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Research Article

Levitin-Polyak Well-Posedness of an Equilibrium-Like Problem in Banach Spaces

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The concept of Levitin-Polyak well-posedness of an equilibrium-like problem in Banach spaces is introduced. Under suitable conditions, some characterizations of its Levitin-Polyak well-posedness are established. Some conditions under which an equilibrium-like problem in Banach spaces is Levitin-Polyak well-posed are also derived.

1. Introduction

In 1966, Tykhonov [1] first established the well-posedness of a minimization problem, which has been known as Tykhonov well-posedness. Since it is important in optimization problems, various concepts of well-posedness have been introduced and studied in past decades. For more about the well-posedness, we refer to [2–4] and the references therein.

The Tykhonov well-posedness of a constrained minimization problem requires that every minimizing sequence should lie in the constraint set. In many situations, the minimizing sequence produced by a numerical optimization method usually fails to be feasible but gets closer and closer to the constraint set. Levitin and Polyak [5] generalized the concept of Tykhnov well-posedness by requiring the existence and uniqueness of minimizer and the convergence of every generalized minimizing sequence toward the unique minimizer, which has been known as Levitin and Polyak well-posedness. There are a lot of results concerned with Tykhonov well-posedness, LP well-posedness, and their generalizations for minimization problems. For details, we refer to [1–3, 5–7].

Recently, the concept of well-posedness has been extended to many other fields, including Nash equilibrium [8], inclusion problems, and fixed point problems [9–13]. Lemaire [12, 13] studied the relations between the well-posedness of minimization problems, inclusion problems, and fixed point problems. Fang et al. [11] proved that the well-posedness of a general mixed variational inequality is equivalent to the existence and the uniqueness of its solution in the Hilbert space. Recently, Ceng and Yao [9] got some results for the well-posedness of the generalized mixed variational inequality, the corresponding inclusion problem, and the corresponding fixed point problem. On the other hand, Li and Xia [14] considered the Levitin-Polyak well-posedness of a generalized variational inequality in Banach space. And they showed that the Levitin-Polyak well-posedness of a generalized variational inequality is equivalent to the uniqueness and existence of its solutions. However, there has been no result for the Levitin-Polyak well-posedness of an equilibrium-like problem.

Motivated and inspired by the research work going on in this field, in this paper, we extend the notion of Levitin-Polyak well-posedness to an equilibrium-like problem in Banach spaces and give some metric characterizations of its Levitin-Polyak well-posedness. Finally, we derive some conditions under which an equilibrium-like problem is Levitin-Polyak well-posed.

2. Preliminaries

Let X be a real reflexive Banach space with its dual X^* and let K be a nonempty, closed, and convex subset of X. Let $F : X \to 2^{X^*}$ be a set-valued mapping, and let $\phi : X^* \times X \times X \to \mathbb{R}$ be a functional. In this paper, we consider the following equilibrium-like problem associated with (F, ϕ, K):

$$\text{ELP}(F, \phi, K): \text{ find } x \in K \text{ such that for some } u \in F(x), \quad \phi(u, x, y) \le 0, \quad \forall y \in K. \tag{1}$$

Definition 1. Let A, B be nonempty subsets of X. The Hausdorff metric $\mathscr{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathscr{H}(A, B) = \max\{e(A, B), e(B, A)\}, \tag{2}$$

where $e(A, B) = \sup_{a\in A} d(a, B)$ with $d(a, B) = \inf_{b\in B}\|a - b\|$.

