\right) = \sum_{i=1}^{n} x_i^2\left(t\right) + \left|\alpha\right| \sum_{i=1}^{n} \int_{t-\tau}^{t} x_{i+1}^2\left(s\right) ds, \tag{A.4}$$

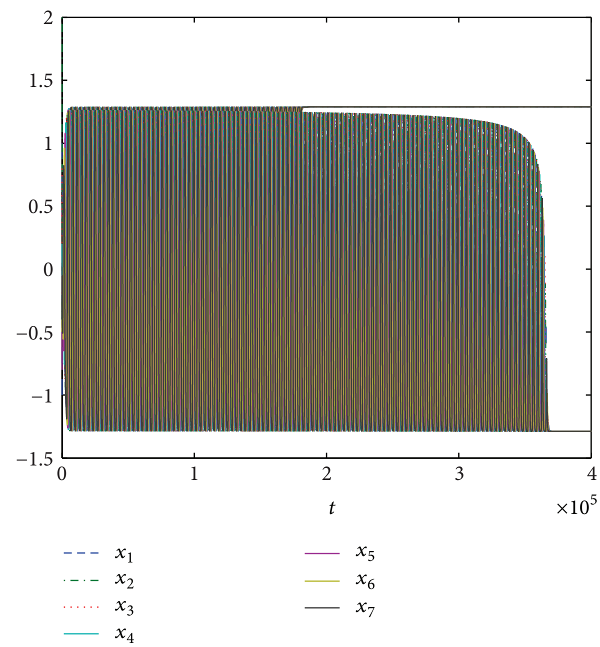


(a)

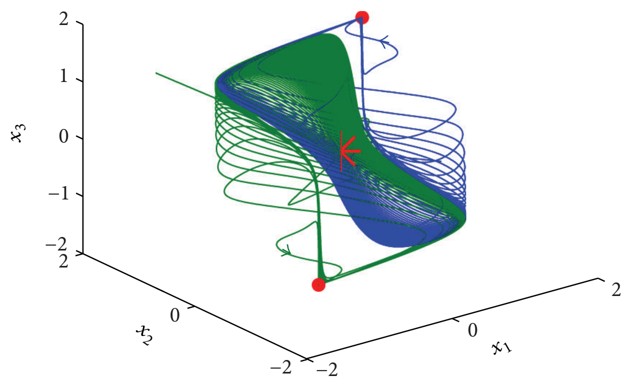


(b)

FIGURE 12: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3 of system (33) at $\tau = 4.5$ far away from the first critical value $\tau_{3,0}$.



(a)



(b)

FIGURE 13: Evolution of system (33) with $\alpha = 1.5, \tau = 2$ starting from different initial values. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components x_1, x_2, and x_3. The rest state x_0 of system (2) is unstable and two nonzero equilibria x_1^* and x_2^* are stable.

we obtain

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(A.3)}
&= 2\sum_{i=1}^{n} x_i(t)\,\dot{x}_i(t) + |\alpha| \sum_{i=1}^{n}\left(x_{i+1}^2(t) - x_{i+1}^2(t-\tau)\right) \\
&= 2\sum_{i=1}^{n} x_i(t)\left(-x_i(t) + \alpha p_{i+1}(t)\, x_{i+1}(t-\tau)\right) \\
&\quad + |\alpha| \sum_{i=1}^{n}\left(x_{i+1}^2(t) - x_{i+1}^2(t-\tau)\right) \\
&\le -2\sum_{i=1}^{n} x_i^2(t) + \sum_{i=1}^{n} 2|\alpha|\,|x_i(t)|\,|x_{i+1}(t-\tau)| \\
&\quad + |\alpha| \sum_{i=1}^{n}\left(x_{i+1}^2(t) - x_{i+1}^2(t-\tau)\right) \\
&\le -2\sum_{i=1}^{n} x_i^2(t) + |\alpha| \sum_{i=1}^{n}\left(x_i^2(t) + x_{i+1}^2(t-\tau)\right) \\
&\quad + |\alpha| \sum_{i=1}^{n}\left(x_{i+1}^2(t) - x_{i+1}^2(t-\tau)\right) \\
&= -2(1-|\alpha|)\sum_{i=1}^{n} x_i^2(t).
\end{aligned} \tag{A.5}$$

For $x(t) \neq 0$ and $|\alpha| < 1$, $dV/dt|_{(A.3)} < 0$, which completes the proof.

B. Proof of Lemma 3

Conclusion (III) follows from the fact that the characteristic equation (8) has a pair of simple purely imaginary roots $\pm i\omega_0$ at $\tau = \tau_{k,j}$ and the transversality condition (15) holds.

Denote by λ_k a zero of $\Delta_k(\tau,\lambda)$ with $\tau = 0$. From (9), we have

$$\operatorname{Re}\lambda_k = \alpha\cos\frac{2k\pi}{n} - 1. \tag{B.1}$$

Clearly, $\operatorname{Re}\lambda_0 = \alpha - 1 > 0$ for $\alpha > 1$. If n is even, then $\operatorname{Re}\lambda_{n/2} = -\alpha - 1 > 0$ for $\alpha < -1$. These, together with (15) and the Rouché-Froebenius theorem, imply that conclusion (I) is true.

If n is odd with $\alpha < \sec((n-1)\pi/n)$, then $\operatorname{Re}\lambda_{(n-1)/2} > 0$. If n is odd with $\sec((n-1)\pi/n) < \alpha < -1$, then

$$\operatorname{Re}\lambda_{(n-1)/2} < 0. \tag{B.2}$$

Just then from (B.1), we have

$$\begin{aligned}\operatorname{Re}\lambda_0 &< \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_{n-1} < \operatorname{Re}\lambda_2 \\ &= \operatorname{Re}\lambda_{n-2} < \cdots < \operatorname{Re}\lambda_{(n-1)/2} \\ &= \operatorname{Re}\lambda_{(n+1)/2}.\end{aligned} \tag{B.3}$$

It follows that the root of $\Delta_k(0,\lambda) = 0$ has negative real parts for any $k \in \{0,1,2,\ldots,n-1\}$. In addition, from (13), we have

$$\tau_{k,j} < \tau_{k-1,j}, \qquad \tau_{0,j} < \tau_{n-1,j+1}, \tag{B.4}$$

which means

$$\begin{aligned}&\underbrace{\tau_{n-1,0} < \cdots < \tau_{(n+1)/2,0} < \tau_{(n-1)/2,0} < \cdots < \tau_{0,0}}_{j=0} \\ &\quad < \underbrace{\tau_{n-1,1} < \cdots < \tau_{(n+1)/2,1} < \tau_{(n-1)/2,1} < \cdots < \tau_{0,1}}_{j=1} < \cdots .\end{aligned} \tag{B.5}$$

From (13), we also have, for n being odd with $\sec((n-1)\pi/n) < \alpha < -1$,

$$\begin{aligned}\tau_{(n-1)/2,0} &= \frac{1}{\omega_0}\left[-\pi + \frac{\pi}{n} + \arccos\frac{1}{\alpha}\right] \\ &= \frac{1}{n\omega_0}\left[\pi - n\left(\pi - \arccos\frac{1}{\alpha}\right)\right] > 0,\end{aligned} \tag{B.6}$$

but

$$\tau_{(n+1)/2,0} = \frac{1}{\omega_0}\left[-\pi - \frac{\pi}{n} + \arccos\frac{1}{\alpha}\right] < 0. \tag{B.7}$$

Consequently, $\tau_{(n-1)/2,0}$ is the first positive critical value for the occurrence of purely imaginary roots $\pm i\omega_0$. Thus, by (15) and the Rouché-Froebenius theorem, conclusion (II) is proved.

C. Proof of Theorem 5

It can be verified that $\lambda = 0$ is a simple zero of $\Delta_k(\tau,\lambda)$ only for $k = 0$ when $\alpha = 1$. When $\alpha = -1$ and n is even, $\lambda = 0$ is a simple zero of $\Delta_k(\tau,\lambda)$ with $k = n/2$.