Lemma 2 (Nadler's theorem [7]). *Let $(X, \|\cdot\|)$ be a normed vector space and let $\mathscr{H}(\cdot,\cdot)$ be the Hausdorff metric on the collection $CB(X)$ of all nonempty, closed, and bounded subsets of X, induced by a metric d in terms of $d(u, v) = \|u - v\|$, which is defined by $\mathscr{H}(U, V) = \max\{e(U, V), e(V, U)\}$, for U and V in $CB(X)$, where $e(U, V) = \sup_{x\in U} d(x, V)$ with $d(x, V) = \inf_{y\in V}\|x - y\|$. If U and V lie in $CB(X)$, then, for any $\epsilon > 0$ and any $u \in U$, there exists $v \in V$ such that $\|u - v\| \le (1 + \epsilon)\mathscr{H}(U, V)$. In particular, whenever U and V are compact subsets in X, one has $\|u - v\| \le \mathscr{H}(U, V)$.*

Definition 3 (see [9]). A nonempty set-valued mapping $F : X \to 2^{X^*}$ is said to be

(i) $\mathscr{H}$-hemicontinuous if, for any $x, y \in X$, the function $t \mapsto \mathscr{H}(F(x + t(y - x)), F(x)))$ from $[0, 1]$ into $\mathbb{R}^+ = [0, +\infty)$ is continuous at 0^+, where $\mathscr{H}(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X)$;

(ii) $\mathscr{H}$-uniformly continuous if, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x - y\| < \delta$, one has $\mathscr{H}(F(x), F(y)) < \epsilon$, where $\mathscr{H}(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X)$.

Definition 4. Let X and Y be two topological spaces and $x \in X$. A set-valued mapping $F : X \to 2^Y$ is said to be upper semicontinuous (u.s.c. in short) at x, if for any neighbourhood V of $F(x)$, there exists a neighbourhood U of x such that $F(y) \subset V$, for all $y \in U$. If F is u.s.c. at each point of X, we say that F is u.s.c. on X.

Definition 5 (see [15]). Let A be a nonempty subset of X. The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i, \operatorname{diam} A_i < \epsilon, \quad i = 1, 2, \ldots, n\right\}, \tag{3}$$

where $\operatorname{diam} A_i$ denotes the diameter of the set A_i, for $i = 1, 2, \ldots, n$.

Definition 6. Let X be a real reflexive Banach space with its dual X^* and let $F : X \to 2^{X^*}$ be a set-valued mapping. A functional $\phi : X^* \times X \times X \to \mathbb{R}$ is said to be monotone with respect to F, if for any $x, y \in X$ and $u \in F(x), v \in F(y)$, $\phi(u, x, y) \ge \phi(v, x, y)$.

Remark 7. If $\phi(u, x, y) = \langle u, x - y\rangle$, for all $x, y \in X$ and $u \in F(x)$, it is easy to know that ϕ is monotone with respect to F which reduces to F being monotone.

We first prove the following proposition.

Proposition 8. *Let K be a nonempty, closed, and convex subset of X and let $F : X \to 2^{X^*}$ be a nonempty compact-valued mapping which is $\mathscr{H}$-hemicontinuous. Let $\phi : X^* \times X \times X \to \mathbb{R}$ be monotone with respect to F, continuous in first argument, and concave in third argument. Moreover, $\phi(u, x, x) = 0$, for all $u \in X^*$, $x \in K$. Then, for a given $x \in K$, the following statements are equivalent:*

(i) *there exists $u \in F(x)$ such that $\phi(u, x, y) \le 0$, for all $y \in K$;*

(ii) *$\phi(v, x, y) \le 0$, for all $y \in K$, $v \in F(y)$.*

Proof. First, we assume that for some $u \in F(x)$, $\phi(u, x, y) \le 0$, for all $y \in K$. Because ϕ is monotone with respect to F, we have

$$\phi(v, x, y) \le 0, \quad \forall y \in K, \; v \in F(y). \tag{4}$$

Conversely, suppose that for all $y \in K$, $v \in F(y)$, we obtain

$$\phi(v, x, y) \le 0. \tag{5}$$

For any given $y \in K$, we define $y_t = ty + (1 - t)x$ for all $t \in (0, 1)$. Replacing y by y_t in the left-hand side of the last inequality, we have that, for each $v_t \in F(y_t)$,

$$\begin{aligned} 0 &\ge \phi(v_t, x, y_t) \\ &= \phi(v_t, x, ty + (1 - t)x) \\ &\ge t\phi(v_t, x, y) + (1 - t)\phi(v_t, x, x) \\ &= t\phi(v_t, x, y). \end{aligned} \tag{6}$$