On the other hand, suppose that $\lambda = u + iv \neq 0$ is a root of $\Delta_k(\tau,\lambda) = 0$. Then from (9)

$$\begin{aligned}u + 1 &= \alpha e^{-\tau u}\cos\left(\frac{2k\pi}{n} - \tau v\right), \\ v &= \alpha e^{-\tau u}\sin\left(\frac{2k\pi}{n} - \tau v\right).\end{aligned} \tag{C.1}$$

For $\alpha = \pm 1$, the latter implies

$$(u+1)^2 + v^2 = e^{-2\tau u}, \tag{C.2}$$

which can be only satisfied for $u < 0$. This completes the proof.

D. Proof of Theorem 7

For $1 \le i \le n$, every equilibrium x of system (2) must satisfy

$$x_i - x_{i+1} = \alpha\left(f\left(x_{i+1}\right) - f\left(x_{i+2}\right)\right). \tag{D.1}$$

Using the monotonicity of f and the assumption $\alpha > 1$, we obtain that if $x_i \neq x_{i+1}$, say $x_i > x_{i+1}$, then $x_{i+1} > x_{i+2}$. Repeating the above procedure, we have

$$x_i > x_{i+1} > x_{i+2} > x_{i+3} > \cdots > x_n > x_1 \cdots > x_{i-1} > x_i, \tag{D.2}$$

which is a contradiction, implying $x_i = x_{i+1}$. Thus, x is an equilibrium of system (2) if and only if x_i, $i = 1,2,\ldots,n$, satisfy the equation $u = \alpha f(u)$. According to the assumption (H2) when $\alpha = 1$ it has just one zero root, and when $\alpha > 1$ it has exactly three roots $u_-, 0, u_+$ and $\alpha f'(u_\pm) < 1$. Consequently, when $\alpha > 1$ system (2) has exactly three equilibria:

$$\begin{aligned}x_- = \left(u_-, u_-, \ldots, u_-\right), \qquad x_0 = (0,0,\ldots,0), \\ x_+ = \left(u_+, u_+, \ldots, u_+\right).\end{aligned} \tag{D.3}$$

In addition, note that the characteristic equation of system (2) at $x_\pm$ is

$$\det M_n(\tau,\lambda) = \prod_{k=0}^{n-1}\Delta_k(\tau,\lambda) = 0, \tag{D.4}$$

where

$$\Delta_k(\tau,\lambda) = \lambda + 1 - \alpha f'\left(u_\pm\right)e^{(2k\pi/n)i}e^{-\lambda\tau}. \tag{D.5}$$

Thus, from Section 3 and the fact that $0 < \alpha f'(u_\pm) < 1$, we have these two equilibria x_- and x_+ both asymptotically stable for all $\tau \ge 0$. This completes the proof.

E. Proof of Theorem 8

Lemma 2 says that system (2) has just the zero equilibrium x_0 for $-1 < \alpha < 1$. In fact, from the proof of Theorem 7 we also obtain that system (2) has just the zero equilibrium when $\alpha = 1$. We now consider the case $\alpha \le -1$. From system (2), $(x_1, x_2, \ldots, x_n)$ is an equilibrium of system (2) if and only if $(x_1, x_2, \ldots, x_n)$ satisfies

$$x_1 = \alpha f(x_2), \qquad x_2 = \alpha f(x_3), \ldots, x_n = \alpha f(x_1). \tag{E.1}$$

This means that x_i, $i = 1, 2, \ldots, n$, satisfy

$$x - \alpha f(\alpha f(\alpha(\cdots \alpha f(\alpha f(x)) \cdots))) = 0. \tag{E.2}$$

Let

$$g(x) = x - \alpha f(\alpha f(\alpha(\cdots \alpha f(\alpha f(x)) \cdots))). \tag{E.3}$$

Then we have $g(0) = 0$ and

$$\begin{aligned} g'(x) = 1 - \alpha^n f'(\alpha f(\alpha(\cdots \alpha f(\alpha f(x)) \cdots))) \\ \cdots f'(\alpha f(x)) f'(x). \end{aligned} \tag{E.4}$$

Note that $f'(x) > 0$ for all $x \in \mathbb{R}$, n is odd, and $\alpha \le -1 < 0$. Thus, we have $g'(x) > 0$ for all $x \in \mathbb{R}$ and hence $x = 0$ is the unique root of $g(x)$. This completes the proof.