This implies that

$$\phi(v_t, x, y) \le 0, \quad \forall v_t \in F(y_t), \; t \in (0, 1). \tag{$*$}$$

Since $F : X \to 2^{X^*}$ is a nonempty compact-valued mapping, $F(y_t)$ and $F(x)$ are nonempty compact and hence lie in $CB(X)$. From Lemma 2, we get that, for each $t \in (0,1)$ and for each fixed $v_t \in F(y_t)$, there exists a $u_t \in F(x)$ such that

$$\|v_t - u_t\| \le (1+t)\mathscr{H}(F(y_t), F(x)). \quad (7)$$

Since $F(x)$ is compact, without loss of generality, we assume that $u_t \to u \in F(x)$ as $t \to 0^+$. Since F is $\mathscr{H}$-hemicontinuous, we get that as $t \to 0^+$,

$$\|v_t - u_t\| \le (1+t)\mathscr{H}(F(y_t), F(x)) \longrightarrow 0. \quad (8)$$

This implies that $v_t \to u \in F(x)$ as $t \to 0^+$. Since ϕ is continuous in first argument, by (∗) we obtain that there exists an $u \in F(x)$ such that

$$\phi(u, x, y) \le 0, \quad \forall y \in K. \quad (9)$$

This completes the proof. □

3. Levitin-Polyak Well-Posedness of ELP(F, ϕ, K)

In this section, we extend the concepts of Levitin-Poylak well-posedness to the equilibrium-like problem and establish its metric characterizations. Let $\alpha \ge 0$ be a given number, and let X, K, F, and ϕ be defined as the previous section.

Definition 9. A sequence $\{x_n\} \subset X$ is called an LP α-approximating sequence for ELP(F,ϕ,K), if there exist $w_n \in X$ with $w_n \to 0$ and $0 < \epsilon_n \to 0$ such that $x_n + w_n \in K$ for all $n \in N$ and there exists $u_n \in F(x_n)$ such that

$$\phi(u_n, x_n, y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in K,\ n \in N. \quad (10)$$

If $\alpha_1 > \alpha_2 \ge 0$, then every LP α_2-approximating sequence is LP α_1-approximating. When $\alpha = 0$, we say that $\{x_n\}$ is an LP approximating sequence for ELP(F,ϕ,K).

Definition 10. ELP(F,ϕ,K) is strongly LP α-well-posed if ELP(F,ϕ,K) has an unique solution and every LP α-approximating sequence converges strongly to the unique solution. In the sequel, strong LP 0-well-posedness is always called as strong LP well-posedness. If $\alpha_1 > \alpha_2 \ge 0$, then strong LP α_1-well-posedness implies strong LP α_2-well-posedness.

Definition 11. ELP(F,ϕ,K) is strongly LP α-well-posed in the generalized sense if ELP(F,ϕ,K) has nonempty solution set S and every LP α-approximating sequence has a subsequence which converges strongly to some point of S. In the sequel, strong LP 0-well-posedness in the generalized sense is always called as strong LP well-posedness in the generalized sense. If $\alpha_1 > \alpha_2 \ge 0$, then strong LP α_1-well-posedness in the generalized sense implies strong LP α_2-well-posedness in the generalized sense.

Remark 12. If $\phi(u,x,y) = \langle u, x-y\rangle + \varphi(x) - \varphi(y)$, for all $x, y \in X$, $u \in F(x)$, then Definitions 10 and 11 reduce to Definitions 3.3 and 3.4 of [14], respectively. Moreover, when X is a Hilbert space, $K = X$, and $w_n \equiv 0$, Definitions 10 and 11 reduce to Definitions 3.2 and 3.3 of [11], respectively.