F. Proof of Theorem 9

Clearly, x_0 is an equilibrium. Let $(x_1, x_2, \ldots, x_n)$ be a nonzero equilibrium of system (2). At first, suppose that $x_1 = x_2$, which, together with (D.1), implies $x_2 = x_3$, then $x_3 = x_4$, and so on. Consequently, we obtain that $x_1 = x_2 = \cdots = x_n$. Thus, x_i, $i = 1, 2, \ldots, n$, satisfy $u = \alpha f(u)$. Using the monotonicity of f and $\alpha < -1 < 0$, we get $x_i = 0$ for any $i = 1, 2, \ldots, n$. This contradiction leads to $x_1 \ne x_2$. In fact, we can also obtain $x_i \ne x_{i+1}$, $i = 1, 2, \ldots, n$, $i(\bmod n)$. By (E.4), we know that $g'(0) < 0$ when n is even and $\alpha < -1$. Therefore, there exists a sufficiently small positive number δ such that $g'(x) < 0$ for $x \in (-\delta, +\delta)$. From $g(0) = 0$ we have $g(x) < 0$ for $x \in (0, +\delta)$ and $g(x) > 0$ for $x \in (-\delta, 0)$. On the other hand, since $f(x)$ is bounded, we have $g(x) \to +\infty$ as $x \to +\infty$ and $g(x) \to -\infty$ as $x \to -\infty$. This means that there exist $u_- < 0$ and $u_+ > 0$ such that $g(u_-) = g(u_+) = 0$, $g(x) > 0$ for $x \in (u_-, 0)$ and $g(x) < 0$ for $x \in (0, u_+)$, and $g'(u_-), g'(u_+) \ge 0$. In addition, by $f''(x) < 0$ for $x > 0$ and $f''(x) > 0$ for $x < 0$, we obtain, for $x > u_+$,

$$\begin{aligned} &f'(x) < f'(u_+), \\ &f'(\alpha f(x)) < f'(\alpha f(u_+)), \ldots, \\ &f'(\alpha f(\alpha(\cdots \alpha f(\alpha f(x)) \cdots))) \\ &\quad < f'(\alpha f(\alpha(\cdots \alpha f(\alpha f(u_+)) \cdots))). \end{aligned} \tag{F.1}$$

So,

$$g'(x) > g'(u_+) \ge 0, \quad \text{for } x > u_+. \tag{F.2}$$

Similarly, we can obtain

$$g'(x) > g'(u_-) \ge 0, \quad \text{for } x < u_-. \tag{F.3}$$

These imply that when n is even and $\alpha < -1$, $g(x)$ has exactly three roots: $u_-, 0, u_+$. Thus, from Lemma 6 and $x_i \ne x_{i+1}$ it follows that system (2) has exactly three roots: the zero equilibrium $x_0 = (0, 0, \ldots, 0)$ and

$$\begin{aligned} x_1^* &= (u_+, u_-, u_+, u_-, \ldots, u_+, u_-), \\ x_2^* &= (u_-, u_+, u_-, u_+, \ldots, u_-, u_+). \end{aligned} \tag{F.4}$$

From (E.1) and (F.4) we can get $u_+ = \alpha f(u_-)$, $u_- = \alpha f(u_+)$, and u_-, u_+ are roots of the following equation:

$$u = \alpha f(\alpha f(u)). \tag{F.5}$$

So,

$$\alpha^2 f'(u_+) f'(u_-) = \alpha^2 f'(\alpha f(u_-)) f'(u_-) = 1. \tag{F.6}$$

Again using the assumption (H1), we have

$$0 < f'(u_+), \qquad f'(u_-) < 1. \tag{F.7}$$

Therefore,

$$\alpha^2 f'^2(u_+) f'^2(u_-) < \alpha^2 f'(u_+) f'(u_-) = 1, \tag{F.8}$$

and then

$$-1 < \alpha f'(u_+) f'(u_-) < 0. \tag{F.9}$$

Letting $v_k = (f'(u_-), f'(u_+)\chi^k, f'(u_-)\chi^{2k}, \ldots, f'(u_+)\chi^{(n-1)k})$, we can obtain that the characteristic equation of the system at x_1^* is

$$\det M_n(\tau, \lambda) = \prod_{k=0}^{n-1} \Delta_k(\tau, \lambda) = 0, \tag{F.10}$$

where

$$\Delta_k(\tau, \lambda) = \lambda + 1 - \alpha f'(u_+) f'(u_-) e^{(2k\pi/n)i} e^{-\lambda\tau}. \tag{F.11}$$

Thus, from Section 3 and (F.9) we get that the equilibrium x_1^* is asymptotically stable for all $\tau \ge 0$. Similarly, we can also obtain that the equilibrium x_2^* is asymptotically stable for all $\tau \ge 0$. This completes the proof.