To obtain the metric characterizations of LP α-well-posedness, we consider the following LP α-approximating solution set of ELP(F,ϕ,K):

$$\Omega_\alpha(\epsilon) = \Big\{x \in \operatorname{dom}\phi : d(x,K) \le \epsilon, \text{ and there exists } u \in F(x) \text{ such that } \forall y \in K, \phi(u,x,y) \le \frac{\alpha}{2}\|x-y\|^2 + \epsilon\Big\}, \quad \forall \epsilon \ge 0. \quad (11)$$

Theorem 13. *Let K be a nonempty, closed, and convex subset of X and let $F : X \to 2^{X^*}$ be a $\mathscr{H}$-hemicontinuous and nonempty compact-valued mapping. Let $\phi : X^* \times X \times X \to \mathbb{R}$ be monotone with respect to F, lower semicontinuous in second argument, and concave in third argument. Moreover, $\phi(u,x,x) = 0$, for all $u \in X^*$, $x \in K$. Then, ELP(F,ϕ,K) is strongly LP α-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \ne \emptyset, \quad \forall \epsilon > 0 \text{ and } \operatorname{diam}(\Omega_\alpha(\epsilon)) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \quad (12)$$

Proof. First, we assume that ELP(F,ϕ,K) is strongly LP α-well-posed and $x^* \in K$ is the unique solution of ELP(F,ϕ,K). It is easy to see that $x^* \in \Omega_\alpha(\epsilon)$. If $\operatorname{diam}(\Omega_\alpha(\epsilon)) \nrightarrow 0$ as $\epsilon \to 0$, then there exist constant $l > 0$ and sequences $\{\epsilon_n\} \subset \mathbb{R}_+$ with $\epsilon_n \to 0$ and $\{x_n^{(1)}\}, \{x_n^{(2)}\}$ with $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$ such that

$$\left\|x_n^{(1)} - x_n^{(2)}\right\| > l, \quad \forall n \in N. \quad (13)$$

Because of $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$, by the definition of $\Omega_\alpha(\epsilon_n)$, for $x_n^{(1)}$, we obtain

$$d\left(x_n^{(1)}, K\right) \le \epsilon_n < \epsilon_n + \frac{1}{n}, \quad (14)$$

and there exists $u_n \in F(x_n^{(1)})$ such that

$$\phi\left(u_n, x_n^{(1)}, y\right) \le \frac{\alpha}{2}\left\|x_n^{(1)} - y\right\|^2 + \epsilon_n, \quad \forall y \in K. \quad (15)$$

Since K is closed and convex, then there exists $\overline{x}_n^{(1)} \in K$ such that $\|x_n^{(1)} - \overline{x}_n^{(1)}\| < \epsilon_n + (1/n)$. Let $w_n = \overline{x}_n^{(1)} - x_n^{(1)}$; we get $w_n + x_n^{(1)} = \overline{x}_n^{(1)} \in K$ and $\|w_n\| = \|x_n^{(1)} - \overline{x}_n^{(1)}\| \to 0$. This implies that $w_n \to 0$. Thus, $\{x_n^{(1)}\}$ is an LP approximating sequence for ELP(F,ϕ,K). By the similar argument, we obtain that $\{x_n^{(2)}\}$ is an LP approximating sequence for ELP(F,ϕ,K). So they have to converge strongly to the unique solution of ELP(F, ϕ, K), which contradicts condition (13).

Conversely, suppose that condition (12) holds. Let $\{x_n\} \subset X$ be an LP α-approximating sequence for ELP(F, ϕ, K). Then, there exists $w_n \in X$ with $w_n \to 0$ such that $x_n + w_n \in K$, and there exist $0 < \epsilon_n' \to 0$ and $u_n \in F(x_n)$ such that

$$\phi(u_n, x_n, y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n', \quad \forall y \in K,\ n \in N. \tag{16}$$

Since $x_n + w_n \in K$, then there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$. It is obvious that $d(x_n, K) \le \|x_n - \overline{x}_n\| = \|w_n\| \to 0$. Suppose that $\epsilon_n = \max\{\epsilon_n', \|w_n\|\}$; we get that $x_n \in \Omega_\alpha(\epsilon_n)$. From (12), we have that $\{x_n\}$ is a Cauchy sequence and converges strongly to a point $\overline{x} \in K$. Since ϕ is monotone with respect to F and lower semicontinuous in second argument, it follows from (16) that, for any $y \in K$, $v \in F(y)$,

$$\begin{aligned} \phi(v, \overline{x}, y) &\le \liminf_{n\to\infty}\{\phi(v, x_n, y)\} \\ &\le \liminf_{n\to\infty}\{\phi(u_n, x_n, y)\} \\ &\le \liminf_{n\to\infty}\left\{\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n'\right\} \\ &= \frac{\alpha}{2}\|\overline{x} - y\|^2. \end{aligned} \tag{17}$$