G. Derivation of Coefficients K_1, K_2 of the Normal Form (19)

The linearized equation at the zero equilibrium of (16) is

$$\dot{z}(t) = L(\tau) z_t, \tag{G.1}$$

where

$$L(\tau)\varphi = -\tau\varphi(0) + \tau\alpha M\varphi(-1). \tag{G.2}$$

The linear operator $L(\tau)$ can be expressed in the integral form

$$L(\tau)\varphi = \int_{-1}^{0} [d\eta_\tau(\theta)]\varphi(\theta), \quad \text{(G.3)}$$

where $\eta_\tau : [-1,0] \to \mathbb{R}^n \times \mathbb{R}^n$ is a function of bounded variation.

Denoting $\tau^* = \tau_{k,j}$ and introducing the new parameter $\mu = \tau - \tau^*$ so that $\mu = 0$ corresponds to the Hopf bifurcation, we rewrite (16) as

$$\dot{z}(t) = L(\tau^*) z_t + \widetilde{F}(z_t, \mu), \quad \text{(G.4)}$$

where

$$\begin{aligned}\widetilde{F}(z_t, \mu) &= L(\mu) z_t \\ &\quad + \gamma\alpha\tau^* M\big(z_1^3(t-1), z_2^3(t-1), \dots, z_n^3(t-1)\big)^T \\ &\quad + \text{h.o.t.}\end{aligned} \quad \text{(G.5)}$$

Let $\omega_* = \omega_0\tau^*$ and $\Lambda_0 = \{i\omega_*, -i\omega_*\}$. It is clear from Lemma 3 that the characteristic equation of $\dot{z}(t) = L(\tau^*)z_t$ has a pair of simple imaginary roots $\pm i\omega_*$ and no other roots in the imaginary axis which are multiple of $\pm i\omega_*$. Thus, the nonresonance conditions relative to Λ_0 are satisfied.

Let $\Phi = (\phi_1, \phi_2)$ be a matrix whose columns form a basis of the center space P of $\dot{z}(t) = L(\tau^*)z_t$ with $\phi_1(\theta) = e^{i\omega_*\theta}v, \phi_2(\theta) = e^{-i\omega_*\theta}\overline{v}$, where the bar means complex conjugation and v is a vector in $\mathbb{C}^n$ such that

$$L(\tau^*)\phi_1 = i\omega_* v. \quad \text{(G.6)}$$

Suppose $\Psi = \text{col}(\psi_1, \psi_2)$ is a matrix whose rows form a basis for the adjoint space P^* with $\langle\Psi(s), \Phi(\theta)\rangle = I$ (the $n\times n$ identity matrix) for the adjoint bilinear form on $C^* \times C$ defined by

$$\begin{aligned}\langle\psi(s), \phi(\theta)\rangle &= \psi(0)\phi(0) \\ &\quad - \int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)\,d\eta_{\tau^*}(\theta)\,\phi(\xi)\,d\xi, \quad \text{(G.7)} \\ &\quad \text{for } \psi \in C^*,\ \phi \in C\end{aligned}$$

with $C^* = C([0,1], \mathbb{R}^{n*})$, where $\mathbb{R}^{n*}$ is the n-dimensional space of row vectors.

Then we have $\Psi(s) = \text{col}(\psi_1(s), \psi_2(s)) = \text{col}(u^T e^{-i\omega_* s}, \overline{u}^T e^{i\omega_* s})$, $s \in [0,1]$, for $u \in \mathbb{C}^n$ such that

$$\langle\psi_1, \phi_1\rangle = 1, \qquad \langle\psi_1, \phi_2\rangle = 0. \quad \text{(G.8)}$$

From (7), (G.6), and (G.7), we can choose

$$v = v_k, \qquad u = \frac{1}{n}(\overline{a})^{-1}\overline{v}_k, \quad \text{(G.9)}$$

where $a = 1 + \tau^* - \omega_* i$, $v_k = (1, \chi^k, \chi^{2k}, \dots, \chi^{(n-1)k})^T$.

Following the procedure [48, 49] very closely, we can obtain the normal form associated with the Hopf singularity

$$\dot{x} = Bx + \frac{1}{2!}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, \mu) + \text{h.o.t.}, \quad \text{(G.10)}$$

where

$$\begin{aligned}\frac{1}{2}g_2^1(x, 0, \mu) &= \begin{pmatrix} i\omega_0(\overline{a})^{-1}x_1\mu \\ -i\omega_0 a^{-1}x_2\mu \end{pmatrix}, \\ \frac{1}{3!}g_3^1(x, 0, 0) &= 3\gamma\tau^* \begin{pmatrix} (\overline{a})^{-1}(1 + i\omega_0)x_1^2 x_2 \\ a^{-1}(1 - i\omega_0)x_1 x_2^2 \end{pmatrix}.\end{aligned} \quad \text{(G.11)}$$

Moreover, the normal form (G.10) can be written in real coordinates w through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$. Transformed to polar coordinates $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this normal form becomes (19), where $K_1 = \text{Re}A_1$ and $K_2 = \text{Re}A_2$.

H. Derivation of the Normal Form (21)

It is convenient to write (16) as the following delay system:

$$\dot{z}(t) = L_0 z_t + \widetilde{F}(z_t, \nu), \quad \text{(H.1)}$$

with the phase space $C = C([-1,0], \mathbb{R}^n)$, where, for $\varphi = (\varphi_1, \varphi_2)^T \in C$,

$$\begin{aligned}L_0(\varphi) &= -\tau\varphi(0) - \tau M\varphi(-1), \\ \widetilde{F}(z_t, \nu) &= L_1(\nu)\varphi(-1) + \tau(-1+\nu)\gamma M(\varphi(-1))^3 \\ &\quad + \text{h.o.t.}, \quad i\,(\text{mod}\,n)\end{aligned} \quad \text{(H.2)}$$

with $L_1(\nu)\varphi(-1) = \tau\nu M\varphi(-1)$.

Let $\Lambda_0 = \{0\}$ and consider the center space P for $\dot{x}(t) = L_0 x_t$ and its dual space P^*, as in the previous appendix. We can choose normalized dual bases Φ of P and Ψ of P^* as follows:

$$\begin{aligned}\Phi(\theta) &= (1, -1, \dots, 1, -1)^T, \quad -1 \le \theta \le 0, \\ \Psi(s) &= \frac{1}{n(1+\tau)}(1, -1, \dots, 1, -1), \quad 0 \le s \le 1,\end{aligned} \quad \text{(H.3)}$$

with $\dot{\Phi} = \Phi B$, $-\dot{\Psi} = B\Psi$, and $B = 0$.

As the procedure introduced in Section 5.1, decomposing z_t in (H.1) according to the decomposition of BC as the form $z_t = \Phi x(t) + y_t$, with $x \in \mathbb{R}$ and $y_t \in \text{Ker}\,\pi \cap D(A) = Q \cap C^1 \overset{\text{def}}{=} Q^1$, we can obtain the normal form of (H.1) on the center manifold of the origin

$$\dot{x} = \frac{1}{2}g_2^1(x, 0, \nu) + \frac{1}{3!}g_3^1(x, 0, \nu) + \text{h.o.t.}, \quad \text{(H.4)}$$

where $x, \nu \in \mathbb{R}$, h.o.t. stands for higher order terms, and $g_2^1(x, 0, \nu)$, $g_3^1(x, 0, \nu)$ are the second and third order terms in (x, ν), respectively. Then it follows from [48, 49] that

$$\begin{aligned}g_2^1(x, 0, \nu) &= \text{Proj}_{(\text{Im}(M_2^1))^c} f_2^1(x, 0, \nu), \\ g_3^1(x, 0, \nu) &= \text{Proj}_{(\text{Im}(M_3^1))^c} \widetilde{f}_3^1(x, 0, \nu),\end{aligned} \quad \text{(H.5)}$$

where $(\text{Im}(M_j^1))^c$ is a complementary space of $\text{Im}(M_j^1)$ in $V_j^2(\mathbb{R})$ with $j = 2, 3$ and $(1/3!)\widetilde{f}_3^1(x, 0, \nu)$ denotes the third order terms after the calculation of the normal form up to the second order terms.