For any $y \in K$, let $y_t = \overline{x} + t(y - \overline{x})$, for all $t \in [0,1]$. Since K is a nonempty, closed, and convex subset, we have that $y_t \in K$. Then, (17) implies that

$$\phi(v_t, \overline{x}, y_t) \le \frac{\alpha}{2}\|\overline{x} - y_t\|^2, \quad \forall v_t \in F(y_t). \tag{18}$$

Since ϕ is concave in third argument and $\phi(u, x, x) = 0$, for all $u \in X^*$, $x \in K$,

$$\phi(v_t, \overline{x}, y) \le \frac{\alpha t}{2}\|\overline{x} - y\|^2, \quad \forall v_t \in F(y_t),\ y \in K. \tag{19}$$

Since F is a nonempty compact-valued mapping and $\mathscr{H}$-hemicontinuous, by Lemma 2, for each fixed $v_t \in F(y_t)$ and each $t \in (0,1)$, there exists a $u_t \in F(\overline{x})$ such that $\|v_t - u_t\| \le \mathscr{H}(F(y_t), F(\overline{x}))$. Since F is $\mathscr{H}$-hemicontinuous, we get that $\|v_t - u_t\| \le \mathscr{H}(F(y_t), F(\overline{x})) \to 0$ as $t \to 0^+$. Since F is compact, without loss of generality, we assume that $u_t \to u \in F(\overline{x})$ as $t \to 0^+$. Thus, we obtain that

$$\begin{aligned} \|v_t - u\| &\le \|v_t - u_t\| + \|u_t - u\| \\ &\le \mathscr{H}(F(y_t), F(\overline{x})) + \|u_t - u\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0^+. \end{aligned} \tag{20}$$

This implies that $v_t \to u$ as $t \to 0^+$. It follows from (19) that

$$\phi(u, \overline{x}, y) \le 0, \quad \forall y \in K. \tag{21}$$

Therefore, $\overline{x}$ solves ELP(F, ϕ, K).

To complete the proof, we only need to prove that ELP(F, ϕ, K) has a unique solution. Suppose that ELP(F, ϕ, K) has two distinct solutions x_1 and x_2. Then, it is obvious that $x_1, x_2 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \le \operatorname{diam}(\Omega_\alpha(\epsilon)) \longrightarrow 0, \tag{22}$$

a contradiction to (12). This completesthe proof. □

Theorem 14. *Let K be a nonempty, closed, and convex subset of X and let $F : X \to 2^{X^*}$ be a $\mathscr{H}$-hemicontinuous and nonempty compact-valued mapping. Let $\phi : X^* \times X \times X \to \mathbb{R}$ be monotone with respect to F and lower semicontinuous in second argument. Moreover, $\phi(u, x, x) = 0$, for all $u \in X^*$, $x \in K$. Then, ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \ne \emptyset, \quad \forall \epsilon > 0 \text{ and } \mu(\Omega_\alpha(\epsilon)) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{23}$$

Proof. Assume that ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense. Let S be the solution set of ELP(F, ϕ, K). Then, S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S. Then, $\{x_n\}$ is an LP α-approximating sequence for ELP(F, ϕ, K). Since ELP(F, ϕ, K) is strongly α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S. Thus, S is compact. It is easy to see that $\Omega_\alpha(\epsilon) \supset S \ne \emptyset$ for all $\epsilon > 0$. Now we show that

$$\mu(\Omega_\alpha(\epsilon)) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{24}$$