Since B is the $n \times n$ zero matrix, it is easy to check that

$$\begin{aligned} \left(\operatorname{Im}(M_2^1)\right)^c &= \operatorname{span}\left\{x^2, x\nu, \nu^2\right\} = V_2^2(\mathbb{R}), \\ \left(\operatorname{Im}(M_3^1)\right)^c &= \operatorname{span}\left\{x^3, x^2\nu, x\nu^2, \nu^3\right\} = V_3^2(\mathbb{R}). \end{aligned} \tag{H.6}$$

From (H.2), we get

$$\begin{aligned} \frac{1}{2!} f_2^1(x, 0, \nu) &= \Psi(0) L_1(\nu)(\Phi x) \\ &= \frac{\tau \nu x}{n(1+\tau)} (1, -1, \dots, 1, -1) \\ &\quad \times \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} \\ &= -\frac{\tau}{1+\tau} \nu x, \end{aligned} \tag{H.7}$$

which, together with (H.4) and (H.5), leads to

$$\frac{1}{2!} g_2^1(x, 0, \nu) = -\frac{\tau}{1+\tau} \nu x. \tag{H.8}$$

For the bifurcation analysis, it is sufficient to compute the coefficient of x^3 in the normal form (see, e.g., [51]). Hence, we write

$$\begin{aligned} &\frac{1}{3!} g_3^1(x, 0, \nu) \\ &\quad = \frac{1}{3!} \operatorname{Proj}_{S_1} \widetilde{f}_3^1(x, 0, 0) + \frac{1}{3!} \operatorname{Proj}_{S_2} \widetilde{f}_3^1(x, 0, \nu), \end{aligned} \tag{H.9}$$

where $S_1 = \operatorname{span}\{x^3\}$, $S_2 = \operatorname{span}\{x^2\nu, x\nu^2, \nu^3\}$. It is clear that $f_2^1(x, 0, 0) = g_2^1(x, 0, 0) = 0$. Thus, we have

$$\begin{aligned} \frac{1}{3!} \widetilde{f}_3^1(x, 0, 0) &= \frac{1}{3!} f_3^1(x, 0, 0) = -\Psi(0) \tau \gamma M(\Phi(-1) x)^3 \\ &= \frac{-\tau \gamma x^3}{n(1+\tau)} (1, -1, \dots, 1, -1) \\ &\quad \times \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} \\ &= \frac{\gamma \tau}{1+\tau} x^3. \end{aligned} \tag{H.10}$$

Consequently, the normal form (H.4) becomes (21).

I. Proof of Lemma 16

It is easy to verify that, for the generator ρ of z_n and $j(\bmod\ n)$,

$$\begin{aligned} \left(\rho \epsilon^1(t)\right)_j &= \epsilon_{j+1}^1(t) \\ &= \cos\left(\omega_0 t + \frac{2jk\pi}{n}\right) \\ &= \cos\left(\omega_0\left(t + \frac{2k\pi}{n\omega_0}\right) + \frac{2(j-1)k\pi}{n}\right) \\ &= \epsilon_j^1\left(t + \frac{2k\pi}{n\omega_0}\right), \\ &= \epsilon_j^1\left(t + \frac{k\omega}{n}\right), \\ \left(\rho \epsilon^2(t)\right)_j &= \epsilon_{j+1}^2(t) \\ &= \sin\left(\omega_0 t + \frac{2jk\pi}{n}\right) \\ &= \sin\left(\omega_0\left(t + \frac{2k\pi}{n\omega_0}\right) + \frac{2(j-1)k\pi}{n}\right) \\ &= \epsilon_j^2\left(t + \frac{2k\pi}{n\omega_0}\right) \\ &= \epsilon_j^2\left(t + \frac{k\omega}{n}\right). \end{aligned} \tag{I.1}$$

This completes the proof.

Conflict of Interests

The authors certify that there is no conflict of interests with any financial organization regarding the material discussed in the paper.

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