It is easy to see that, for every $\epsilon > 0$,

$$\begin{aligned} \mathscr{H}(\Omega_\alpha(\epsilon), S) &= \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} \\ &= e(\Omega_\alpha(\epsilon), S). \end{aligned} \tag{25}$$

Taking into account the compactness of S, we obtain

$$\mu(\Omega_\alpha(\epsilon)) \le 2\mathscr{H}(\Omega_\alpha(\epsilon), S) + \mu(S) = 2e(\Omega_\alpha(\epsilon), S). \tag{26}$$

To prove (23), it is sufficient to show that

$$e(\Omega_\alpha(\epsilon), S) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{27}$$

Indeed, if $e(\Omega_\alpha(\epsilon), S) \nrightarrow 0$ as $\epsilon \to 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$, and $x_n \in \Omega_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in N, \tag{28}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. By the definition of $\Omega_\alpha(\epsilon_n)$, we know that $d(x_n, K) \le \epsilon_n < \epsilon_n + (1/n)$, and there exists $u_n \in F(x_n)$ such that

$$\phi(u_n, x_n, y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in K. \tag{29}$$

Thus, there exists $\overline{x}_n \in K$ such that $\|\overline{x}_n - x_n\| < \epsilon_n + (1/n)$. Let $w_n = \overline{x}_n - x_n$; then, we have $w_n + x_n \in K$ with $w_n \to 0$. So $\{x_n\}$ is an LP α-approximating sequence for ELP(F, ϕ, K). Since ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to some point of S. This contradicts (28) and so

$$e(\Omega_\alpha(\epsilon), S) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{30}$$

Conversely, suppose that (23) holds. We first show that $\Omega_\alpha(\epsilon)$ is closed for all $\epsilon > 0$. Let $\{x_n\} \subset \Omega_\alpha(\epsilon)$ with $x_n \to x$; then, there exists $u_n \in F(x_n)$ such that $d(x_n, K) \le \epsilon$ and

$$\phi(u_n, x_n, y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon, \quad \forall y \in K,\ n \in N. \tag{31}$$

Since F is an upper semicontinuous and nonempty compact-valued mapping, there exist a sequence $\{u_{n_k}\}$ of $\{u_n\}$ and some $u \in F(x)$ such that $u_{n_k} \to u$. Therefore, it follows from (31) and the lower semicontinuity of ϕ that

$$\phi(u, x, y) \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \quad \forall y \in K. \tag{32}$$

It is obvious that $d(x, K) \leq \epsilon$. This implies that $x \in \Omega_\alpha(\epsilon)$ and so $\Omega_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Observe that

$$S = \bigcap_{\epsilon > 0} \Omega_\alpha(\epsilon). \tag{33}$$

Since $\mu(\Omega_\alpha(\epsilon)) \to 0$, the theorem in page 412 of [15] can be applied and one concludes that S is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathscr{H}(\Omega_\alpha(\epsilon), S) \longrightarrow 0. \tag{34}$$

Let $\{\widehat{x}_n\} \subset X$ be an LP α-approximating sequence for ELP(F, ϕ, K). Then, there exists $w_n \in X$ with $w_n \to 0$ such that $\widehat{x}_n + w_n \in K$, and there exist $\widehat{u}_n \in F(\widehat{x}_n)$ and $0 < \epsilon_n' \to 0$ such that

$$\phi(\widehat{u}_n, \widehat{x}_n, y) \leq \frac{\alpha}{2}\|\widehat{x}_n - y\|^2 + \epsilon_n', \quad \forall y \in K,\ n \in N. \tag{35}$$

Since $\widehat{x}_n + w_n \in K$, then there exists $\overline{x}_n \in K$ such that $\widehat{x}_n + w_n = \overline{x}_n$. It follows that

$$d(\widehat{x}_n, K) \leq \|\widehat{x}_n - \overline{x}_n\| = \|w_n\| \longrightarrow 0. \tag{36}$$

Set $\epsilon_n = \max\{\|w_n\|, \epsilon_n'\}$; we get $\widehat{x}_n \in \Omega_\alpha(\epsilon_n)$. From (23) and the definition of $\Omega_\alpha(\epsilon_n)$, we obtain

$$d(\widehat{x}_n, S) \leq e(\Omega_\alpha(\epsilon_n), S) \longrightarrow 0. \tag{37}$$

Since S is compact, there exists $p_n \in S$ such that

$$\|p_n - \widehat{x}_n\| = d(\widehat{x}_n, S) \longrightarrow 0. \tag{38}$$

From the compactness of S, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which converges strongly to $\overline{p} \in S$. Hence, the corresponding subsequence $\{\widehat{x}_{n_k}\}$ of $\{\widehat{x}_n\}$ converges strongly to $\overline{p} \in S$. Thus, ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense. The proof is complete. □

4. Conditions for Levitin-Polyak Well-Posedness

In this section, we get some conditions under which the ELP(F, ϕ, K) in Banach spaces is Levitin-Polyak well-posed.

For any $\delta_0 \geq 0$, we denote $M(\delta_0) = \{x \in X : d_K(x) \leq \delta_0\}$. We have the following result.

Theorem 15. *Let K be a nonempty, closed, and convex subset of X and let $F : X \to 2^{X^*}$ be a $\mathscr{H}$-hemicontinuous and nonempty compact-valued mapping. Let $\phi : X^* \times X \times X \to \mathbb{R}$ be monotone with respect to F, lower semicontinuous in first and second arguments, and concave in third argument. Moreover, $\phi(u, x, x) = 0$, for all $u \in X^*$, $x \in K$. If there exists some δ_0 with $\delta_0 > 0$ such that $M(\delta_0)$ is compact, then ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be an LP approximating sequence for ELP(F, ϕ, K). Then, there exist $0 < \epsilon_n' \to 0$ and $w_n \in X$ with $w_n \to 0$ such that

$$x_n + w_n \in K, \tag{39}$$

and there exists $u_n \in F(x_n)$ satisfying

$$\phi(u_n, x_n, y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n', \quad \forall y \in K,\ n \in N. \tag{40}$$

Since $x_n + w_n \in K$, then there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$. Thus,

$$d(x_n, K) \leq \|x_n - \overline{x}_n\| = \|w_n\| \longrightarrow 0. \tag{41}$$

Let $\epsilon_n = \max\{\epsilon_n', \|w_n\|\}$; we can get $d(x_n, K) \leq \epsilon_n$. Without loss of generality, suppose that $\{x_n\} \subset M(\delta_0)$ for n is sufficiently large. By the compactness of $M(\delta_0)$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\overline{x} \in M(\delta_0)$ such that $x_{n_k} \to \overline{x}$. It is easy to see that $\overline{x} \in K$. Furthermore, by the u.s.c. of F at $\overline{x}$ and compactness of $F(\overline{x})$, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and some $\overline{u} \in F(\overline{x})$ such that $u_{n_k} \to \overline{u}$. Since ϕ is lower semicontinuous in first and second arguments, it follows from (40) that

$$\phi(\overline{u}, \overline{x}, y) \leq \frac{\alpha}{2}\|\overline{x} - y\|^2, \quad \forall y \in K. \tag{42}$$

For any $y \in K$, let $y_t = \overline{x} + t(y - \overline{x})$, for all $t \in (0, 1)$; it is obvious that $y_t \in K$. Now, from (42), we have

$$\phi(\overline{u}, \overline{x}, y_t) \leq \frac{\alpha}{2}\|\overline{x} - y_t\|^2. \tag{43}$$

By the convexity of ϕ, it follows that, for each $t \in (0, 1)$, we obtain

$$\phi(\overline{u}, \overline{x}, y) \leq \frac{\alpha t}{2}\|\overline{x} - y\|^2, \quad \forall y \in K. \tag{44}$$

Let $t \to 0^+$ in the last inequality; then, we have

$$\phi(\overline{u}, \overline{x}, y) \leq 0, \quad \forall y \in K. \tag{45}$$

This shows that $\overline{x}$ solves ELP(F, ϕ, K). Thus, ELP(F, ϕ, K) is strongly LP α-well-posed in the generalized sense. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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